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Part II: Experiments

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In a recent work (JOSEPH, 1976), ideas from the theory of domain perturbations were used to develop an algorithm for the computation of unsteady motions of a simple fluid. In this algorithm, the rest state is perturbed with an unsteady motion. The solution is expressed in powers of the amplitude ε of the unsteady *data*; the stress is expanded into a Fréchet series; the Fréchet stresses are represented as multiple integrals and the multiple integrals are reduced to canonical form appropriate for the solution of practical problems of rheological fluid mechanics. In Part I of this paper we apply JOSEPH's algorithm to find the shape of the free surface on a simple fluid between cylinders undergoing torsional oscillations. In Part II we describe an experiment testing the predictions of Part I. Good agreement is demonstrated for all experimental tests of the theory when the amplitude of the oscillation is small.

Part I: Theory

1. Mathematical Formulation

JOSEPH & FOSDICK (1973) formulated the free surface problem between cylinders when the data, the prescribed angular frequency of the cylinder walls, is steady. We treat the same problem when the angular frequency of the inner cylinder varies sinusoidally in time (see Fig. 1). In this paper we shall be concerned only with those aspects of the problem which bear upon the solution of rheological problems with unsteady data.

Interesting features of the solution which also occur in the steady problem [JOSEPH & FOSDICK, 1973; JOSEPH, BEAVERS & FOSDICK, 1973; BEAVERS & JOSEPH, 1975] are not emphasized in this paper.

The region occupied by the incompressible fluid under the free surface $z=h(r,t,\varepsilon)$ is designated as

$$\mathscr{V}_\varepsilon=[r,\theta,z\colon a\leqq r\leqq b,\ z\leqq h(r,t,\varepsilon)].$$

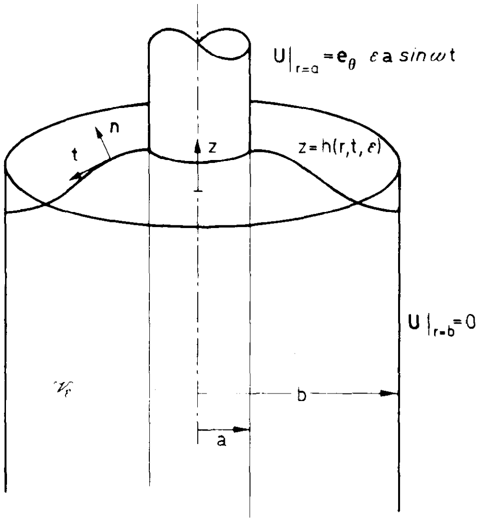


Fig. 1. The free surface on a fluid between cylinders in torsional oscillation.

The stress is denoted by

$$T = -p\mathbf{1} + S$$

where $-p\mathbf{1}$ is the constitutively indeterminate part of T and S is the extra stress. In NOLL's (1958) theory of the simple fluid,

$$S = \mathscr{F}_{s=0}^{\infty}[G(s)] \tag{1.1}$$

where $\mathscr{F}$ is a response functional whose argument functions are histories:

$$G(s) = C_t(t-s) - \mathbf{1}, \tag{1.2}$$

$$C_t(\tau) = F_t^T(\tau)\ F_t(\tau), \tag{1.3}$$

$$F_t(\tau) = \nabla_x \chi_t(x, \tau) \tag{1.4}$$

where

$$\xi = \chi_t(x, \tau) \qquad (\chi_t(x, t) = x)$$

is the position of the particle which is presently at x at an earlier time $\tau < t$. It is convenient to introduce a symbol

$$\Phi = p + \rho g z$$

for the head.

Other symbols used in the analysis are listed below.

(r, θ, z), (e_r, e_θ, e_z)	Polar cylindrical coordinates, coordinate base vectors.
$U, (U, V, W)$	Velocity, physical components of velocity.
$\omega, T, \rho, \mu, p_a, g = -e_z g$	Frequency, surface tension, density, viscosity, atmospheric pressure and gravity.
$\Omega(r, z, t, \varepsilon) = V(r, z, t, \varepsilon)/r$	Angular frequency function.
$\Omega(a, z, t, \varepsilon) = \varepsilon \sin \omega t$	Definition of ε.

$\boldsymbol{n}, \boldsymbol{t}$	Outward unit normal to the free surface, tangent vector in the intersection of the free surface and the plane $\theta = \text{const.}$
$f_{,r}; f_{,z}; f_{,t}$	Partial derivatives of $f(r, z, t)$.
$\boldsymbol{X}, (R, \theta, Z)$	Position vector of the particles in the rest state when $\varepsilon = 0$. Coordinates in the rest state.
$\mathscr{V}_\varepsilon, \mathscr{V}_0$	Volume occupied by the fluid. Volume occupied by the fluid in the rest state.
$f^{[n]}(\boldsymbol{X}, t), f^{\langle n \rangle}(\boldsymbol{X}, t)$	Defined for functions $f(\boldsymbol{x}, t, \varepsilon)$: see (2.4) and (2.5).
$t, \tau < t, s = t - \tau$	Times.
$\boldsymbol{\xi}(\boldsymbol{X}, \tau, \varepsilon) = \boldsymbol{\chi}_t(\boldsymbol{x}, \tau)$ $\boldsymbol{\xi}(\boldsymbol{X}, t, \varepsilon) = \boldsymbol{x}$	Position vectors for the particle $\boldsymbol{X}$: (see (2.1) and (2.2)).
$((\boldsymbol{\xi}^{[n]})) = \boldsymbol{\xi}^{[n]}(\boldsymbol{X}, \tau) - \boldsymbol{\xi}^{[n]}(\boldsymbol{X}, t)$	Derivatives of the position vector difference.
$\boldsymbol{G}(s), \boldsymbol{G}(s, \varepsilon), G(s)$	History, history with parameter ε, shear relaxation modulus.
$\boldsymbol{A}(s) = \boldsymbol{A}_1[\boldsymbol{U}^{\langle 1 \rangle}(\boldsymbol{X}, t-s)] \equiv \nabla_{\boldsymbol{X}} \boldsymbol{U}^{\langle 1 \rangle}(\boldsymbol{X}, t-s) + \nabla_{\boldsymbol{X}} \boldsymbol{U}^{\langle 1 \rangle}(\boldsymbol{X}, t-s)^T$	First Rivlin-Ericksen tensor regarded as a functional of $\boldsymbol{U}$ with $\boldsymbol{U} = \boldsymbol{U}^{\langle 1 \rangle}(\boldsymbol{X}, t-s)$.
$\nabla_{\boldsymbol{X}}, \nabla_{\boldsymbol{\xi}}, \nabla$	Gradient in $\mathscr{V}_0$, gradient in $\mathscr{V}_\varepsilon$ with respect to coordinates ξ_i, gradient in $\mathscr{V}_\varepsilon$ with respect to coordinates x_i but, after (5.9), we take $\nabla \equiv \nabla_{\boldsymbol{X}}$.

The following quantities appear in the study of rod-climbing starting with Section 5.

$\overline{(\cdot)} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} (\cdot)\, dt$	Definition of the mean.
$\Theta, \varepsilon = \omega\Theta/2, \overline{\overline{\Omega^2}}$	Angle of twist, amplitude (6.8), mean-square angular frequency of the rod (6.9).
$\bar{h}(R, \omega, \varepsilon), \bar{\hbar}(R, \omega)$	Mean height rise functions.
$H = \bar{h}/\Theta^2, H_{(N,M)}$	Normalized height rise (6.10), normalized height rise for the (N, M) generalized Maxwell model.
$\hat{\beta}_\Lambda(\omega); \boldsymbol{A}^2(\omega); \Lambda^2(\omega), \lambda^2; \Lambda_r(\omega), \Lambda_i(\omega)$	(7.10); (5.6d); (7.5); (7.8).

Now we shall list the equations which govern the motion (see JOSEPH & FOSDICK (1973) and BEAVERS & JOSEPH (1974) for a more detailed derivation in the steady case). The motion is assumed to be axisymmetric.

(I) Field equations, $\boldsymbol{x} \in \mathscr{V}_\varepsilon$:

$$\operatorname{div} \boldsymbol{U} = 0, \tag{1.5a}$$

$$\rho \left[\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} \right] = -\nabla \Phi + \nabla \cdot \boldsymbol{S}. \tag{1.5b}$$

(II) Cylinder wall equations:

$$\boldsymbol{U}|_{r=a} = \boldsymbol{e}_\theta \varepsilon a \sin \omega t, \tag{1.5c}$$

$$\boldsymbol{U}|_{r=b} = 0. \tag{1.5d}$$

(III) Asymptotic conditions, $z \to -\infty$:

$\Phi(r, z, t, \varepsilon)$ and the components of $\boldsymbol{U}$ and $\boldsymbol{S}$ are independent of z. (1.5e)

(IV) Interface conditions, $z = h(r, t, \varepsilon)$:

(a) The circumferential component ($S_{n\theta}$) of the shear stress vanishes,

$$S_{z\theta} - h_{,r} S_{r\theta} = 0. \tag{1.5f}$$

(b) The tangential component (S_{nt}) of the shear stress vanishes,

$$h_{,r}(S_{zz} - S_{rr}) + (1 - h_{,r}^2)\, S_{rz} = 0. \tag{1.5g}$$

(c) The kinematic condition holds:

$$W = h_{,t} + U h_{,r}. \tag{1.5h}$$

(d) The jump in the normal stress is balanced by surface tension:

$$p_a - \Phi + S_{zz} - h_{,r} S_{rz} + \rho g h = \frac{T}{r}\left(r h_{,r}/\sqrt{1 + h_{,r}^2}\right)_{,r}. \tag{1.5i}$$

(V) Mean height condition:

$$\frac{2}{b^2 - a^2} \int_a^b r h(r, t, \varepsilon)\, dr = 0. \tag{1.5j}$$

The mean height condition arises from the requirement that the total volume of the incompressible fluid is unchanged during the motion.

(VI) Boundary conditions for $h(r, t, \varepsilon)$ at $r = a, b$:
We require boundary conditions compatible with a flat free surface in the rest state $h(r, t, 0) = h^{\langle 0 \rangle} \equiv 0$. For example, we could specify that the contact angle should oscillate around a flat *mean* angle of contact fixed on the line $z = \bar{h}$. We shall require that a flat *mean* angle of contact be prescribed*

$$\bar{h}_{,r}(a, \varepsilon) = \bar{h}_{,r}(b, \varepsilon) = 0 \tag{1.5k}$$

where

$$\bar{h}(r, \varepsilon) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} h(r, t, \varepsilon)\, dt. \tag{1.6}$$

When $\varepsilon = 0$, there is no motion. Then, designating variables evaluated when $\varepsilon = 0$ by a superscript [0], we find that

$$\boldsymbol{U}^{[0]},\ \boldsymbol{S}^{[0]},\ p_a - \Phi^{[0]} \text{ and } h^{[0]} \tag{1.7}$$

all vanish. This defines the rest state. Cylindrical polar coordinates (R, Θ, Z) are introduced to label the particles in

$$\mathscr{V}_0 = [R, Z\colon a \leqq R \leqq b,\ Z \leqq 0].$$

The position vector $\boldsymbol{X}$ of these particles will be used for particle labels.

* This condition holds in the experiments reported in Part II.

2. Kinematics for Perturbation of the Rest State

The position vector of the particle X is designated as

$$\xi = \xi[X, \tau, \varepsilon] \tag{2.1a}$$

where

$$x = \xi[X, t, \varepsilon] \tag{2.1b}$$

is the position of the particle X at the present time t. The relative position vector

$$\chi_t(x, \tau, \varepsilon) = \xi[X, \tau, \varepsilon], \tag{2.2a}$$

$$\chi_t(x, t, \varepsilon) = x \tag{2.2b}$$

is obtained from (2.1) by replacing X in (2.1a) with the function $X(x, t, \varepsilon)$ obtained by solving (2.1b). When $\varepsilon = 0$,

$$\xi[X, \tau, 0] = X. \tag{2.3}$$

It follows from (2.2) that x depends on ε and functions $f(x(\varepsilon), t, \varepsilon)$ depend on ε explicitly and inplicitly through x when X is fixed. We define two derivatives:

$$f^{[n]}(X, t) = \frac{1}{n!} \frac{\partial^n f}{\partial \varepsilon^n}\bigg|_{\varepsilon=0} \quad \text{holding } X \text{ fixed} \tag{2.4}$$

and

$$f^{\langle n \rangle}(X, t) = \frac{1}{n!} \frac{\partial^n f}{\partial \varepsilon^n}\bigg|_{\varepsilon=0} \quad \text{holding } x \text{ fixed.} \tag{2.5}$$

We shall now suppose that the solution of the problem is analytic in ε when ε is small. This is a conceptually simple assumption. But we can make do with less. In our computation, which carries the analysis to second order, we need two derivatives with respect to ε.

