# The Bifurcation of T-periodic Solutions into nT-periodic Solutions and TORI

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With 3 Figures

My lecture on bifurcation and stability of solutions which branch from forced T-periodic solutions is based on the recent work of G. IOOSS and myself [1] and on my forthcoming paper on factorization theorems [2]. In general, forced T-periodic solutions bifurcate into subharmonic solutions with a fixed period $\tau(\tau=nT;\ n=1,2,3,4)$ independent of the amplitude or into a torus [1,3,4,5,6] containing solutions whose analytic properties are not yet fully understood. The subharmonic bifurcating solutions with n=1 are the T-periodic equivalent of a symmetry-breaking bifurcation of steady solutions with other steady solutions. The symmetry breaking flower instability of the axisymmetric climb of a viscoelastic fluid on an oscillating rod [7] which is shown in the movie "Novel Weissenberg effects" by G. S. BEAVERS and myself is one example of such a symmetry breaking T-periodic bifurcation. The solutions on the torus are very roughly the T-periodic equivalent of a Hopf bifurcation, of a steady solution into a periodic solution; like the Hopf bifurcation the solutions on the torus possess frequencies which depend on the amplitude but in the nonautonomous, T-periodic case the variation of these frequencies need not be smooth. A good example of smooth variation of frequencies on a two-dimensional torus appears to describe the observations of SWINNEY, FENSTERMACHER and GOLLUB [8,9] of the oscillatory regimes of flow which follow wavy vortices in the Taylor problem when the Reynolds number is increased.

## 1. Stability and Repeated Bifurcation of Solutions of Nonlinear Evolution Equations in a Single Variable [2,10]

To clarify some aspects of the mathematical nature of bifurcation theory we shall first construct a simple theory for evolution equations in $R_1$. In this theory we give complete and rigorous results for stability and repeated branching which actually apply to the repeated bifurcation of steady solutions in a Banach space at a simple eigenvalue. In the general problem of bifurcation of steady solutions we have in mind steady equilibrium solutions to nonlinear equations possessing different patterns of spatial symmetry. These solutions are points in a Banach space and the families with different symmetries may be projected as plane curves (bifurcation curves). In $R_1$ the projections and the solutions coincide and the theory simplifies enormously.

We are going to study the repeated bifurcation and stability of solutions of the evolution problem

$$V_t + F(\mu,V) = 0 \tag{1.1}$$

where $F(\mu,0) = 0$, $F(0,V) \neq 0$ when $V \neq 0$ and F together with its first two partial derivatives are continuous functions of $\mu$, $V \in R_1$; in particular we have

$$F(\mu,V+V') = F(\mu,V) + F_V(\mu,V)V' + \frac{1}{2} F_{VV}(\mu,V)V'^2 + R(\mu,V,V')V'^3 \quad (1.2)$$

for any $V,V' \in R_1$.

Bifurcating solutions arising from autonomous evolution equations in $R_1$ are necessarily steady. This means that the study of bifurcation in $R_1$ is equivalent to finding branches of solutions of the equation

$$F(\mu,V) = 0 \ . \quad (1.3)$$

Suppose that $V = \varepsilon$ and $\mu = \mu(\varepsilon)$ is a solution of (1.3). Then

$$F(\mu,\varepsilon) = 0 = F_\mu(\mu,\varepsilon)d\mu + F_V(\mu,\varepsilon)d\varepsilon \quad (1.4)$$

We define a point of bifurcation to be a double point of (1.4); that is, a point through which there are two solutions of (1.4), possessing distinct tangents. At such a point

$$F_\mu = F_V = 0 \quad (1.5)$$

and

$$F_{V\mu}^2 - F_{VV}F_{\mu\mu} > 0 \ . \quad (1.6)$$

A disturbance w of $V = \varepsilon$ with $\mu = \mu(\varepsilon)$ satisfies the equation

$$\frac{dw}{dt} + F_V(\mu(\varepsilon),\varepsilon)w + 0(w^2) = 0 \ .$$

Linearizing for small disturbances $w = e^{-\gamma t}w'$, we find that $\gamma(\varepsilon) = F_V(\mu(\varepsilon),\varepsilon)$. The function $\gamma(\varepsilon)$ is nicely described by a <u>factorization theorem for the stability of the solution $V = \varepsilon$, $\mu = \mu(\varepsilon)$</u>:

$$\gamma(\varepsilon) = F_V(\mu(\varepsilon),\varepsilon) = -\mu_\varepsilon F_\mu(\mu(\varepsilon),\varepsilon) \equiv \mu_\varepsilon \hat{\gamma}(\varepsilon) - \mu_\varepsilon\{-\varepsilon F_{V\mu}(0,0) + 0(\varepsilon^2)\} \quad (1.7)$$

The second equality in (1.7) follows from $(1.4)_2$. The third equality is a definition of $\hat{\gamma}(\varepsilon)$ and the fourth follows from expanding $F(\mu(\varepsilon),\varepsilon)$ in powers of $\varepsilon$. We note that the stability of the solution $v = 0$ of (1.1) is governed by $V_t + F_V(\mu,0)V = 0$. Then, with $V = e^{-\sigma t}V'$, we find that $\sigma = F_V(\mu,0)$, so that $\sigma_\mu = F_{V\mu}(0,0)$. If $V = 0$ loses stability strictly as $\mu$ increases past zero then $F_{V\mu}(0,0) < 0$. It follows from (1.7) with $F_{V\mu}(0,0) < 0$ that locally, near $\varepsilon = 0$, subcritical bifurcating solutions $\mu_\varepsilon < 0$ are unstable, $\gamma(\varepsilon) < 0$, and supercritical solutions $\mu_\varepsilon > 0$ are stable, $\gamma(\varepsilon) > 0$.

A point at which $\mu_\varepsilon = 0$ is a stationary point of the bifurcation curve. A point at which $\mu_\varepsilon$ changes sign is a <u>critical point</u> of the bifurcation curve. If $\hat{\gamma}(\varepsilon) \neq 0$ at a critical point then $\gamma(\varepsilon)$ changes sign when $\mu_\varepsilon$ does. If $\hat{\gamma}(\varepsilon) \neq 0$ at a critical point, then $F_\mu(\mu,\varepsilon) \neq 0$ and $(\mu,\varepsilon)$ is not a point of bifurcation (see Fig. 1.1)

Now we show that <u>points at which $\hat{\gamma}(\varepsilon_0) = 0$ and $\gamma_\varepsilon(\varepsilon_0) \neq 0$ are points of bifurcation</u>. At such points (1.7) shows that (1.5) holds and

$$\gamma_\varepsilon(\varepsilon_0) = F_{VV}(\mu(\varepsilon_0),\varepsilon_0) + \mu_\varepsilon(\varepsilon_0)F_{V\mu}(\mu(\varepsilon_0),\varepsilon_0) = -\mu_\varepsilon(\varepsilon_0)F_{\mu V}(\mu(\varepsilon_0),\varepsilon_0) - \mu_\varepsilon^2 F_{\mu\mu}(\mu(\varepsilon_0),\varepsilon_0) \ . \quad (1.8)$$

It follows that at $\varepsilon = \varepsilon_0$

$$d\varepsilon^2 F_{VV} + 2\, d\varepsilon d\mu\, F_{V\mu} + d\mu^2 F_{\mu\mu} = 0 \,. \tag{1.9}$$

The discriminant of (1.9)

$$F_{V\mu}^2 - F_{VV} F_{\mu\mu} \geq 0 \tag{1.10}$$

is not negative. Suppose equality holds in (1.10). Then the second or third term in (1.8) must vanish which is impossible since $\gamma_\varepsilon(\varepsilon_0) \neq 0$. It follows that the inequality is strict and $(\mu(\varepsilon_0), \varepsilon_0)$ is a point of bifurcation

As one example of the foregoing, consider the equation

$$V_t + V(\mu V - 9)(\mu + 2V - V^2) = 0 \tag{1.11}$$

The bifurcating solutions are

(i) $V = 0 \quad \mu \in R_1$,

(ii) $\mu = V^2 - 2V$

and

(iii) $\mu = 9/V$.

The curve (ii) has two bifurcation points and one critical point at $(\mu, v) = (-1, 1)$. The stability of various branches of (1.11) are indicated in Fig. 1.1. It is almost a miracle that examples of secondary bifurcation in $R_1$ as simple as the one just given seem not to have been discussed in the literature on bifurcation. Of course, everyone knows that $F(\mu, V) = 0$ can have multiple solutions.

Our understanding is that bifurcation in $R_1$ is equivalent to the continuous branching of solutions $F(\mu, V) = 0$. This conventional use of the term "bifurcation" is restrictive since it excludes isolated solutions of $F(\mu, V) = 0$ which are not ultimately connected to the solution $V = 0$. The hyperbola $\mu = 9/V$ in the third quadrant of Fig. 1.1 is just one type of isolated solution which can occur.

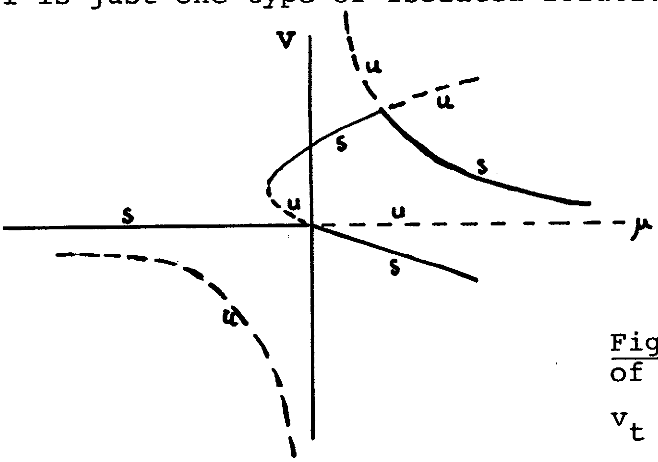



Fig. 1.1 Stability and bifurcation of equilibrium solutions of $v_t = v(9 - \mu v)(\mu + 2v - v^2)$