Assuming analyticity, we have

$$\xi(X, \tau, \varepsilon) = X + \sum_{n=1} \varepsilon^n \xi^{[n]}(X, \tau)$$

and

$$x = \xi(X, t, \varepsilon) = X + \sum_{n=1}^{\infty} \varepsilon^n \xi^{[n]}(X, t).$$

Hence,

$$\xi(X, \tau, \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n ((\xi^{[n]}))$$

where

$$((\xi^{[n]})) = \xi^{[n]}(X, \tau) - \xi^{[n]}(X, t).$$

We may also expand the relative position vector $\chi_t(x, \tau, \varepsilon)$ in powers of ε, holding x fixed

$$\chi_t(x, \tau, \varepsilon) = \chi_t^{\langle 0 \rangle}(x, \tau) + \sum_{n=1}^{\infty} \varepsilon^n \chi_t^{\langle n \rangle}(x, \tau). \tag{2.6a}$$

Evaluation of (2.6a) when $\varepsilon = 0$ shows that $\chi_t(x, \tau) = \chi_t^{\langle 0 \rangle}(x, \tau)$ for all $\tau < t$. Taking $\tau = t$, we find that

$$\chi_t^{\langle 0 \rangle}(x, \tau) = x. \tag{2.6b}$$

It follows from (2.6a, b) and (2.2b) that

$$\chi_t(x, \tau, \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n \chi_t^{\langle n \rangle}(x, \tau)$$

and

$$\chi_t^{\langle n \rangle}(x, t) = 0 \quad \text{when} \quad n > 1.$$

Collecting results we have

$$\xi(X, \tau, \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n ((\xi^{[n]})) = \chi_t(x, \tau, \varepsilon) = x + \sum_{n=1}^{\infty} \varepsilon^n \chi_t^{\langle n \rangle}(x, \tau). \tag{2.7}$$

Since x depends in ε and $((\xi^{[n]}))$ depends on X, the coefficients in the two series of (2.7) need not be equal.

To compare the coefficients in the two series of (2.7) and to facilitate our later calculation of canonical forms for the stress tensor, we take note of simple rules for computing derivatives with respect to ε when x is fixed:

(I) When x is fixed, $dx = d\xi(X, t, \varepsilon) = 0$ and

$$dX = -\frac{\partial X}{\partial x_l} \left(\frac{\partial x_l}{\partial \varepsilon}\right)_X d\varepsilon$$

where the subscript X means that X is held constant when $x_l = \xi_l(X, \tau, \varepsilon)$ is differentiated with respect to ε.

(II)

$$\left(\frac{\partial X_i}{\partial \varepsilon}\right)_x = -\frac{\partial X_i}{\partial x_l} \left(\frac{\partial x_l}{\partial \varepsilon}\right)_X, \tag{2.8a}$$

(III)

$$\left(\frac{\partial \xi(X, \tau, \varepsilon)}{\partial \varepsilon}\right)_x = -\frac{\partial \chi_t}{\partial x_l} \left(\frac{\partial x_l}{\partial \varepsilon}\right)_X + \left(\frac{\partial \xi}{\partial \varepsilon}\right)_X, \tag{2.8b}$$

(IV)

$$\left(\frac{\partial g[\xi(X, \tau, \varepsilon), \tau, \varepsilon]}{\partial \varepsilon}\right)_x = \frac{\partial g}{\partial \xi_i} \left(\frac{\partial (\chi_t)_i}{\partial \varepsilon}\right)_x + \left(\frac{\partial g}{\partial \varepsilon}\right)_\xi. \tag{2.8c}$$

Higher derivatives of the function $g(\xi, \tau, \varepsilon) = g(\chi_t, \tau, \varepsilon) = \tilde{g}(x, \tau, \varepsilon)$ with respect to ε at constant x may be computed by repeated application of (2.8c). In the limit $\varepsilon \to 0$ we find from (2.8a), (2.8b) and (2.8c) that

$$X^{\langle 1 \rangle} = -x^{[1]}, \qquad \chi_t^{\langle 1 \rangle} = ((\xi^{[1]})) \tag{2.9, 10}$$

and

$$\left(\frac{\partial \tilde{g}}{\partial \varepsilon}\right)_x \equiv \tilde{g}^{\langle 1 \rangle} = \chi_t^{\langle 1 \rangle} \cdot \nabla_X g^{\langle 0 \rangle}(X, \tau) + g^{\langle 1 \rangle}(X, \tau). \tag{2.11}$$

If, as in our applications, $g^{\langle 0 \rangle} = 0$, then

$$\frac{1}{2} \left(\frac{\partial^2 \tilde{g}}{\partial \varepsilon^2}\right)_x \equiv \tilde{g}^{\langle 2 \rangle} = g^{\langle 2 \rangle}(X, \tau) + ((\xi^{[1]})) \cdot \nabla_X g^{\langle 1 \rangle}(X, \tau). \tag{2.12}$$

Formulas relating the coefficients in the two series of (2.7) may be obtained by differentiating (2.7) with respect to ε when $\varepsilon = 0$ at constant x. The first differentiation leads again to (2.10). The second differentiation shows that

$$\chi_t^{\langle 2\rangle}(\boldsymbol{x},\tau)=\left(\frac{\partial((\boldsymbol{\xi}^{[1]}))}{\partial\varepsilon}\right)_{\boldsymbol{x}}+((\boldsymbol{\xi}^{[2]}))=\left(\frac{\partial \boldsymbol{X}}{\partial\varepsilon}\right)_{\boldsymbol{x}}\cdot\nabla_{\boldsymbol{X}}\chi^{\langle 1\rangle}+((\boldsymbol{\xi}^{[2]}))$$
$$=-\boldsymbol{x}^{[1]}\cdot\nabla_{\boldsymbol{X}}((\boldsymbol{\xi}^{[1]}))+((\boldsymbol{\xi}^{[2]})). \tag{2.13}$$

The most important kinematic quantities to be used in the perturbation of the rest state with unsteady motions are the position vector differences $((\boldsymbol{\xi}^{[n]}))$. These quantities are used ultimately to monitor the history of the strain. We shall show that $((\boldsymbol{\xi}^{[n]}))$ may be computed if $\boldsymbol{\xi}^{[l]}(\boldsymbol{X},\tau)$ and $\boldsymbol{U}^{[m]}(\boldsymbol{X},\tau)$ are known when $l<m$ and $m\leqq n$. We first note that

$$\boldsymbol{U}(\boldsymbol{\xi},\tau,\varepsilon)=\dot{\boldsymbol{\xi}}(\boldsymbol{X},\tau,\varepsilon)=\sum_{n=1}^{\infty}\varepsilon^n\dot{\boldsymbol{\xi}}^{[n]}(\boldsymbol{X},\tau)=\sum_{n=1}^{\infty}\varepsilon^n\boldsymbol{U}^{[n]}(\boldsymbol{X},\tau). \tag{2.14}$$

It follows that

$$\dot{\boldsymbol{\xi}}^{[n]}=\frac{d\boldsymbol{\xi}^{[n]}}{d\tau}=\boldsymbol{U}^{[n]}(\boldsymbol{X},\tau).$$

Hence

$$((\boldsymbol{\xi}^{[n]}))=\int_t^\tau \boldsymbol{U}^{[n]}(\boldsymbol{X},\tau')\,d\tau'. \tag{2.14}$$

Moreover,

$$\boldsymbol{U}^{[1]}(\boldsymbol{X},\tau)=\boldsymbol{U}^{\langle 1\rangle}(\boldsymbol{X},\tau), \tag{2.16}$$

$$\boldsymbol{U}^{[2]}(\boldsymbol{X},\tau)=\boldsymbol{U}^{\langle 2\rangle}(\boldsymbol{X},\tau)+\boldsymbol{\xi}^{[1]}(\boldsymbol{X},\tau)\cdot\nabla_{\boldsymbol{X}}\boldsymbol{U}^{\langle 1\rangle}, \tag{2.17}$$

$$\boldsymbol{U}^{[3]}(\boldsymbol{X},\tau)=\boldsymbol{U}^{\langle 3\rangle}(\boldsymbol{X},\tau)+\boldsymbol{\xi}^{[2]}(\boldsymbol{X},\tau)\cdot\nabla_{\boldsymbol{X}}\boldsymbol{U}^{\langle 1\rangle}+\boldsymbol{\xi}^{[1]}\cdot\nabla_{\boldsymbol{X}}\boldsymbol{U}^{\langle 2\rangle}(\boldsymbol{X},\tau)+\tfrac{1}{2}\xi_i^{[1]}\xi_j^{[1]}\frac{\partial^2\boldsymbol{U}^{\langle 1\rangle}}{\partial X_i\,\partial X_j}.$$

In the computational algorithm for perturbations of the rest state, we first compute $\boldsymbol{U}^{\langle 1\rangle}(\boldsymbol{X},\tau)$ as the solution of a well-posed boundary value problem. We may then compute $((\boldsymbol{\xi}^{[1]}))$. Given $\boldsymbol{U}^{\langle 1\rangle}$ and $((\boldsymbol{\xi}^{[1]}))$ it is possible to solve another boundary value problem for $\boldsymbol{U}^{\langle 2\rangle}(\boldsymbol{X},\tau)$. Then we can calculate $((\boldsymbol{\xi}^{[2]}))$. In this way we may generate the functions $\boldsymbol{U}^{\langle n\rangle}$ and $((\boldsymbol{\xi}^{[n]}))$ sequentially as in other well-behaved perturbation theories.

3. Series Expansions and Fréchet Expansions of the Stress

Having established the required kinematic preliminaries, we are now ready to solve (1.1) as a perturbation of the rest state. As in the steady case treated by JOSEPH & FOSDICK (1973) and BEAVERS & JOSEPH (1974), the solution is expanded in powers of ε; the azimuthal component of velocity V and the shear stresses $S_{r\theta}$ and $S_{z\theta}$ are odd functions of ε; and Φ, U, W, S_{rz}, S_{rr}, $S_{\theta\theta}$, S_{zz} and $h(r,t,\varepsilon)$ are even functions of ε. There are different equivalent ways to represent the same solution (see JOSEPH & STURGES, 1975). The basic generating functions for these solutions are the functions $\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{X},\tau)$. When the motion is unsteady, we must also compute the $((\boldsymbol{\xi}^{[n]}))$. Now we shall derive the equations governing the functions $\boldsymbol{U}^{\langle 1\rangle}(\boldsymbol{X},\tau)$, $((\boldsymbol{\xi}^{[1]}))$, $\boldsymbol{U}^{\langle 2\rangle}$, $\Phi^{\langle 2\rangle}$, $h^{[2]}$ and $((\boldsymbol{\xi}^{[2]}))$. The equations which govern at higher orders can be derived by the same algorithms.