## 2. Bifurcation and Stability of Solutions Branching from T-periodic Forced Solutions [1]

We turn now to the problem of stability, bifurcation and repeated bifurcation of the nonlinear, nonautonomous, evolution problem

$$\frac{dV}{dt} + F(t,\mu,V) = 0 \tag{2.1}$$

Here, $F(t,\mu,V)$ is a nonlinear, T-periodic $(F(t,\cdot,\cdot) = F(t+T,\cdot,\cdot))$, map from $R \times \mathbb{C} \times H$ into H, where H is a Hilbert space with natural scalar product $(u,v)_H = \overline{(u,v)}_H$, which carries real vectors $V \in H$ into real vecters when $\mu \in \mathbb{C}$ is real. It is further assumed that $F(t,\mu,V)$ is an analytic operator with a Fréchet expansion

$$F(t,\mu,V+w) = F(t,\mu,V) + F_V(t,\mu,V|w) + \frac{1}{2} F_{VV}(t,\mu,V|w,w) + 0(w^3) \tag{2.2}$$

for $t \in R$, V, $w \in X$ = domain $(F_V) \supset H$(compactly) and $\mu$ in a neighborhood of a real interval of $\mathbb{C}$. It is assumed that $F(t,\mu,0) = 0$ so that $V = 0$ is a solution of (2.1).

We may identify (2.1) with a partial differential equation in which V is the difference between two solutions driven by prescribed T-periodic data. One of the two solutions is T-periodic and it accounts for the appearance of t in $F(t,\mu,V)$. The solution $V = 0$ of (2.1) corresponds to a forced T-periodic solution of the original problem. When the data is steady, the same type of analysis leads to an autonomous problem

$$\frac{dV}{dt} + F(\mu,V) = 0 \ . \tag{2.3}$$

The evolution problems (2.2) and (2.3) have very different properties [2].

The stability of the solution $V = 0$ of (2.1) to small disturbances may be determined by Floquet analysis of the variational equations

$$\frac{dz}{dt} + F_V(t,\mu,0|z) = 0 \ . \tag{2.4}$$

According to Floquet theory we may determine the stability of zero by studying the Floquet exponents $-\sigma(\mu)$ of the representation $z = e^{-\sigma(\mu)t}\zeta(t)$ where $\zeta(t) = \zeta(t+T)$ and $\sigma(\mu) = \xi(\mu) + i\omega(\mu)$. These exponents are eigenvalues of the operator $J(\mu)$; that is,

$$-\sigma\zeta + J(\mu)\zeta = 0 \tag{2.5}$$

where $J = d/dt + F_V(t,\mu,0|\ )$. The operator $J(\mu)$ is a Fredhom operator (with a compact resolvent) taking T-periodic vectors in X into T-periodic vectors in H; that is $J(\mu): X_T \to H_T$. The scalar product on $H_T$ is

$$[u,v]_T = \frac{1}{T}\int_0^T (u,v)_H dt. \tag{2.6}$$

The Floquet exponents are the exponents of the Floquet multiplier $\lambda(\mu) = \exp(-\sigma(\mu)T) = \exp(-\sigma(\mu)T+2\pi ik)$. The Floquet multiplier is an eigenvalue of the monodromy operator. This operator is the analogue of the monodromy matrix for ordinary differential equations (see, for example section 7 of [7]). The monodromy matrix is the fundamental solution matrix whose values at $t = 0$ coincide with the unit matrix. This matrix can be regarded as a map $z(0) \to z(T)$. The same type of map has been defined by IOOSS [5] for evolution equations on Banach spaces. In this case, $\lambda(\mu)$ are the eigenvalues of linear compact operator $S_\mu(T)$

in X mapping $z(0)$ into $z(T)$.

When $\xi(\mu) > 0$ for all eigenvalues of $J(\mu)$, then $V = 0$ is stable. We assume (H.1) that $V=0$ loses stability strictly as $\mu$ is increased past zero: $\sigma(0) = \sigma_0 = i\omega_0$, $\xi,_\mu(0) < 0$. At criticality ($\mu=0$), $J(0) \equiv J_0$ and $\lambda(0) \equiv \lambda_0$ is of modulus one. We may represent the Floquet exponent $-\sigma_0 = -i\omega_0$ at criticality with

$$\omega_0 = \frac{2\pi}{T}(r+k),\ 0 \le r < 1,\ k = 0, \pm 1, \pm 2, \ldots$$

Without losing generality, we may set $k = 0$ [1] because repeated points on the imaginary axis of the complex $\sigma$ plane correspond to unique points

$$\lambda_0 = e^{-2\pi i r/T} \tag{2.7}$$

on the unit circle of the complex $\lambda$ plane. Problems with $k \neq 0$ may always be reduced to the one with $k=0$; if $\hat{\zeta}(t) = \hat{\zeta}(t+T)$ is an eigenfunction of (2.5) with $k \neq 0$ the $\hat{\zeta}(t) = \hat{\zeta}(t+T) = e^{2\pi ikt/T}\zeta(t)$ is also an eigenfunction of (2.5) with $k = 0$.

We now study bifurcation under the hypothesis (H.1) and (H.2): Assume that $-2\pi ir/T$ is an algebraically simple eigenvalue of $J_0$. (Then $\lambda_0$ is a simple eigenvalue of $S_0(T)$.) If $\zeta(t)$ is the eigenvalue of $J_0$ belonging to $\sigma_0 = 2\pi ir/T$, then $\overline{\zeta}(t)$ is another eigenvector of $J_0$ belonging to $\overline{\sigma}_0 = -2\pi ir/T$. Hence $z(t)=e^{-2\pi irt/T}\zeta(t)$ and $\overline{z}(t)$ both solve (2.4) when $\mu = 0$. In the analysis of bifurcation we must consider all values of $\lambda_0$ on the unit circle; that is, all values $r$, $0 \le r < 1$. If $r$ is irrational then $z(t)$ and $\overline{z}(t)$ are independent and zero is a semi-simple double eigenvalue of the operator

$$\mathbf{J}_0 = \frac{d(\cdot)}{dt} + F_V(t,0,0|\cdot) \tag{2.8}$$

in a space of doubly-periodic functions $f(\frac{2\pi t}{T}, \frac{2\pi rt}{T})$. In this case, we get a bifurcating two-dimensional torus. When the amplitude $\varepsilon$ of the torus small, the principal part of the solution on the torus is doubly periodic function with a frequency $2\pi/T$ and a frequency $\omega(\varepsilon)$ which varies with $\varepsilon$ and is such that $\omega_0 = 2\pi r/T$ [5,11]. The variation of the frequency $\omega(\varepsilon)$ of the solutions on the torus need not be smooth. The analytical properties of solutions on bifurcating tori in Navier-Stokes and other problems are not well understood.

The complement of the set of irrational numbers on $0 \le r < 1$ is the set of rational fractions $m/n$, $m<n$. Of course these fractions are dense on $(0,1)$. It would be unfortunate if the bifurcation results depended in any important way on the difference between the irrational values of $r$ and the rational fractions. Fortunately, this difference does not exist as a general feature; only the values $r = 0, 1/2, 1/3, 2/3, 1/4, 3/4$ are special; they lead to subharmonic periodic solutions with the property that their period is a fixed multiple of $T$, independent of $\varepsilon$.

We can organize the motivate the study of subharmonic bifurcation at a rational fraction in the following way. A subharmonic solution $z(t)$ is an $nT$-periodic function; hence

$$z(t+nT)=e^{\frac{-2\pi ir}{T}(t+nT)}\zeta(t+nT) = e^{-2\pi irn}z(t) = z(t).$$

if and only if

$$e^{-2\pi irn} = 1 = \lambda_0^n = \bar{\lambda}_0^n \ . \tag{2.9}$$

It follows that the subharmonic solution z(t) can exist with $0 \leq r < 1$ if and only if $\lambda_0$ is the nth root of unity and $r = m/n$, $m = 0,1,2,\ldots;$ $n = 1,2,\ldots,$ $m < n$.

To construct subharmonic bifurcating solutions we introduce the domain space $X_{nT}$ = domain $\mathcal{J}_0$ and the range space $H_{nT}$ with scalar product defined by (2.6) with nT replacing T. There is an adjoint $\mathcal{J}_0^*$ relative to $[\cdot,\cdot]_{nT}$ and, if

$$z^* = {}^{-2\pi imt/n}\ \zeta^*(t), \tag{2.10}$$

where

$$2\pi i\frac{m}{n}\ \zeta^* + J_0^*\ \zeta^* = 0,\ \zeta^*(t) = \zeta^*(t+T),$$

is complex, then $z^*$ and $\bar{z}^*$ span the null space of $\mathcal{J}_0^*$. It is easy to verify that $[z,\bar{z}^*]_{nT} = 0$ and we may take $[z,z^*]_{nT} = 1$.

We may take z and z* as real-valued when $\lambda_0 = e^{-2\pi im/n}$ is real; that is when $n = 0$, $\lambda_0 = 1$ and $m/n = 1/2$, $\lambda_0 = -1$. When $m = 0$, $z = \zeta(t+T)$ is real and T-periodic; that is $n=1$. When $n/m = 1/2$, $z = e^{-\pi it/T}\ \zeta = e^{\pi it/T}\bar{\zeta}$ is real and 2T-periodic. In these two cases, and only these two, $\mathcal{J}_0$ has a one-dimensional null space.