Equations (1.5a) through (1.5e) are supposed to hold identically in $\mathscr{V}_\varepsilon$ when ε is in some open interval $I(\varepsilon)$ of the origin. These equations are in the form

$$\mathscr{G}(\boldsymbol{x}, t, \varepsilon) = 0, \qquad \boldsymbol{x} \in \mathscr{V}_\varepsilon.$$

Since $\mathscr{G} = 0$ is an identity in ε,

$$\mathscr{G}^{[n]}(\boldsymbol{X}, t) = 0, \qquad \boldsymbol{X} \in \mathscr{V}_0$$

where

$$\boldsymbol{\xi}(\boldsymbol{X}, \tau, 0) = \boldsymbol{X} \qquad \text{for all } \tau < t.$$

Since $\mathscr{G} = 0$ is an identity in $\boldsymbol{x} \in \mathscr{V}_\varepsilon$,

$$\mathscr{G}^{[1]}(\boldsymbol{X}, t) = \mathscr{G}^{\langle 1 \rangle}(\boldsymbol{X}, t) + \boldsymbol{x}^{[1]} \cdot \nabla_{\boldsymbol{X}} \mathscr{G}^{\langle 0 \rangle}(\boldsymbol{X}, t) = \mathscr{G}^{\langle 1 \rangle}(\boldsymbol{X}, t) = 0.$$

We find, by induction, that

$$\mathscr{G}^{\langle n \rangle}(\boldsymbol{X}, t) = 0. \tag{3.1}$$

A somewhat more complicated result holds for equations (1.5f) through (1.5k). These equations are identities in ε, but they hold only on the interface and not identically in $\mathscr{V}_\varepsilon$. A typical interface condition may be represented as

$$\mathscr{f}(\boldsymbol{x}, t, \varepsilon) = 0$$

for all $\boldsymbol{x}$ such that $d\boldsymbol{x} = (\boldsymbol{e}_r + \boldsymbol{e}_z h_{,r})\, dr$, $a \leqq r \leqq b$. It follows that

$$\mathscr{f}^{[n]} = 0, \qquad n > 0, \tag{3.2}$$

$$\mathscr{f}^{[1]} = \mathscr{f}^{\langle 1 \rangle}(\boldsymbol{X}, t) + \boldsymbol{x}^{[1]} \cdot \nabla_{\boldsymbol{X}} \mathscr{f}^{\langle 0 \rangle}, \tag{3.3}$$

$$\mathscr{f}^{[2]} = \mathscr{f}^{\langle 2 \rangle}(\boldsymbol{X}, t) + \boldsymbol{x}^{[1]} \cdot \nabla_{\boldsymbol{X}} \mathscr{f}^{\langle 1 \rangle} + \tfrac{1}{2} x_i^{[1]} x_j^{[1]} \frac{\partial^2 \mathscr{f}^{\langle 0 \rangle}}{\partial X_i \, \partial X_j} + \boldsymbol{x}^{[2]} \cdot \nabla_{\boldsymbol{X}} \mathscr{f}^{\langle 0 \rangle}. \tag{3.4}$$

The formulas (3.2) through (3.4) simplify considerably because $\mathscr{f}^{\langle 0 \rangle} = 0$ on the rest state.

Equations (3.1) and (3.2) were originally derived for the theory of domain perturbations in problems for which it is not necessary to compute histories (see JOSEPH & STURGES (1975) for the most recent and complete version of this theory). The present problem is more complicated because it requires the computation of the history of the strain from the prescribed data. Mathematically, the difference between the earlier theory and the present one is in the choice of the function which maps $\mathscr{V}_\varepsilon \leftrightarrow \mathscr{V}_0$. In the present problem the mapping function is a material mapping associated with the position vector $\boldsymbol{\xi}(\boldsymbol{X}, t, \varepsilon)$. Our use here of a material mapping allows us to meet two objectives. As in the earlier problem we can pose problems on a domain $\mathscr{V}_0$ of simple configuration. In addition, the material mapping is natural to our main purpose, the sequential computation of the history which is required to specify the present motion. For simple fluids it is possible to derive the canonical forms of the stress without introducing material coordinates. The present derivation, however, should carry over directly to simple materials in which the present state of the stress does depend on the reference configuration.

We are now ready to treat the most important part of the problem: the canonical representation of the Fréchet stresses which arise in the perturbation.

By Fréchet stresses we mean the forms of the stress which arise when $\mathscr{F}[G(s,\varepsilon)]$ is differentiated with respect to ε holding x fixed. We are going to derive the canonical forms for $\mathscr{F}^{\langle 1\rangle}$ and $\mathscr{F}^{\langle 2\rangle}$. The canonical form of $\mathscr{F}^{\langle 3\rangle}$ is given in JOSEPH's (1976) book on stability. The starting point for this aspect of the analysis is the assumption that the stresses in the simple fluids treated here can be expressed by a series of multiple integrals (GREEN & RIVLIN, 1956; COLEMAN & NOLL, 1961) of polynomials in the histories $G(s,\varepsilon)$. TRUESDELL & NOLL (1965) call the polynomial forms "stress tensors for fluids of integral type of order N". The stress tensor for the fluid of order two (COLEMAN & NOLL, 1961) is

$$\begin{aligned}\mathscr{F}[G(s,\varepsilon)]=\int_0^\infty \zeta(s)G(s,\varepsilon)\,ds+\int_0^\infty\int_0^\infty \{\beta(s_1,s_2)G(s_1,\varepsilon)G(s_2,\varepsilon)\\ +\alpha(s_1,s_2)[\operatorname{tr}G(s_1,\varepsilon)]\,G(s_2,\varepsilon)\}\,ds_1\,ds_2\end{aligned}\tag{3.5}$$

where $\zeta(s)=dG/ds$, $\beta(s_1,s_2)=\beta(s_2,s_1)$ and $\alpha(s_1,s_2)$ are material functions and $G(s)$ is the shear relaxation modulus. In the fluid of order one, the quadratic terms of (3.5) are set equal to zero. In the fluid of order three (PIPKIN, 1964), it is necessary to add cubic terms in $G(s,\varepsilon)$, in the form of triple integrals over trilinear products, to the terms already present in (3.5). Fluids of order N contain N-linear terms of the same type.

In carrying out perturbations to order N in ε, we retain complete generality within the class of fluids of integral type by restricting attention to the N^{th} order fluid. The $N+m$-linear products ($m>0$) vanish identically in the N^{th} order approximation. Here, we consider $N=2$.

4. Canonical Forms of the Fréchet Stresses at Orders One and Two

COLEMAN & NOLL (1961) in their study of first and second order viscoelasticity note that for small strains

$$G(s,\varepsilon)=\varepsilon\,G_1(s)+O(\varepsilon^2)$$

where

$$\varepsilon G_1(s)=2[E(t-s)-E(t)]\tag{4.1}$$

and $E_1(\tau)$ is the infinitesimal strain tensor relative to a fixed reference configuration which here may be taken as $\mathscr{V}_0$ the configuration of the rest state. PIPKIN (1964) has noted that for incompressible materials, tr $[G(s,\varepsilon)]=O(\varepsilon^2)$ because tr $G_1(s)=0$. It follows that the coefficient of $\alpha(s_1,s_2)$ in (3.5) is $O(\varepsilon^2)$ and may be dropped from consideration in any second order theory. In the second order theory, there are only two material functions, $\zeta(s)$ and $\beta(s_1,s_2)$. We assume that $-\zeta(s)$ is positive and that $\zeta(s)$ and $\beta(s_1,s_2)$ decay rapidly to zero for large values of s and s_1 or s_2.

One more preliminary transformation of (3.5) is useful. We introduce into (3.5) the shear modulus $G(s)$ and the material function $\gamma(s_1,s_2)=\gamma(s_2,s_1)$,

$$\zeta(s)=\frac{dG(s)}{ds},\qquad \beta(s_1,s_2)=\frac{\partial^2\gamma(s_1,s_2)}{\partial s_1\,\partial s_2}.\tag{4.2}$$

$G(s)$ is assumed to be strictly positive and $G(s)$ and $\gamma(s_1,s_2)$ decay rapidly with s and s_1 or s_2. After setting to zero the redundant last term of (3.5), we integrate the remaining two terms by parts and find that

$$\mathscr{F}[G(s,\varepsilon)] = \int_0^\infty G(s)\, F_t^T(\tau,\varepsilon)\, A_1[U(\xi,\tau,\varepsilon)]\, F_t(\tau,\varepsilon)\, ds$$
$$+ \int_0^\infty \int_0^\infty \gamma(s_1,s_2)\, \{F_t^T(\tau_1,\varepsilon)\, A_1[U(\xi,\tau_1,\varepsilon)]\, F_t(\tau_1,\varepsilon)\} \qquad (4.3)$$
$$\cdot\, \{F_t^T(\tau_2,\varepsilon)\, A_1[U(\xi,\tau_2,\varepsilon)]\, F_t(\tau_2,\varepsilon)\}\, ds_1\, ds_2$$

where

$$A_1[U(\xi,\tau)] = \nabla_\xi U + \nabla_\xi U^T. \qquad (4.4)$$

Now we come to the last step in the reduction of $\mathscr{F}$ to canonical form for perturbations of the rest state. We need to compute $(\mathscr{F}[G(s,\varepsilon)])^{\langle 1\rangle}$ and $(\mathscr{F}[G(s,\varepsilon)])^{\langle 2\rangle}$ from (4.3). We first write (4.3) in component form relative to a Cartesian basis:

$$\mathscr{F}_{ij}[G(s,\varepsilon)] = \int_0^\infty G(s)\, Q_{ij}(\tau,\varepsilon)\, ds + \int_0^\infty \int_0^\infty \gamma(s_1,s_2)\, Q_{il}(\tau_1,\varepsilon)\, Q_{lj}(\tau_2,\varepsilon)\, ds_1\, ds_2 \qquad (4.5)$$

where

$$Q_{ij}(\tau,\varepsilon) = \frac{\partial \xi(\tau,\varepsilon)}{\partial x_i} \cdot \frac{\partial U[\xi(X,\tau,\varepsilon)\,\tau,\varepsilon]}{\partial x_j} + \text{transpose}.$$

We take the first and second derivatives of (4.5) with respect to ε at $\varepsilon = 0$ holding x fixed. Using the differentiation formulas (2.10) with $g^{\langle 0\rangle} = 0$ and (2.11), we find that

$$(\mathscr{F}[G(s,\varepsilon)])^{\langle 1\rangle} = \int_0^\infty G(s)\, A(s)\, ds \qquad (4.6)$$

where

$$A(s) = A_1[U^{\langle 1\rangle}(X,t-s)] \qquad (4.7)$$

and

$$(\mathscr{F}[G(s,\varepsilon)])^{\langle 2\rangle} = \int_0^\infty G(s)\, A_1[U^{\langle 2\rangle}(X,t-s)]\, ds + \int_0^\infty G(s)\, \{((\xi^{[1]}))\cdot \nabla_X A(s) + A(s)\, \nabla_X((\xi^{[1]}))$$
$$+ (A(s)\, \nabla_X((\xi^{[1]})))^T\}\, ds + \int_0^\infty \int_0^\infty \gamma(s_1,s_2)\, A(s_1)\, A(s_2)\, ds_1\, ds_2 \qquad (4.8)$$

where

$$((\xi^{[1]})) = \xi^{[1]}(X,t-s) - \xi^{[1]}(X,t). \qquad (4.9)$$

The canonical form (4.6) has been known for a long time. The canonical form (4.8) and the method used to derive it are given first in JOSEPH's book (1976) on stability. In that book (Chapter XIII) it is shown that the fluids of grade N which express the stress in terms of Rivlin-Ericksen tensors A_n evaluated at present time are a special case of the Fréchet expansion of $\mathscr{F}$ into order fluids of the integral type. In particular, the stress tensors which arise when the retardation expansion is specialized to steady motion may be obtained from order fluids of the integral type when the history is specialized to steady flow. The canonical form for the third order fluid of integral type and algorithms for computation at higher orders are given in JOSEPH's book.

Having now established our constitutive assumptions, we return to the computations of the free surface on a fluid between cylinders in torsional oscillation.

5. The Primary Shearing Motion

We now seek a solution of (1.5) at first order. To obtain the equations governing the first partial derivatives of the unknown functions, we differentiate (1.5) with respect to ε at $\varepsilon=0$ holding $\boldsymbol{x}$ fixed. After differentiating, we set $\boldsymbol{x}=\boldsymbol{X}$ according to the algorithms which were discussed in Section (3). The symmetries of the problem with respect to sign changes in ε are such that we may take

$$\boldsymbol{U}^{\langle 1\rangle}(\boldsymbol{X},t)=\boldsymbol{e}_\theta V(R,t), \tag{5.1a}$$

$$V(R)=R\Omega(R,t), \tag{5.1b}$$

$$\Phi^{\langle 1\rangle}=h^{\langle 1\rangle}=0. \tag{5.2}$$

Then

$$\begin{aligned} \boldsymbol{A}_1[\boldsymbol{U}^{\langle 1\rangle}]&=[\boldsymbol{e}_r\boldsymbol{e}_\theta+\boldsymbol{e}_\theta\boldsymbol{e}_r](V_{,R}-V/R)\\ &=[\boldsymbol{e}_r\boldsymbol{e}_\theta+\boldsymbol{e}_\theta\boldsymbol{e}_r]\,R\Omega_{,R} \end{aligned} \tag{5.3}$$

and

$$\mathscr{F}^{\langle 1\rangle}=[\boldsymbol{e}_r\boldsymbol{e}_\theta+\boldsymbol{e}_\theta\boldsymbol{e}_r]\int_0^\infty G(s)\,R(V(R,t-s)/R)_{,R}\,ds. \tag{5.4}$$

The unknown function $V(r,t)$ satisfies the following problem:

$$\rho\frac{\partial V}{\partial t}=D^2\int_0^\infty G(s)\,V(R,t-s)\,ds \tag{5.5a}$$

where

$$D^2=\frac{1}{R}\frac{d}{dR}\left(R\frac{d}{dR}\right)-\frac{1}{R^2}$$

and

$$V(a,t)=a\sin\omega t,\qquad V(b,t)=0. \tag{5.5b}$$

The solution of problem (5.5) is

$$\begin{aligned} V(R,t)&=v(R)\,e^{i\omega t}+\bar{v}(R)\,e^{-i\omega t},\\ \Omega(R,t)&=\tilde{\omega}(R)\,e^{i\omega t}+\bar{\tilde{\omega}}(R)\,e^{-i\omega t},\\ \tilde{\omega}(R)&\equiv v(R)/R \end{aligned} \tag{5.6a}$$

where $\bar{v}(R)$ is the complex conjugate of $v(R)$,

$$v(R)=\frac{a}{2i}\mathscr{C}_1(R\mathbb{A})/\mathscr{C}_1(a\mathbb{A}) \tag{5.6b}$$

where

$$\mathscr{C}_1(R\mathbb{A})=K_1(R\mathbb{A})\,I_1(b\mathbb{A})-K_1(b\mathbb{A})\,I_1(R\mathbb{A}), \tag{5.6c}$$

$$\mathbb{A}^2=i\rho\omega/\int_0^\infty G(s)\,e^{-i\omega s}\,ds \tag{5.6d}$$

and $K_1(x)$ and $I_1(x)$ are the modified Bessel functions; $K_1(x)\to\infty$ as $x\to 0$ and $I_1(x)\to\infty$ as $x\to\infty$.

If there is no outer cylinder and the fluid extends radially to infinity, then (5.6c) reduces to

$$\mathscr{C}_1(R\mathbb{A})=K_1(R\mathbb{A}).$$

Particularly simple forms of the solution, involving only circular functions, may be obtained when the gap is narrow.

Now we shall compute the history of the first order motion. Combining (2.15), (2.16) and (5.1a), we find that

$$((\xi^{[1]})) = e_\theta \int_t^\tau V(R, \tau')\, d\tau' \tag{5.7}$$

where e_θ is a unit vector in $\mathscr{V}_0$ and is independent of t. Since

$$\frac{d}{d\tau} e_r \cdot \xi^{[1]}(R, \theta, Z, \tau) = \frac{d}{d\tau} e_z \cdot \xi^{[1]}(R, \theta, Z, \tau) = 0, \tag{5.8}$$

we conclude that at first order the particles move in circles and

$$\begin{aligned} e_\theta \cdot ((\xi^{[1]})) = R((\theta^{[1]})) &= \int_t^\tau [v(R)\, e^{i\omega\tau'} + \text{conjugate}]\, d\tau' \\ &= (e^{i\omega(t-s)} - e^{i\omega t})\, v(R)/i\omega + \text{conjugate}. \end{aligned} \tag{5.9}$$

The following quantities are needed in the calculation of the motion and stress at 2nd order (from here on, $\nabla \equiv \nabla_X$):

$$\begin{aligned} \nabla((\xi^{[1]})) &= -e_r e_\theta ((\theta^{[1]})) + e_\theta e_r \{R((\theta^{[1]}))\}_{,R}, \\ ((\xi^{[1]})) \cdot \nabla A &= ((\theta^{[1]})) \frac{\partial}{\partial\theta} A = 2R\Omega_{,R} ((\theta^{[1]}))\, (e_\theta e_\theta - e_r e_r), \end{aligned} \tag{5.10}$$

$$A \nabla((\xi^{[1]})) + (A \nabla((\xi^{[1]})))^T = 2R\Omega_{,R} [e_r e_r \{R((\theta^{[1]}))\}_{,R} - e_\theta e_\theta ((\theta^{[1]}))] \tag{5.11}$$

$$((\xi^{[1]})) \cdot \nabla A + A \nabla((\xi^{[1]})) + (A \nabla((\xi^{[1]})))^T = 2R^2 \Omega_{,R} ((\theta^{[1]}))_{,R}\, e_r e_r, \tag{5.12}$$

$$A(s_1)\, A(s_2) = (e_r e_r + e_\theta e_\theta)\, R^2 (\Omega(R, t-s_1))_{,R} (\Omega(R, t-s_2))_{,R}, \tag{5.13}$$

$$U^{\langle 1 \rangle} \cdot \nabla U^{\langle 1 \rangle} = -e_r R \Omega^2. \tag{5.14}$$

6. Motion and Stress at 2nd Order

The first deviation of the free surface from flatness occurs at 2nd order. The field equations, wall conditions and asymptotic conditions which govern at 2nd order are obtained by differentiating (1.5a) through (1.5e) according to the differentiation rule (3.1) (with $n=2$):

$$\operatorname{div} U^{\langle 2 \rangle} = 0, \tag{6.1a}$$

$$\begin{aligned} \rho U^{\langle 2 \rangle}_{,t} + \nabla \Phi^{\langle 2 \rangle} - \nabla^2 \int_0^\infty G(s) U^{\langle 2 \rangle}(X, t-s)\, ds &= \nabla \cdot \int_0^\infty G(s) [((\xi^{[1]})) \cdot \nabla A(s) \\ &\quad + A(s)\, \nabla((\xi^{[1]})) + (A(s)\, \nabla((\xi^{[1]})))^T]\, ds \\ &\quad + \nabla \cdot \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, A(s_1)\, A(s_2)\, ds_1\, ds_2 - \rho U^{\langle 1 \rangle} \cdot \nabla U^{\langle 1 \rangle} \end{aligned}$$

$$= \boldsymbol{e}_r \left\{ \frac{2}{R} \int_0^\infty G(s) \left[R^3 \Omega_{,R}(R, t-s) ((\theta^{[1]}))_{,R} \right]_{,R} ds \right.$$

$$\left. + \int_0^\infty \int_0^\infty \gamma(s_1, s_2) \left[R^2 \Omega_{,R}(R, t-s_1) \, \Omega_{,R}(R, t-s_2) \right]_{,R} ds_1 \, ds_2 + \rho R \Omega^2(R,t) \right\}, \quad (6.1\text{b})$$

$$\boldsymbol{U}^{\langle 2 \rangle}(a, Z, t) = \boldsymbol{U}^{\langle 2 \rangle}(b, Z, t) = 0, \quad (6.1\text{c})$$

$$[\Phi^{\langle 2 \rangle}(R, Z, t), \boldsymbol{U}^{\langle 2 \rangle}(R, Z, t)] \xrightarrow[Z \to -\infty]{} [\Phi^{\langle 2 \rangle}(R, t), \boldsymbol{U}^{\langle 2 \rangle}(R, t)]. \quad (6.1\text{d})$$

When the flow is steady, the functions A, $\boldsymbol{U}^{\langle 1 \rangle}$ and Ω do not depend on t and we find, using (5.7), that

$$((\xi^{[1]})) = -\boldsymbol{e}_\theta s V(r) = -s \boldsymbol{U}^{\langle 1 \rangle}.$$

Using the COLEMAN-MARKOVITZ (1964) formulas

$$\mu = \int_0^\infty G(s) \, ds,$$

$$\alpha_1 = -\int_0^\infty s G(s) \, ds$$

and

$$\alpha_2 = \int_0^\infty \int_0^\infty \gamma(s_1, s_2) ds_1 \, ds_2,$$

we may rewrite (6.1b) as

$$\nabla \Phi^{\langle 2 \rangle} - \mu \nabla^2 \boldsymbol{U}^{\langle 2 \rangle} = \nabla \cdot \{ \alpha_1 [\boldsymbol{U}^{\langle 1 \rangle} \nabla \boldsymbol{A} + \boldsymbol{A} \nabla \boldsymbol{U}^{\langle 1 \rangle} + (\boldsymbol{A} \nabla \boldsymbol{U}^{\langle 1 \rangle})^T] + \alpha_2 \boldsymbol{A} \cdot \boldsymbol{A} \} - \rho \boldsymbol{U}^{\langle 1 \rangle} \cdot \nabla \boldsymbol{U}^{\langle 1 \rangle}.$$

This is a familiar formula from the theory of perturbations of the rest state of a simple fluid with a steady motion. It forms the basis of the 2nd order theory of rod climbing (JOSEPH & FOSDICK, 1973).

The equations which govern at the interface are obtained by differentiating (1.5g) through (1.6) with respect to ε following the differentiation rules (3.3) and (3.4). Since $\not{f}^{\langle 0 \rangle} = 0$ where $\not{f}$ is any of the variables defined in (1.5f) through (1.6), we have $\not{f}^{[1]} = \not{f}^{\langle 1 \rangle}$. Moreover, (5.8) shows that $\boldsymbol{x}^{[1]}$ is proportional to $\boldsymbol{e}_\theta$ so that $\boldsymbol{x}^{[1]} \cdot \nabla$ is a derivative with respect to θ. This derivative vanishes because $f^{\langle 1 \rangle}(R, t)$ is independent of Θ. It follows that $\not{f}^{[2]} = \not{f}^{\langle 2 \rangle}$.

We shall work out the differentiations which are required on (1.5g) as an example. We then list the rest of the interface conditions. Since $h^{\langle 1 \rangle} = 0$, the equation (1.5g), which after differentiation

$$\{ h_{,r}(S_{zz} - S_{rr}) + (1 - h_{,R}^2) S_{rz} \}^{\langle 2 \rangle} = 0$$

holds on $Z = 0$, may be reduced to

$$S_{rz}^{\langle 2 \rangle} = \boldsymbol{e}_r \boldsymbol{e}_z : (\mathscr{F}[\boldsymbol{G}(s, \varepsilon)])^{\langle 2 \rangle} = 0$$

where $\mathscr{F}^{\langle 2 \rangle}$ is given by (4.8). Using (5.10), (5.11) and (5.12), we find that

$$\int_0^\infty G(s) [U_{,Z}^{\langle 2 \rangle}(R, Z, t-s) + W_{,R}^{\langle 2 \rangle}(R, Z, t-s)] \, ds = 0. \quad (6.1\text{e})$$

Evaluating the second derivative with respect to ε at constant $\boldsymbol{x}$ of equations (1.5h) through (1.5k), we find that

$$W^{\langle 2\rangle}=h^{\langle 2\rangle}_{,t}, \tag{6.1f}$$

$$-\Phi^{\langle 2\rangle}+2\int_0^\infty G(s)\,W^{\langle 2\rangle}_{,Z}(R,Z,t-s)\,ds+\rho g h^{\langle 2\rangle}=\frac{T}{R}\,(R h^{\langle 2\rangle}_{,R})_{,R}, \tag{6.1g}$$

$$\overline{\overline{h}}^{\langle 2\rangle}_{,R}(a)=\overline{\overline{h}}^{\langle 2\rangle}_{,R}(b)=0, \tag{6.1h}$$

and

$$\frac{2}{b^2-a^2}\int_a^b R\,h^{\langle 2\rangle}(R,t)\,dR=0. \tag{6.1i}$$

The symmetries of our problem with respect to a change in the sign of ε is such that $V^{\langle 2\rangle}=0$ in $\mathscr{V}_0$. With $V^{\langle 2\rangle}=0$, equation (1.5f) is satisfied identically. Since $\boldsymbol{U}^{\langle 2\rangle}$ is an axisymmetric field and $V^{\langle 2\rangle}=0$, we may represent $\boldsymbol{U}^{\langle 2\rangle}$ with a stream function

$$\boldsymbol{U}^{\langle 2\rangle}=\nabla\Psi\wedge\boldsymbol{e}_\theta/R=\operatorname{curl}(\boldsymbol{e}_\theta\Psi/R) \tag{6.1j}$$

where $\operatorname{curl}=\nabla\wedge$ is with respect to rest coordinates in $\mathscr{V}_0$.