Now I shall indicate how the subharmonic solutions can be constructed by analytic perturbation theory. The detailed demonstrations are given in [1]. Assume that there are subharmonic bifurcating solutions $V = U(t,\varepsilon) = U(t+nT,\varepsilon)$, $\mu = \mu(\varepsilon)$ which are analytic in some neighborhood $I(\varepsilon)$ of the origin. Then introducing the notation $(\cdot)_n = d^n(\cdot)/d\varepsilon^n$ and for short, $F_V(\mu,U|U_1) = F_V(U_1)$, etc, which suppresses the dependence of the operators on t, $U(t,\varepsilon)$ and $\mu(\varepsilon)$, we find that

$$\frac{dU}{dt} + F(t,\mu,U) = 0\ , \tag{2.11}$$

$$\frac{dU_1}{dt} + F_V(U_1) + \mu_1\ F_\mu = 0\ , \tag{2.12}$$

$$\frac{dU_2}{dt} + F_V(U_2) + F_{VV}(U_1,U_1) + 2\mu_1\ F_{V\mu}(U_1) + \mu_1^{\ 2}\ F_{\mu\mu} + \mu_2 F_\mu = 0\ , \tag{2.13}$$

$$\frac{dU_3}{dt} + F_V(U_3) + F_{VVV}(U_1,U_1,U_1) + 3\ F_{VV}(U_1,U_2) \tag{2.14}$$

$$+ 3\mu_1\ F_{V\mu}(U_2) + 3\mu_1\ F_{VV\mu}(U_1,U_1) + 3\mu_2\ F_{\mu V}(U_1)$$

$$+ 3\mu_1^{\ 2}\ F_{V\mu\mu}(U_1) + \mu_2\mu_1\ F_{\mu\mu} + \mu_3\ F_\mu = 0\ .$$

Existence and stability properties of the subharmonic bifurcating solutions near $\varepsilon = 0$ are determined by these equations and the equation for $U_4$.

When $\varepsilon = 0$, $F_\mu(0,0) = F_{\mu\mu}(0,0) = 0$ and the first two terms of

(2.12), (2.13) and (2.14) may be replaced by $\mathcal{J}_0 U_n$, n = 1,2,3. Since, $J_0 U_1 = 0$ and zero is a simple eigenvalue of $\mathcal{J}_0$ (when n=1 or n=2) or a semi-simple double eigenvalue of $\mathcal{J}_0$ we have

$$U_1 = az + \overline{az} \ . \tag{2.15}$$

When n=1 or 2, $z=\overline{z}$ and $a = \overline{a}$ is determined by the normalization associated with the definition of $\varepsilon$. In all other cases this normalization gives one relation between a and $\overline{a}$ and the ultimate values of these quantities is determined by the conditions for solvability. These conditions arise from the Fredholm alternative for $\mathcal{J}_0$:

Let (H.1) and (H.2) hold. Then there is $u \in X_{nT}$ solving

$$\mathcal{J}_0 \, u = f \in H_{nT} \tag{2.16}$$

if and only if

$$[f,z^*]_{nT} = [f,\overline{z}^*]_{nT} \ , \tag{2.17}$$

If f is real-valued one of the conditions (2.17) implies the other and

$$u = bz + \overline{bz} + \omega \tag{2.18}$$

where $\omega = \mathcal{J}_0^{-1} f$ is unique.

It follows from this statement of the Fredholm alternative that

$$U_\ell = a_\ell z + \overline{a}_\ell \overline{z} + w_\ell, \ \ell > 1$$

and, using (2.15),

$$\mathcal{J}_0 w_2 + F_{VV}(az + \overline{az},\ az + \overline{az}) + 2\mu_1 \ F_{V\mu}(az + \overline{az}) = 0 \ . \tag{2.19}$$

(2.19) is solvable if and only if

$$a^2[F_{VV}(z,z),z^*]_{nT} + \overline{a}^2[F_{VV}(\overline{z},\overline{z}),z^*]_{nT} + 2|a|^2[F_{VV}(z,\overline{z}),z^*]_{nT}$$

$$+ 2\mu_1 a \ \sigma_\mu = 0 \tag{2.20}$$

where, by a standard perturbation result using (2.5) and (2.17)

$$\sigma_{,\mu} = [F_{V\mu}(z),z^*]_{nT} \ . \tag{2.21}$$

We are assuming that the real part $\xi_{,\mu}$ of $\sigma_{,\mu}$ is negative at criticality; that is, V = 0 loses stability strictly as $\mu$ crosses zero.

To evaluate (2.2), we make use of the following computational lemma. Let $g(t) = f(\varepsilon)\exp(2\pi imt/n) = g(t+mT)$. Then

$$[f,z^*]_{nT} = [g,\zeta^*]_{nT} = 0 \ . \tag{2.22}$$

(2.22) may be proved by direct computation. Application of (2.22) to (2.20) leads to the conclusion that

$$\mu_1 = 0 \text{ for all n except } n = 1 \text{ and } n = 3 \ . \tag{2.23}$$

We may conclude that if solutions with n = 2,4,5,6,7,... bifurcate and $\mu_2 \neq 0$ they bifurcate on one side of criticality. In general,

$$\mu_1 \neq 0 \text{ when } n = 1 \text{ or } n = 3. \qquad (2.24)$$

The T-periodic and 3T-periodic solutions bifurcate on both sides of criticality.

Supposing now that (2.23) holds we find that (2.14) is solvable if and only if

$$3\mu_2\, \sigma_\mu\, a + 3[F_{VV}(U_1,U_2),z^*]_{nT} + 3[F_{VVV}(U_1,U_1,U_1),z^*]_{nT} = 0. \qquad (2.25)$$

When $n \geq 5$ (2.25) we find, using (2.22), that (2.25) is in the form

$$\mu_2\sigma_\mu + |a|^2 f = 0$$

which has no solution except for the special case in which $f/\sigma_\mu$ is real. Hence, in general, there is no nT periodic bifurcation for $n \geq 5$.