The motion is driven by the inhomogeneous terms on the right of (6.1b). Two important properties of these terms are:

(1) They have a potential:

$$\operatorname{curl}\boldsymbol{e}_r f(R)=0$$

for every function $f(R)$ which is independent of Θ and Z.

(2) They may be decomposed into a mean term which is independent of t and a time-periodic term which oscillates with a frequency 2ω:

$$\begin{aligned}\rho[\operatorname{curl}(\boldsymbol{e}_\theta\Psi/R)]_{,t}&-\int_0^\infty G(s)\nabla^2\operatorname{curl}(\boldsymbol{e}_\theta\Psi/R)\,ds+\nabla\Phi^{\langle 2\rangle}\\ =\Big\{&-\frac{4}{R}[R^3\tilde\omega_{,R}\bar{\tilde\omega}_{,R}]_{,R}\int_0^\infty G(s)\frac{\sin\omega s}{\omega}\,ds\\ &+2[R^2\tilde\omega_{,R}\bar{\tilde\omega}_{,R}]_{,R}\int_0^\infty\int_0^\infty\gamma(s_1,s_2)\cos\omega(s_1-s_2)\,ds_1\,ds_2\\ &+2\rho R|\tilde\omega|^2+e^{2i\omega t}F(R,\omega)+e^{-2i\omega t}\bar F(R,\omega)\Big\}\boldsymbol{e}_r\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}F(R,\omega)=&\frac{2}{i\omega R}[R^3(\tilde\omega_{,R})^2]_{,R}\int_0^\infty G(s)[e^{-2i\omega s}-e^{-i\omega s}]\,ds\\ &+[R^2(\tilde\omega_{,R})^2]_{,R}\int_0^\infty\int_0^\infty\gamma(s_1,s_2)e^{-i\omega(s_1+s_2)}\,ds_1\,ds_2+\rho R\tilde\omega^2.\end{aligned}$$

The solution of equations (6.2) and (6.1c) through (6.1j) also may be decomposed into a mean term and a time-periodic term which oscillates with a frequency 2ω:

$$\begin{aligned}\Psi&=\tfrac12\overline{\overline{\psi}}+\tfrac14 e^{i2\omega t}\psi_1+\tfrac14 e^{-i2\omega t}\bar\psi_1,\\ \Phi^{\langle 2\rangle}&=\tfrac12\overline{\overline{\phi}}+\tfrac14 e^{i2\omega t}\phi_1+\tfrac14 e^{-i2\omega t}\bar\phi_1,\\ h^{\langle 2\rangle}&=\tfrac12\overline{\overline{\hbar}}+\tfrac14 e^{i2\omega t}\hbar_1+\tfrac14 e^{-i2\omega t}\bar\hbar_1.\end{aligned} \tag{6.3}$$

To obtain the equations governed by the mean quantities, we insert the representations (6.3) into the governing equations. We find, from the curl of (6.2), that in $\mathscr{V}_0$,

$$\mathscr{L}^2\bar{\bar{\psi}}=0, \quad \mathscr{L}\equiv\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}+\frac{\partial^2}{\partial Z^2}.$$

From (6.1 c) we find that

$$\bar{\bar{\psi}} \text{ and } \bar{\bar{\psi}}_{,R} \text{ vanish at } R=a,b.$$

From (6.1 e) and (6.1 f) we find that on $Z=0$,

$$\left(\frac{\bar{\bar{\psi}}_{,R}}{R}\right)_{,R}-\left(\frac{\bar{\bar{\psi}}_{,Z}}{R}\right)_{,Z}=0$$

and

$$\bar{\bar{\psi}}_{,R}=0.$$

It follows, from standard arguments, that

$$\bar{\bar{\psi}}=0 \text{ in } \mathscr{V}_0.$$

The remaining part of the problem for the mean has a simple structure

$$\begin{aligned}\frac{1}{2}\frac{d\bar{\bar{\phi}}}{dR}=&-\frac{4}{R}[R^3|\tilde{\omega}_{,R}|^2]_{,R}\int_0^\infty G(s)\frac{\sin\omega s}{\omega}ds\\&+2[R^2|\tilde{\omega}_{,R}|^2]_{,R}\int_0^\infty\int_0^\infty\gamma(s_1,s_2)\cos\omega(s_1-s_2)\,ds_1\,ds_2+2\rho R|\tilde{\omega}|^2,\end{aligned}\tag{6.4}$$

$$-\bar{\bar{\phi}}(R)+\rho g\bar{\bar{h}}(R)=\frac{T}{R}(R\bar{\bar{h}}_{,R})_{,R},\tag{6.5}$$

$$\bar{\bar{h}}_{,R}(a)=\bar{\bar{h}}_{,R}(b)=0,\tag{6.6}$$

$$\frac{2}{b^2-a^2}\int_a^b R\bar{\bar{h}}(R)\,dR=0.\tag{6.7}$$

Equation (6.4) determines $\bar{\bar{\phi}}$ to within an arbitrary additive constant. The value of this constant is determined by (6.7). When $T=0$, the slope of the free surface is given by

$$\frac{d\bar{\bar{h}}}{dR}=\frac{1}{\rho g}\frac{d\bar{\bar{\phi}}}{dR}$$

where $\bar{\bar{\phi}}(R)$ is given by (6.4).

The time-periodic motion is driven by the oscillating pressure field associated with the function ($\phi_1 e^{2i\omega t}$+conjugate). The oscillating pressure field forces a periodic motion from the boundary. (6.1 f) shows that if there is a periodic interior motion, then the shape of the free surface must change with the same frequency. (6.1 g) shows that if the pressure field changes periodically, then so does the motion and the shape of the free surface. The time-periodic part of the motion at second order is governed by a fourth order differential equation which reduces when the gap is narrow to an equation for the stream function with biharmonic plus harmonic terms. This problem is presently under study by STURGES & JOSEPH. When the gap is narrow, the free surface may be regarded as generated in response to motion induced by oscillating planes. This problem may be solved exactly, with the mean rise in height expressed in terms of circular functions and the secondary motion expressed in terms of a "Fourier series" of biorthogonal eigenfunctions. The mean rise in height is proportional to the second order equivalent of second-normal stress difference. The analysis of the narrow gap problem and of the time-periodic secondary motions will be given in Part III. In the experiments described in Part II, the periodic motions are very small even when the mean climb is large.

In closing, we stress the importance of the amplitude ε as a basis for our expansion and for the interpretation of experiments. We first note that

$$\Omega(a,t)=\varepsilon\sin\omega t=\frac{d\theta}{dt}.$$

Hence

$$\theta(t)-\theta_0=-\frac{\varepsilon}{\omega}[\cos\omega t-1].$$

The maximum angular displacement $\theta_m-\theta_0$ of the rod occurs when $\omega t=\pi$. Hence

$$\varepsilon=\frac{\omega\Theta}{2} \tag{6.8}$$

where $\Theta=\theta_m-\theta_0$ is the *angle of twist*. The amplitude may also be expressed in terms of the mean square angular velocity

$$\overline{\overline{\Omega^2}}(a)=\frac{2\pi}{\omega}\int_0^{\omega/2\pi}\Omega^2(a,t)\,dt=\frac{\varepsilon^2}{2}=\frac{\omega^2\Theta^2}{8}. \tag{6.9}$$

The mean rise in height is given by

$$\overline{h}(R,\omega,\varepsilon)=\varepsilon^2\overline{h}^{\langle 2\rangle}(R,\omega)+O(\varepsilon^4)=\overline{\overline{\Omega^2}}\,\overline{\hbar}(R,\omega)+O(\overline{\overline{\Omega^4}}). \tag{6.10}$$

In the second order theory, the normalized rise in height

$$H=\overline{h}/\Theta^2\sim\omega^2\,\overline{\hbar}(R,\omega)/8 \tag{6.11}$$

is a universal function of the oscillation frequency ω for a given rod and fluid. The second order theory requires that $|\omega\Theta|^2$ be small; the frequency ω can be arbitrarily large if the angle of twist is made very small. It is not hard to show that

$$\lim_{\omega\to 0}\overline{\hbar}(R,\omega)\to\tfrac{1}{2}h^{[2]}(R)$$

where $h^{[2]}(R)$ is defined by equation (12.10) in the paper by JOSEPH & FOSDICK (1973). The problem of steady climbing emerges in the limit in which $\overline{\overline{\Omega^2}}$ is small and fixed and $\omega\to 0$.

7. Approximate Solutions for the Oscillating Rod

We understand that the limit $b\to\infty$ denotes the problem in which a single rod undergoes torsional oscillations in an infinite sea of fluid. Though it is possible to solve completely the rod problem in terms of Bessel functions, we prefer to express the solution in approximate, but strictly elementary, terms. The exact equations to be solved are:

$$\frac{1}{R}(R v_{,R})_{,R}-(1+R^2\mathbb{A}^2)\frac{v}{R^2}=0, \qquad v(a)=a/2i, \qquad (v(\infty), v_{,R}(\infty))=(0,0) \tag{7.1}$$

where $\mathbb{A}$ is a complex number given by (5.6d). The mean rise in height at 2nd order is governed by

$$\frac{T}{R}(R\bar{\bar{h}}_{,R})_{,R}-\rho g\bar{\bar{h}}(R)+\bar{\bar{\phi}}(R)=0, \qquad \bar{\bar{h}}_{,R}(a)=0, \qquad (\bar{\bar{h}}(\infty), \bar{\bar{h}}_{,R}(\infty))=(0,0) \tag{7.2}$$

where $\bar{\bar{\phi}}$ is given by (6.4) with $\tilde{\omega}(R)=v(R)/R$.

To obtain a simple approximate solution, we follow a method of successive approximations introduced by JOSEPH, BEAVERS & FOSDICK (1973). In the first approximation we replace $R^2\mathbb{A}^2$ in the last term of (7.1a) with $a^2\mathbb{A}^2$ and $1/R$ in the first term of (7.2) with R/a^2:

$$R(R v_{,R})_{,R}-\Lambda^2 v=0, \qquad v(a)=a/2i, \qquad (v(\infty), v_{,R}(\infty))=(0,0) \tag{7.3}$$

and

$$R(R\bar{\bar{h}}_{,R})_{,R}-\lambda^2\bar{\bar{h}}+a^2\bar{\bar{\phi}}/T=0, \qquad \bar{\bar{h}}_{,R}(a)=0, \qquad (\bar{\bar{h}}(\infty), \bar{\bar{h}}_{,R}(\infty))=(0,0) \tag{7.4}$$

where

$$\Lambda^2=(1+a^2\mathbb{A}^2), \qquad \lambda^2=\rho g a^2/T \tag{7.5}$$

and $\bar{\bar{\phi}}$ is given by (6.4) with $\tilde{\omega}(R)=v(R)/R$. The difference between the approximating problems (7.3) and (7.4) and the exact problem (7.1) and (7.2) vanishes at $R=a$. JOSEPH, BEAVERS & FOSDICK (1973) showed that the neglected terms may be restored through successive approximations. The first term, as given by the solution of (7.3) and (7.4), is already a good approximation (within a few percent, see JOSEPH, BEAVERS & FOSDICK, 1973) when the parameters have values like those we encounter in our experiments.

The solution to (7.3) is

$$v(R)=a^{1+\Lambda}/2iR^{\Lambda} \tag{7.6}$$

where Λ is the complex root of (7.5) with a positive real part. An explicit form for Λ, suitable for comparison with experiments, is given by (8.4) and (8.5). The velocity corresponding to (7.6) is given by (5.6a):

$$V(R,t)=a\left(\frac{a}{R}\right)^{\Lambda_r}\sin\left(\omega t+\Lambda_i\log\frac{a}{R}\right) \tag{7.7}$$

where

$$\Lambda_r=\mathrm{re}\sqrt{1+a^2\mathbb{A}^2}$$

and

$$\Lambda_i=\mathrm{im}\sqrt{1+a^2\mathbb{A}^2}. \tag{7.8}$$

It follows from (7.7) that the azimuthal velocity is in the form of a decaying wave. Equation (5.6d) shows that Λ_i and $V(R,t)$ tend to zero with ω.

Using (7.6), with $\tilde{\omega}(R) = v(R)/R$, we find that

$$\bar{\bar{\phi}}(R) = a^{2(\Lambda_r+1)} \left\{ \frac{|\Lambda+1|^2 \hat{\beta}_\Lambda}{(\Lambda_r+1) R^{2(\Lambda_r+1)}} - \frac{\rho}{2\Lambda_r R^{2\Lambda_r}} \right\} \tag{7.9}$$

where

$$\begin{aligned} \hat{\beta}_\Lambda = &-(2\Lambda_r+1) \int_0^\infty G(s) \frac{\sin \omega s}{\omega} ds \\ &+(\Lambda_r+1) \int_0^\infty \int_0^\infty \gamma(s_1, s_2) \cos \omega(s_1 - s_2)\, ds_1\, ds_2 . \end{aligned} \tag{7.10}$$

When $\omega \to 0$, $\Lambda \to 1$ and the problem defined by (7.4) and (7.9) is just exactly the one treated on pages 386–387 of the paper by JOSEPH, BEAVERS & FOSDICK (see Eq. 15.18). The solution of this problem is

$$\begin{aligned} \bar{\bar{\mathscr{h}}}(R, \omega) = \frac{a^{2\Lambda_r+4}}{T} &\left\{ \frac{|\Lambda+1|^2 \hat{\beta}_\Lambda}{(\Lambda_r+1)[4(\Lambda_r+1)^2 - \lambda^2]} \left(\frac{2(\Lambda_r+1) a^{\lambda - 2\Lambda_r - 2}}{\lambda R^\lambda} - \frac{1}{R^{2(\Lambda_r+1)}} \right) \right. \\ &\left. + \frac{\rho}{2\Lambda_r [4\Lambda_r^2 - \lambda^2]} \left(\frac{1}{R^{2\Lambda_r}} - \frac{2\Lambda_r a^{\lambda - 2\Lambda_r}}{\lambda R^\lambda} \right) \right\}. \end{aligned} \tag{7.11}$$

At $R = a$,

$$\bar{\bar{\mathscr{h}}}(a, \omega) = \frac{a}{\sqrt{\rho g T}} \left\{ \frac{|\Lambda+1|^2 \hat{\beta}_\Lambda}{(\Lambda_r+1)[2(\Lambda_r+1)+\lambda]} - \frac{\rho a^2}{2\Lambda_r (2\Lambda_r+\lambda)} \right\}. \tag{7.12}$$

The mean rise in height is given by

$$\bar{h}(R, \omega, \varepsilon) = \bar{\bar{\Omega}}^2 \bar{\bar{\mathscr{h}}}(R, \omega) + O(\bar{\bar{\Omega}}^2)^2 \tag{7.13}$$

when $\omega \to 0$, $\Lambda \to 1$, $\Lambda_r \to 1$ and $\bar{\bar{\mathscr{h}}}(R, \omega) \to \frac{1}{2} h_0(R)$ where $h_0(R) = a^4 H_0(R)$ and $H_0(R)$ is given by Equation (15.18) of JOSEPH, BEAVERS & FOSDICK (1973). Higher approximations may also be obtained by the method followed in that paper.

8. The Material Functions

At the lowest significant order the mean rise in height is given by the first term on the right of Equation (7.13). To give explicit form to this expression for the rise, it is necessary to approximate the material functions $G(s)$ and $\gamma(s_1, s_2)$. In this approximation we adopt the following point of view: The functions $G(s)$ and $\gamma(s_1, s_2)$ are required to specify the response of any simple fluid of integral type in all motions of small amplitude*. This leads to a *restricted problem of viscometry* for small amplitude motions which perturb the rest state: find the form of the material functions $G(s)$ and $\gamma(s_1, s_2)$.

Problems of viscometry involve the interaction between theory and experiment. In the present case we postulate forms for $G(s)$ and $\gamma(s_1, s_2)$, compute the mean rise, and compare the computed rise with experimental measurements.

* Small amplitude steady motions fit into this framework. For steady motions, small amplitude motions are slow motions. Usually a distinction is made between small amplitude motions in which the strain is small and slow motions in which the velocity is small. This distinction is not important in the present theory. The appropriate definition of ε is set by the data and the computational algorithm leads automatically to the correct forms of the stress.

The "postulating" part of this procedure of viscometry is an unfortunate but necessary procedure in the experimental determination of material functions.

To keep the analysis simple we shall define a sequence of approximating functions for the material functions. The approximating functions $G_N(s)$ and $\gamma_M(s_1, s_2)$ are taken as generalized Maxwell models depending on a finite number $(2N+2M)$ of constants which are chosen for consistency with the Coleman-Markovitz formulas following (6.1d):

$$G_N(s)=\frac{-\mu^2}{\alpha_1}\sum_1^N \frac{a_n^2}{b_n} e^{\frac{\mu}{\alpha_1}\frac{a_n}{b_n}s}, \tag{8.1a}$$

$$\gamma_M(s_1, s_2)=\alpha_2 \sum_1^M c_n k_n^2 e^{-k_n(s_1+s_2)} \tag{8.1b}$$

where

$$\sum_1^N a_n=\sum_1^N b_n=\sum_1^M c_n=1. \tag{8.1c}$$

There are $2N+2M-3$ unknown constants if the constants μ, α_1 and α_2 of the fluid of second grade be known.

We neither claim nor believe that the postulated forms (8.1) are true representations of the material functions in our fluid or in any other physical fluid. We are going to show that by increasing the number of parameters, we can fit the theoretical mean rise in height at second order to the measured rise in height. In the simplest approximation, $(N, M)=(1, 1)$, we get a good fit when $\omega^2<30$; in the next approximation, $(N, M)=(1, 2)$, we get a good fit when $\omega^2<450$. This "curve fitting" is offered as an example of a plausible procedure for the solution of the restricted problem of viscometry for simple fluids in all motions which perturb the rest state. We would say that the postulate (8.1) is good if the constants we determine from curve fitting in this problem determine the response of the same fluid in other motions which perturb the rest state.

When the postulated forms for $G(s)$ and $\gamma(s_1, s_2)$ are inserted into (7.8) and (7.10), we find that $\bar{\bar{h}}(R, \omega)$ and $\bar{\bar{h}}(a, \omega)$ are given by (7.11) and (7.12) with

$$\boldsymbol{A}^2=i\omega A^2-\omega^2 B^2, \tag{8.2}$$

$$A^2=\frac{\dfrac{\rho}{\mu}\displaystyle\sum_1^N \frac{a_n}{\xi_n^2}}{\left[\displaystyle\sum_1^N \frac{a_n}{\xi_n^2}\right]^2+\dfrac{\omega^2\alpha_1^2}{\mu^2}\left[\displaystyle\sum_1^N \frac{b_n}{\xi_n^2}\right]^2},$$

$$B^2=\frac{\dfrac{-\alpha_1\rho}{\mu^2}\displaystyle\sum_1^N \frac{b_n}{\xi_n^2}}{\left[\displaystyle\sum_1^N \frac{a_n}{\xi_n^2}\right]^2+\dfrac{\omega^2\alpha_1^2}{\mu^2}\left[\displaystyle\sum_1^N \frac{b_n}{\xi_n^2}\right]^2},$$

$$\hat{\beta}_\Lambda=(2\Lambda_r+1)\,\alpha_1\sum_1^N \frac{b_n}{\xi_n^2}+(\Lambda_r+1)\,\alpha_2\sum_1^N \frac{C_n}{1+(\omega^2/k_n^2)} \tag{8.3}$$

where

$$\xi_n^2 = 1 + \left(\frac{\alpha_1 \omega}{\mu}\right)^2 \left(\frac{b_n}{a_n}\right)^2.$$

It is necessary to distinguish between the low frequency case $a^2 \omega^2 B^2 \leqq 1$ and the high frequency case $a^2 \omega^2 B \geqq 1$. In the low frequency case,

$$A_r = \frac{1}{\sqrt{2}} \{1 - a^2 \omega^2 B^2 + [1 - a^2 \omega^2 B^2 + a^4 \omega^2 A^4]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{8.4a}$$

and

$$A_i = a^2 \omega A^2 / 2A_r. \tag{8.4b}$$

In the high frequency case,

$$A_i = \frac{1}{\sqrt{2}} \{a^2 \omega^2 B^2 - 1 + [a^2 \omega^2 B^2 - 1 + a^4 \omega^2 A^4]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{8.5a}$$

and

$$A_r = a^2 \omega A^2 / 2A_i. \tag{8.5b}$$

When $N = 1$,

$$A^2 = \frac{\rho}{\mu} \quad \text{and} \quad B^2 = \frac{-\alpha_1 \rho}{\mu^2}. \tag{8.6}$$

The simplest approximation is associated with the values $(N, M) = (1, 1)$. For these values (8.6) holds and

$$\hat{\beta}_A = \frac{(2A_r + 1)\alpha_1}{1 + \left(\frac{\alpha_1 \omega}{\mu}\right)^2} + \frac{(A_r + 1)\alpha_2}{1 + \left(\frac{\omega}{k_1}\right)^2}. \tag{8.7}$$

In the simplest approximation the response of the fluid is determined when the values of four constants, μ, α_1, α_2 and k_1, are known. Three of these constants, μ, α_1 and α_2, may be regarded as known from experiments with slow steady flow. The constant k_1 may be determined from the experiment described in Part II.

The next approximation is associated with the values $(N, M) = (1, 2)$. For these values, (8.6) holds and

$$\hat{\beta}_A = \frac{(2A_r + 1)\alpha_1}{1 + \left(\frac{\alpha_1 \omega}{\mu}\right)^2} + (A_r + 1)\alpha_2 \left[\frac{C_1}{1 + \left(\frac{\omega}{k_1}\right)^2} + \frac{1 - C_1}{1 + \left(\frac{\omega}{k_2}\right)^2}\right]. \tag{8.8}$$

In the (1, 2) approximation the response of the fluid is determined when the values of six constants, μ, α_1, α_2, C_1, k_1 and k_2, are known. The constants C_1, k_1 and k_2 may be determined from the experiment in Part II.

In the (2, 2) approximation the response of the fluid is determined when eight constants are known, and so on.

Part II: Experiments

9. Experiments

The experiments described in this section were designed to test the predictions of the second order theory. We restricted our study to measurements of the mean rise at the rod. The scope of these experiments is therefore too limited to be regarded as a complete test of the second order theory. We are satisfied, however, that there is convincing agreement between the theory and the experiments. In particular, the experiments establish the following points:

(a) There is a mean climb which completely dominates the total climb at each and every instant and at all oscillation frequencies within the operating range of our apparatus.

(b) There is a distinct observable region in which the second order theory holds. In this region we have found no discrepancy between the theory and our experiments of limited scope. The normalized rise in height $\bar{h}(a,\omega)/\Theta^2$ in this region is a universal function of the oscillation frequency (Fig. 2), and depends on the rod radius and the fluid but not on the angle of twist.

(c) For each fixed value of the angular frequence $\bar{\Omega}^2=\omega^2\Theta^2/8$ the height of climb increases with decreasing oscillation frequency ω. The maximum height of climb is attained for steady flow, $\omega=0$.