Equation (2.25) must be studied when n=2 and n=4. Equation (2.20) must be studied when n=1 and n=3. I have already noted when n=1 or n=2, z is real and $a=\bar{a}$ is determined by normalization. In this case, the analysis of bifurcation and the stability of bifurcation follows along by now classical lines. For n=3, (2.20) is a cubic equation and for n=4, (2.25) is a quartic equation for the real or imaginary part of a. (We may eliminate, say, the imaginary part by enforcing a normalizing condition connected to the definition of the amplitude $\varepsilon$.) Having once determined the number of real roots of the cubic equation (2.20) or the quartic equation (2.25) for allowed values of a we must then verify that the higher order perturbation equations are solvable. This will be true at all orders if it is true at one order beyond (2.20) or (2.25).

The results of the study of bifurcation and stability of solutions branching from T-periodic ones is given in Fig. 2.1, see page 9.

## 3. Factorization Theorems for the Stability of nT-bifurcating Solutions

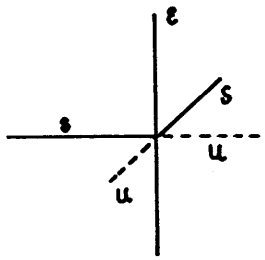
We now relax the assumption that the amplitude $\varepsilon$ of the $\tau$ = nT periodic bifurcating solutions is small and undertake the study of the stability of these solutions under the hypothesis H.3: $U(\tau,\varepsilon) = U(t+nT,\varepsilon)$, $\mu(\varepsilon)$ is a $\tau$ = nT-periodic bifurcating solution which is analytic on some possibly large interval $I(\varepsilon)$ . I do not require that $I(\varepsilon)$ be a neighborhood of the origin so that the factorization theorems will apply to isolated solution branches and to branches of solutions which arise from repeated bifurcations.

The stability of $U(t,\varepsilon)$ is governed, in the linearized approximation, by

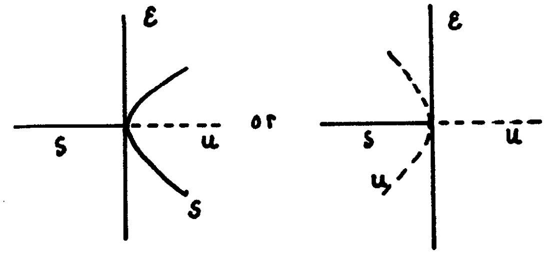
$$\frac{dw}{dt} + F_V(t,\mu(\varepsilon),U(t,\varepsilon)|w) = 0 \qquad (3.1)$$

where $F_V(t,\cdot,\cdot|\cdot)$ is T-periodic and $U(t,\varepsilon)$ is $\tau$ periodic. The spectral problem associated with (3.1) may be obtained from the Floquet representation $w = e^{-\gamma t}\Gamma(t)$ where $\Gamma(t)$ is $\tau$ = nT periodic and

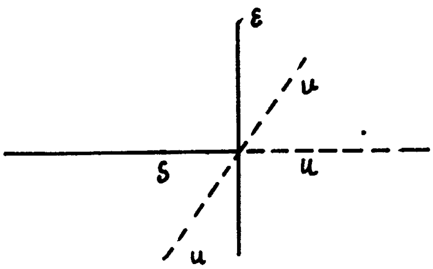
$$-\gamma(\varepsilon)\Gamma + \mathbf{J}(\varepsilon)\Gamma = 0 \qquad (3.2)$$



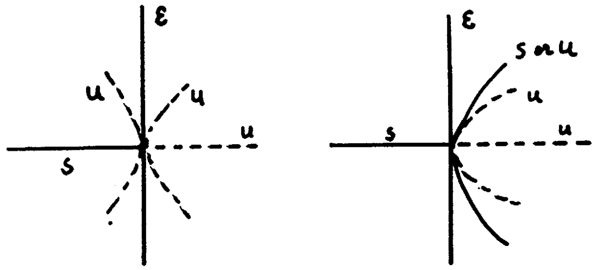

n=1. A single one-paramerter family of T-periodic solutions bifurcate on both sides of criticality at a simple eigen-value




n=2. A single one-parameter family of 2T-periodic solutions bifurcate on one side of criticality at a simple eigenvalue




n=3. A single one-parameter family of 3T-periodic solu-tions bifurcates on both sides of criticality at a semi-simple double eigenvalue




n=4. There are three alternatives for the bifurcation of 4T-periodic solution at a semi-simple double eigenvalue: (a) Two solutions bifur-cate, each on a different side of criticality and both are unstable, (b) Two solutions bifurcate on the same side criticality and one is un-stable, the stability of the other being determined by details varying from problem to problem, (c) no 4T-periodic solutions bifurcate