We discuss these points in more detail in Section 10, where we compare the predictions of the general theory of Sections 6 and 7 with the experiments. We also compare the predictions of the first two members, (1, 1) and (1, 2), of the family of generalized Maxwell models with the observed universal function of the oscillation frequency. In the remainder of this section we describe briefly

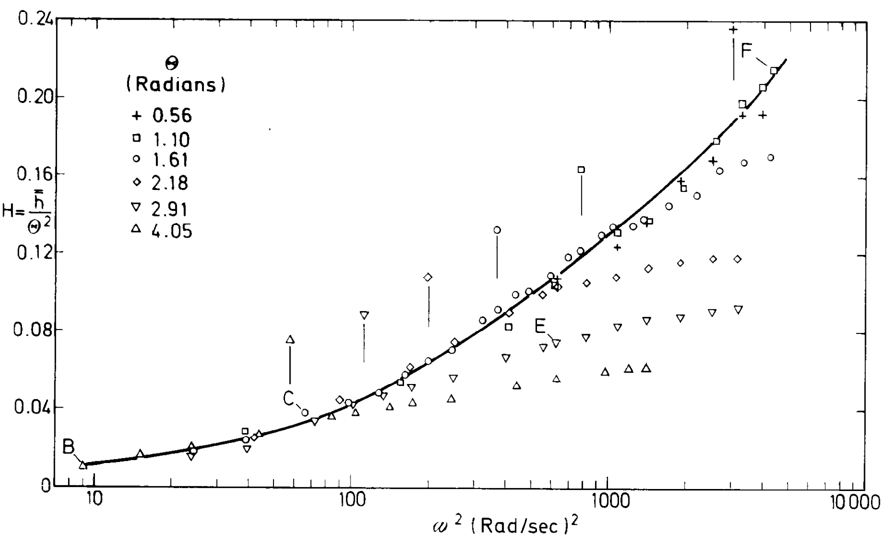


Fig. 2. The normalized height rise $\bar{h}/\Theta^2$ as a function of the frequency ω of oscillation. Experimental points belonging to six different angles of twist are shown. The experimental normalized rise at second-order is shown as a solid line. The vertical bars are values ω_c^2 associated with the criterion (10.3). The free surfaces corresponding to the points (B, C) and (E, F) are shown in Figs. 5 and 6 respectively.

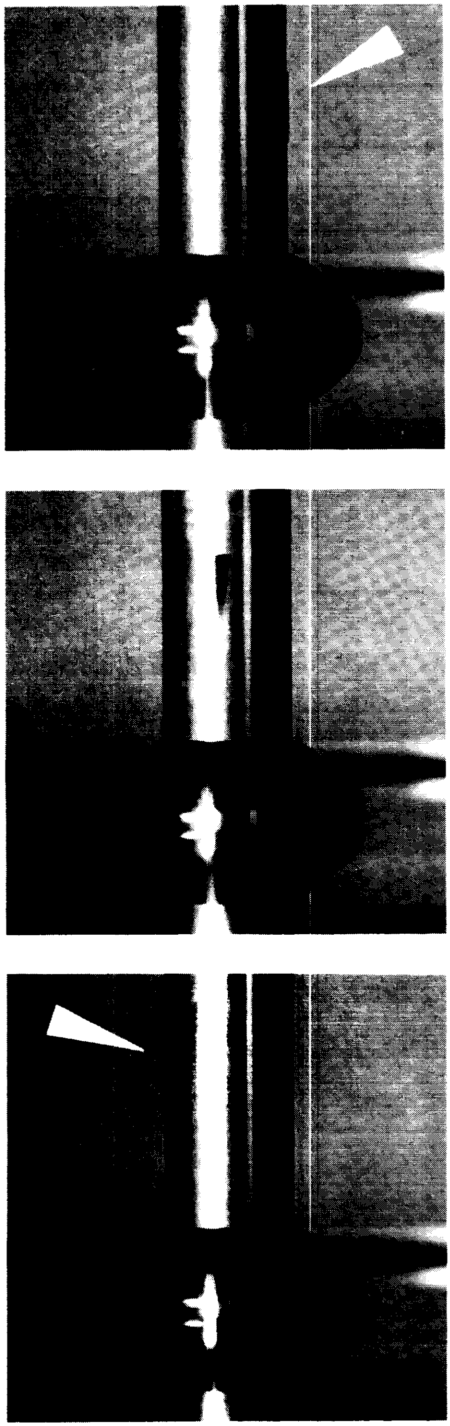


Fig. 3. (The caption for this figure appears on page 345.)

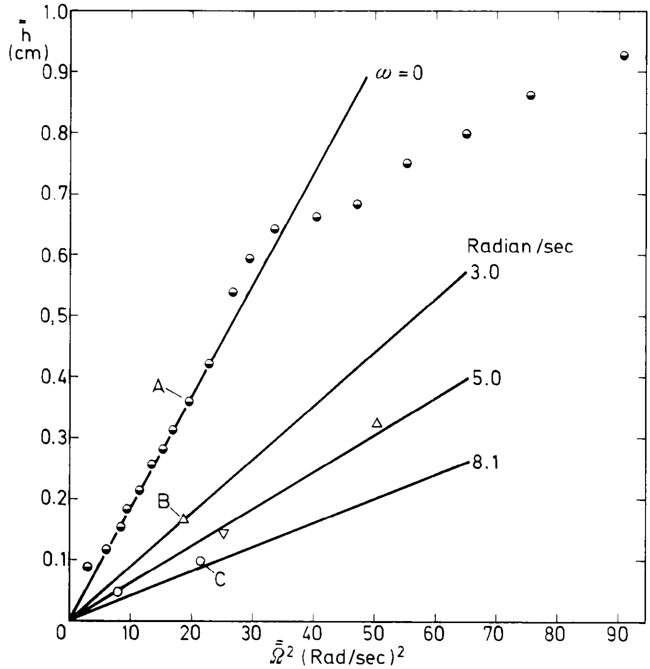


Fig. 4. Graph of the equation $\bar{h} = \bar{\Omega}^2 \bar{h}(a, \omega)$ for $a = 0.636$ cm. The line $\omega = 0$ gives the height rise when the rotation is steady. Photographs of the climbing at the points (A, B, C) are shown in Fig. 5.

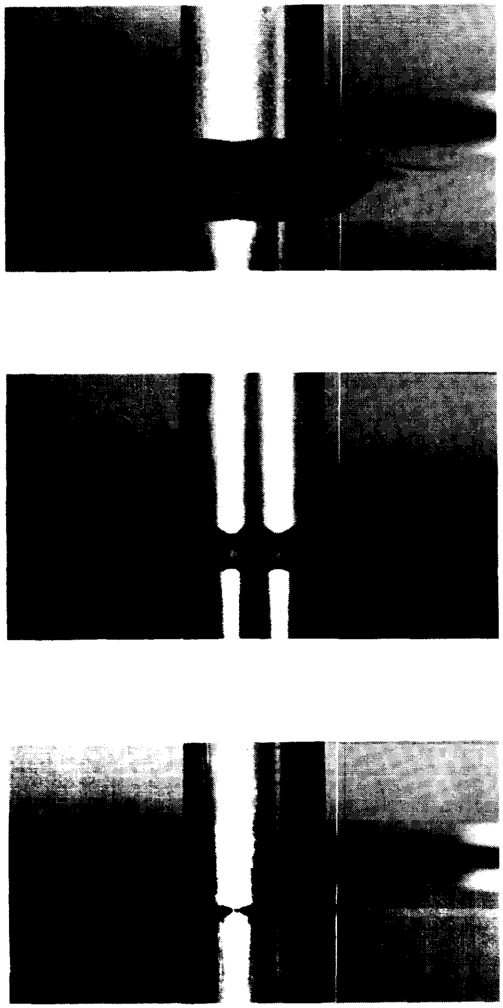
the experimental apparatus, the operating procedures used to make the experiments conform as closely as possible to the conditions assumed in the theory, and the methods used to characterize the fluid properties in steady flow.

The experimental apparatus was designed to oscillate a circular rod with a sinusoidal motion about its axis in a large volume of fluid. The rod was aligned vertically in the center of a circular container of radius $b = 15.2$ cm and depth 7.7 cm. The rod was driven from below through a rack and pinion mechanism. A spring-loaded cone bearing at the upper end restricted the motion of the rod to oscillations about its axis. The rack was connected to a variable-speed electric motor by means of a crank consisting of an adjustable slider. With this arrangement the amplitude of the rack motion, and thus the angle of twist Θ of the rod, could be set to any predetermined value. In the experiments Θ was measured directly by observing the motion of a radial line on the upper end of the rod using a traveling microscope fitted with a protractor eyepiece. The oscillation frequency ω

Fig. 3. The mean climb. The angular frequency of the rod is given by

$$\Omega(a, t) = \frac{\omega \Theta}{2} \sin \omega t$$

where $(a, \omega, \Theta) = (0.636\text{ cm}, 26.4\text{ rad/sec}, 2.91\text{ rad})$. The black mark on the rod shows three different angular displacements of the rod during an excursion through a half cycle. The mean climb dominates the whole climb. The oscillatory part of the climb is not visible.



A

B

C

Fig. 5. Mean rise in height for three different values of ω, $(\omega_A, \omega_B, \omega_C) = (0, 3.0, 8.2$ rad/sec), at (roughly) the same value of $\bar{\Omega}^2 (\approx 20)$. These three points lie in the region of validity of the second-order theory and are on the solid line in Fig. 2 and are identified in Fig. 4.

was measured by means of a photocell and light source in conjunction with ten reflecting surfaces equally spaced around the circumference of the motor drive-shaft. The apparatus was designed so that rods of any radius could be used, although in these experiments we used only one rod of radius $a = 0.636$ cm.

The fluid used for these experiments was a concentrate of a methacrylate copolymer in oil*. The fluid filled the container to its full depth of 7.7 cm. No

* TL-227, manufactured by the Texaco Oil Company. The fluid (STP) used in the experiments of JOSEPH, BEAVERS & FOSDICK (1973) and of BEAVERS & JOSEPH (1975) on the climb on a rod in steady rotation, exhibits the same general response to the oscillating rod as TL-227 but the climb is about twenty times smaller.

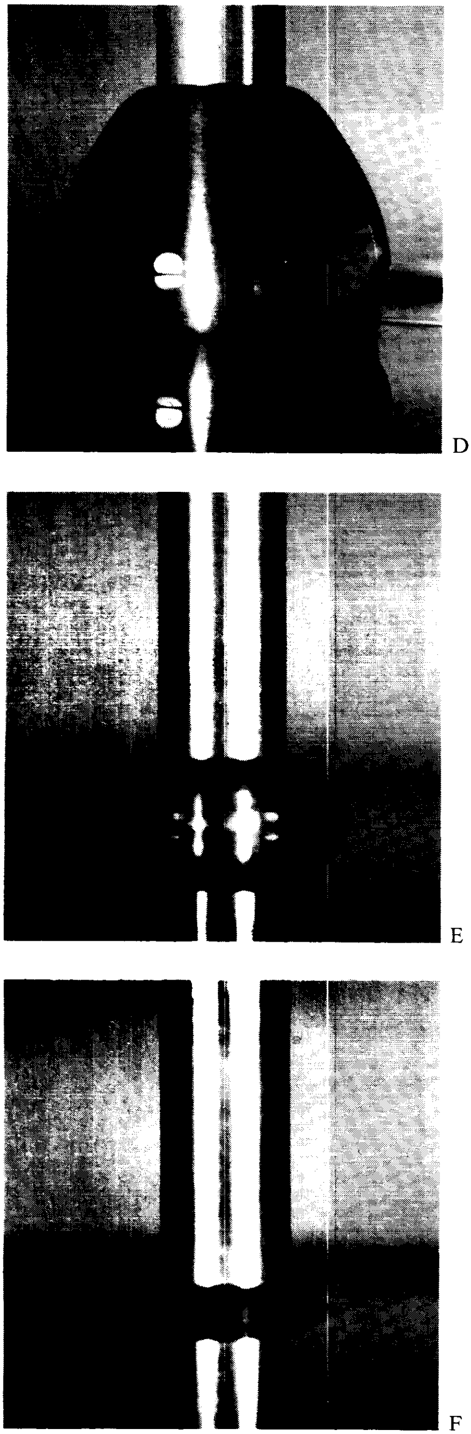


Fig. 6. Mean rise in height for three different values of ω, $(\omega_D,\ \omega_E,\ \omega_F) = (0,\ 25.0,\ 65.9$ rad/sec) at (roughly) the same value of $\bar{\Omega}^2$ (≈ 667). One of these points, F, is in the region of validity of the second-order theory. The other two points are in the region where the rise is diminished by higher order effects which are neglected at order two. The points (E, F) are identified in Fig. 2.

special precautions were taken to control the temperature of the fluid during the course of the experiments. We did monitor the fluid temperature, however, and for all measurements the fluid temperature was in the range $80.5 \pm 1°$ F.