<u>Fig. 2.1</u> Bifurcation of forced T-periodic solutions. The period of nT-periodic solutions with n=1,2,3,4 is independent of the amplitude. For all values of $r \neq 0$, 1/2, 1/3, 2/3, 1/4, 3/4 a torus bifurcates on one side of criticality. The supercritical torus is stable and the subcritical torus is unstable. The frequencies of the solutions on the torus vary continuously with amplitude but the variation need not be differentiable.

where $\mathcal{J}(\varepsilon)(\cdot) = (\cdot)_{,t} + F_V(t,\mu(\varepsilon),U(t,\varepsilon)\ (\cdot))$ maps $X_{nT}$ into $H_{nT}$ and satisfies the hypothesis

<u>H.4: $\mathcal{J}(\varepsilon)$ is a Fredholm operator with a compact resolvent from $X_{nT}$ into itself.</u>

Since $\mathcal{J}(\varepsilon)$ has a compact resolvent its spectrum is of eigenvalues of at most finite multiplicity and there is an adjoint $\mathcal{J}^*(\varepsilon)$ into $H_{nT}$ such that

$$-\bar{\gamma}(\varepsilon)\Gamma^* + \mathcal{J}^*\ \Gamma^* = 0 \tag{3.3}$$

where $\bar{\gamma}$ is the conjugate of $\gamma$. The number of independent eigenvectors $\Gamma_i$ belonging to $\gamma(\varepsilon)$ is the dimension of the null space of $-\gamma(\varepsilon) + \mathcal{J}(\varepsilon)$

which is the same as the dimension of the null space of the adjoint operator $-\gamma(\varepsilon) + \mathbb{J}^*$.

The last hypotheses which we need to prove the factorization is

H.5: $\gamma(\varepsilon)$ is an algebraically simple eigenvalue of $\mathbb{J}(\varepsilon)$ for all $\varepsilon \in I(\varepsilon)$ except possibly on an exceptional set of isolated points across which $\Gamma(t,\varepsilon)$ and $\Gamma^*(t,\varepsilon)$ are continuous. H.4 implies that $\gamma(\varepsilon)$ is continuous across points in the exceptional set. We may normalize

$$[\Gamma(\varepsilon),\Gamma^*(\varepsilon)]_{nT} = 1 \tag{3.4}$$

at all points where $\gamma(\varepsilon)$ is simple, and by continuity also on points in the exceptional set.

Suppose H.3, H.4, H.5 hold and assume that

$$[U_\varepsilon(\varepsilon),\Gamma^*]_{nT} \neq 0 \; . \tag{3.5}$$

Then there is a unique continuous function $\hat{\gamma}(\varepsilon)$ defined on all $I(\varepsilon)$ such that

$$\gamma(\varepsilon) = \mu_\phi(\varepsilon)\hat{\gamma}(\varepsilon) \tag{3.6}$$

where

$$\hat{\gamma}(\varepsilon) = - [F_\mu(\mu(\varepsilon),U(\varepsilon),\Gamma^*]_{nT}/[U_\varepsilon,\Gamma^*]_{nT} \; . \tag{3.7}$$

Moreover,

$$\Gamma = b(\varepsilon)(U_\varepsilon + \mu_\varepsilon q) \tag{3.8}$$

where $b(\varepsilon)$ is a normalizing factor for $\Gamma$ and

$$q(t,\varepsilon) = q(t + nT,\varepsilon) \tag{3.9}$$

is uniquely determined by

$$\hat{\gamma}U_\varepsilon + F_\mu(\mu(\varepsilon),U(\varepsilon)) + \{\gamma-\mathbb{J}\}\, q = 0 \tag{3.10}$$

and

$$[q,\Gamma^*]_{nT} = 0 \; . \tag{3.11}$$

Proof: (2.12) may be written as

$$\mathbb{J}U_\varepsilon + \mu_\varepsilon F_\mu(\mu(\varepsilon),U(\varepsilon)) = 0 \, . \tag{3.12}$$

Since $\Gamma^*$ satisfies (3.3) we have

$$-\mu_\varepsilon [F_\mu,\Gamma^*]_{nT} = [\mathbb{J}U_\varepsilon,\Gamma^*]_{nT} = [U_\varepsilon,\mathbb{J}^*\Gamma^*]_{nT} = \gamma(\varepsilon)[U_\varepsilon,\Gamma^*]_{nT} \; . \tag{3.13}$$

Equation (3.12) holds at all points where $\gamma(\varepsilon)$ is an algebraically simple eigenvalue and also, by continuity, across points in the exceptional set where $\gamma(\varepsilon)$ is not algebraically simple. Solving (3.13) for $\gamma(\varepsilon)$ we find (3.7). Now combining (3.2) and (3.8) we get

$$-\mu_\varepsilon\hat{\gamma}(\varepsilon)(U_\varepsilon+\mu_\varepsilon q) + \mathbb{J}U_\varepsilon + \mu_\varepsilon \mathbb{J}q = 0$$

Elimination of $JU_\varepsilon$ with (3.12) leads to

$$\mu_\varepsilon\{\hat{\gamma}U_\varepsilon + F_\mu(\mu,U) + [\gamma-\mathbb{J}]_q\} = 0 \tag{3.14}$$

The coefficient of $\mu_\varepsilon$ in (3.14) vanishes when $\mu_\varepsilon \neq 0$ and, by continuity, even when $\mu_\varepsilon = 0$ at a point. This proves (3.10). Since $-\gamma+\mathbb{J}$ is Fredholm and (3.13) holds, (3.14) is uniquely solvable with $[q,\Gamma^*] = 0$ wherever $\gamma(\varepsilon)$ is algebraically simple. So we get a unique q when $\gamma(\varepsilon)$ is algebraically simple and, by continuity, at isolated points in the exceptional set. This proves the factorization theorem for nT-periodic bifurcating solutions.

Remark 1: If re$\hat{\gamma}(\varepsilon_0)\neq 0$ at a point $\varepsilon_0$ where $\mu(\varepsilon)$ changes sign (such points are called critical points or turning points), then re $\gamma(\varepsilon)$ changes sign as $\varepsilon$ crosses $\varepsilon_0$. This remark suggests that in most problems the bifurcating solutions will gain or lose stability across a critical point

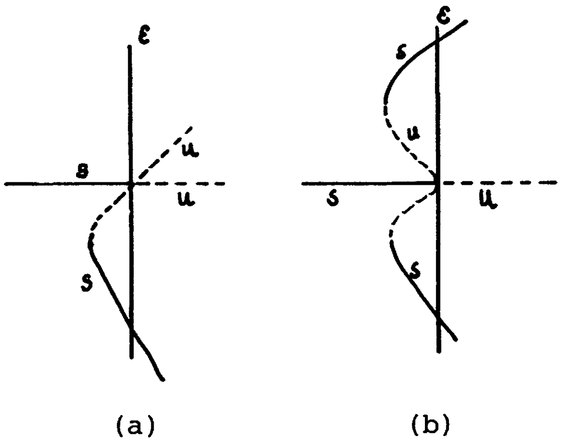



Fig. 3.1 Recovery of stability at critical points of bifurcation curve. (a) is the conjectured form of the global extension of 3T-periodic bifurcating solutions (b) is the conjectured form of any unstable subcritical one-sided bifurcating solution; say, n=4.

Remark 2: In most problems $\gamma(\varepsilon)$ is an algebraically simple eigenvalue of $\mathbb{J}(\varepsilon)$ for nearly all values of $\varepsilon$. The stability of the bifurcating solution $U(t,\varepsilon)$ is controlled by the eigenvalue $\gamma(\varepsilon)$ with the smallest re $\gamma(\varepsilon)$. The factorization may be used to calculate the stability of the bifurcating solution at $\varepsilon=0$ when $\gamma_0 = 0$ is a simple eigenvalue of $\mathbb{J}_0$; that is, when n=1 or n=2. In this case we find that when $\varepsilon \to 0$; $\mu \to 0$,

$$(\Gamma_0,\Gamma_0^*)\to(z(t),z^*(t)) = (z(t+nT),z^*(t+nT)),$$

$$U(t,\varepsilon) \to \varepsilon z + O(\varepsilon^2),$$

and

$$F_\mu(t,\mu,U) \to \varepsilon F_{\mu V}(t,0,0\ z).$$

Since $[z,z^*]_{nT} = 1$, $[U_\varepsilon,\Gamma^*]_{nT} \to 1$ and using (3.7) and (2.21), we find that when $\varepsilon$ is small

$$\hat{\gamma}(\varepsilon) \simeq -\varepsilon[F_{\mu V}(t,0,0\ z),z^*]_{nT} = -\varepsilon\sigma_{,\mu}\ . \tag{3.15}$$

Equations (3.6) and (3.15) implies the local ($\varepsilon \to 0$) statement of stability for T-periodic and 2T-periodic bifurcating solutions: subcritical solutions are unstable and supercritical solutions are stable.

When n=3 or n=4 the analysis of the stability of the bifurcating solutions requires that one construct a perturbation analysis of a semi-simple double eigenvalue to separate the branches of $\gamma(\varepsilon)$. Without such an analysis it would not be possible to specify the linear combinations of $(z,\bar{z})$ and $(z^*,\bar{z}^*)$ which give the limiting $\varepsilon \to 0$ value of $\Gamma(\varepsilon)$ and $\Gamma^*(\varepsilon)$.

Factorization theorems can be used to characterize points of secondary and repeated bifurcation [2] at a simple eigenvalue. Factorization theorems for autonomous problems may also be proved [2,7]. It is interesting that the factorization theorem for periodic bifurcating solutions of the Hopf type show that the eigenvalue $\gamma=0$ is always an algebraically double eigenvalue of the appropriate operator. This algebraically double eigenvalue is geometrically simple in the general case. In the special case the derivative of the frequency $\dot{\omega}(\varepsilon)$ of the Hopf solution with respect to $\varepsilon$ vanishes when $\gamma(\dot{\varepsilon})$ does. ($\varepsilon=0$ is a special case). If $\omega,_{\varepsilon}(\varepsilon) = \gamma(\varepsilon) = 0$ then $\gamma(\varepsilon)$ is a semi-simple double eigenvalue [2].

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