An experimental run consisted of selecting the angle of twist Θ and varying ω through the range of values within the capacity of the apparatus ($0 < \omega < 65$ rad/sec, approximately). To obtain horizontal contact at the fluid-rod-air interface we coated the rod with Scotchgard, and took data for increasing values of ω so that the interface always advanced on to a dry part of the rod. We could not detect any departure from flat contact, even when ω was large (Figs. 3, 5 and 6).

The methods used to obtain fluid properties $(\rho, T, \mu, \alpha_1, \alpha_2) = (0.896\,\text{gm/cm}^3$, 30.5 dynes/cm, 200 poise, -50 gm/cm, 83.9 gm/cm) were essentially those described by BEAVERS & JOSEPH (1975). The values of μ and α_1 were obtained from measurements made on a Rheometrics Mechanical Spectrometer using cone and plate flow. The value of $\hat{\beta} = 3\alpha_1 + 2\alpha_2$ was computed from the slope of the line labeled $\omega = 0$ in Fig. 4, using Equation (6.5) of BEAVERS & JOSEPH. The value of α_2 was obtained from the values of $\hat{\beta}$ and α_1. We believe that $\hat{\beta}(= 17.9$ gm/cm) and μ were accurately determined. The value $\alpha_1 = -50$ gm/cm is uncertain because of the inherent limitation of the cone and plate apparatus at low rates of shear, and also because in TL-227 the value of the normal stress difference increases very rapidly as the shear rate decreases to zero.

10. Results

We turn now to a description of the main results of the experiment. Figure 3 shows how the mean climb dominates the whole climb. No change in the height of climb can be detected for the three angular positions shown. In the experiment, the oscillatory part of the climb was not visible to the unaided eye except at the highest values of ω and Θ.

Figures 4, 5 and 6 show that the mean climb is a decreasing function of ω when $\bar{\Omega}^2 = \omega^2\Theta^2/8$ is fixed. The largest climb for a fixed value of $\bar{\Omega}^2$ occurs for steady flow, $\omega = 0$. At each fixed value of ω, the mean climb is an increasing function of the angle of twist. In Fig. 4 the lines for $\omega = 3.0$, 5.0 and 8.1 were drawn to correspond to points on the universal curve of Fig. 2, while the line $\omega = 0$ represents the best fit through the steady flow experimental data for $\bar{\Omega}^2 < 25$. The justification for labeling the steady flow data as $\omega = 0$ is given in Fig. 7, where the ratio R of the height rise in steady flow to the height rise in unsteady flow at the same value of $\bar{\Omega}^2$ is plotted against ω^2. The solid line corresponds to the normalized rise curve of Fig. 2. The theory requires $R \to 1$ as $\omega \to 0$, which appears to be substantiated by the experimental data.

The main experimental results of this study are the data and the solid line shown in Fig. 2. To describe this figure it is first necessary to specify the region in which the second order theory is valid. JOSEPH, BEAVERS & FOSDICK (1973) developed a criterion to identify the region of validity of the second order theory. We are going to use the same argument to identify the region of validity for the second order theory in our experiments. We first note that the rise in height may be regarded as given by the leading term of a power series in the Froude number

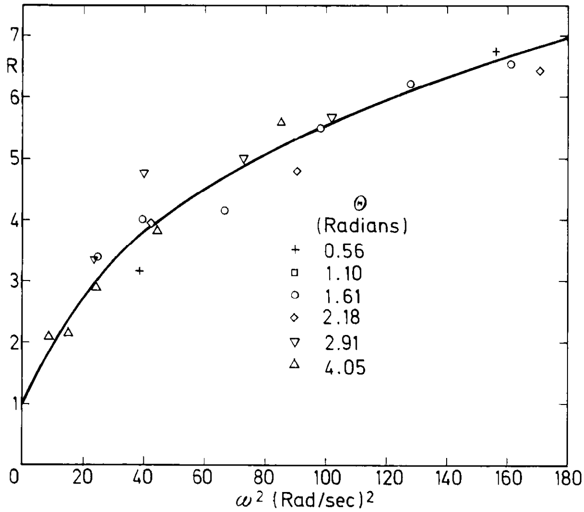


Fig. 7. Limiting $\omega \to 0$ values of the mean rise. R is the ratio of the height rise in steady flow $h_s \simeq k_s \bar{\bar{\Omega}}^2$ when $\bar{\bar{\Omega}}^2$ is small to the mean height rise in unsteady flow $\bar{\bar{h}} = k(\omega)\bar{\bar{\Omega}}^2$. The data, in agreement with the theory, show that

$$R = \frac{h_s}{\bar{\bar{h}}} = \frac{k_s}{k(\omega)} \to 1$$

as $\omega \to 0$.

$F = \bar{\bar{\Omega}}^2 l/g$:

$$\bar{\bar{h}}(R, \omega) = \frac{g}{l}\,\bar{\bar{h}}(R, \omega)\, F + O(F^2). \tag{10.1}$$

Here l is a characteristic length depending on the radius of the rod and the fluid. Though this length is not uniquely given, and in any case must depend on unknown characterizing parameters for the fluid, we may guess, using (10.1), that

$$l^2 = O[g\,\bar{\bar{h}}(a, \omega)] \quad \text{and} \quad g\,\bar{\bar{h}}(a, \omega) \sim \hat{\beta}_A(\omega)\, a \left(\frac{g}{\rho T}\right)^{\frac{1}{2}}$$

The criterion for the validity of the second-order theory used by Joseph, Beavers & Fosdick (p. 385) is that the Froude number F should be less than one:

$$F = \frac{\bar{\bar{\Omega}}^2 l}{g} \sim \frac{\omega^2 \Theta^2}{8g} \left[\hat{\beta}_A(\omega)\, a \left(\frac{g}{\rho T}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} < 1.$$

This criterion may be written as

$$\omega^2 < \frac{8g}{\Theta^2} \left[\frac{1}{a\hat{\beta}_A(\omega)} \left(\frac{\rho T}{g}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{10.2}$$

Our experiments show that $\hat{\beta}_A(\omega)$ is a decreasing function of ω with a maximum value $\hat{\beta}_A(0) = 3\alpha_1 + 2\alpha_2$. Hence the criterion

$$\omega^2 < \frac{8g}{\Theta^2} \left[\frac{1}{a(3\alpha_1 + 2\alpha_1)} \left(\frac{\rho T}{g}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{10.3}$$

is a conservative estimate of the region of validity of the second order theory. The discrepancy between the criterion (10.2) and the criterion (10.3) is an increasing function of ω. Hence, if the criterion (10.2) is valid, then (10.3) will give increasingly conservative estimates of the ω interval of validity of the second-order theory.

JOSEPH, BEAVERS & FOSDICK (1973) showed that the criterion (10.3) was consistent with their observations of the height rise of STP in the steady case. This criterion is also applicable in the present experiments, as can be seen from inspection of Fig. 2. In Fig. 2 we have plotted the observed normalized height rise for six different angles of twist Θ, and have indicated with vertical bars the corresponding values of

$$\omega_c^2 = \frac{8g}{\Theta^2}\left[\frac{1}{a(3\alpha_1+2\alpha_2)}\left(\frac{\rho T}{g}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} = \frac{758}{\Theta^2\sqrt{a}} \qquad \text{(in our experiment)}$$

given below:

Θ	4.05	2.91	2.18	1.61	1.10	0.56	radians
symbol	△	▽	◇	○	□	◇	
ω_c^2	58	112	200	368	786	3045	(rad/sec)^2

The experiments support the argument of the previous paragraph and lead to the conclusion that (10.3) is a conservative estimate of the region of validity of the second-order theory, which becomes more and more conservative as ω is increased (because $\hat{\beta}_A(\omega)$ decreases).

To relate Fig. 2 to the second-order theory the reader should mentally delete all of the experimental points which violate the criterion (10.3). For example, all of the points marked △ for which $\omega^2 > 58$ should be deleted. When this is done the remaining points define the curve shown as a solid line in Fig. 2. We interpret this curve as the experimental realization of the universal function

$$H(a,\omega) \sim \omega^2 \bar{\bar{h}}(a,\omega)/8$$

whose values are given theoretically by (6.11) and (7.12). Returning now to Fig. 2, we note that many of the deleted points also lie on the curve. This feature becomes increasingly pronounced as the angle of twist is increased. We interpret this feature as a demonstration that (10.2) is a more correct and less conservative estimate than (10.3).

Our conclusion then is that, as in the steady case, the points which break away from the universal function are a manifestation of effects which are induced by the higher order terms in the expansion (10.1).

In Fig. 8 we compare the normalized mean height rise curves given by experiments (the dashed line) with the theoretical height rise curves for the generalized (N, M) Maxwell models when $(N, M) = (1, 1)$ and $(1, 2)$. The values of the characterizing constants for these generalized models are determined by requiring that the theoretical and experimental rise curves should match over the largest possible interval $[0, \omega)$ of oscillation frequencies. We did not compute values of the five parameters which appear in the next member $(2, 2)$ of the sequence. We expect that when the five parameters of the $(2, 2)$ member of the sequence of

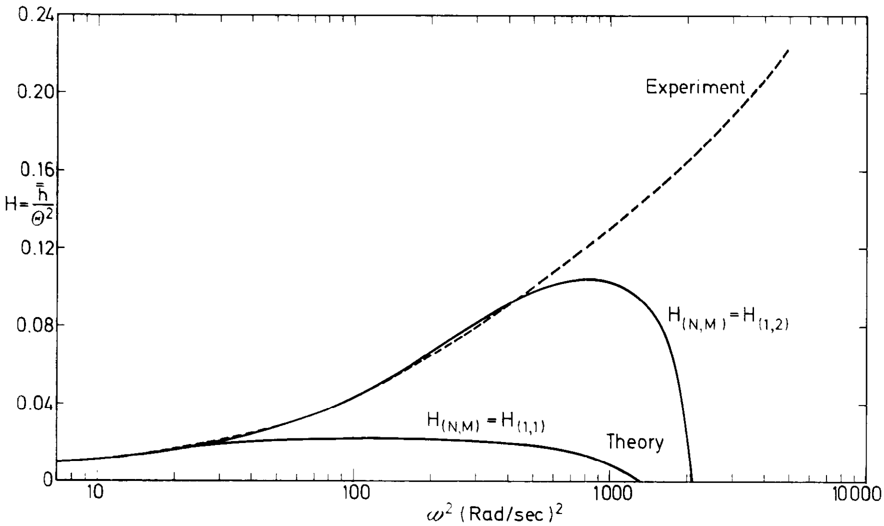


Fig. 8. Comparisons of the predictions of the generalized (N, M) Maxwell model with the observed normalized rise curve for $(N, M) = (1, 1)$ and $(N, M) = (1, 2)$.

Maxwell models are chosen optimally, the agreement between theory and experiment will be extended to much larger values of ω.

It is necessary here to explain what is meant by agreement between theory and experiment. The (N, M) Maxwell model has $2N + 2M$ parameters. The values of three of these parameters are fixed from experiments with steady flow. There are therefore $2N + 2M - 3$ parameters to be determined. The equations for the height rise is such that we may fit $2N + 2M - 3$ points of the normalized rise curve by an appropriate selection of the $2N + 2M - 3$ parameters. This does not guarantee that theoretical and experimental points between $2N + 2M - 3$ fitted points will lie close together. If the fitted points are widely spaced, theoretical and experimental values at intermediate points will not lie close together. Now consider Fig. 8. For the (1, 1) fluid there is just one disposable constant k_1^2. We fit the point at $\omega^2 = 25$. There is a close fit for $I_{(1,1)}(\omega) = [\omega \colon 0 \leqq \omega^2 < 30]$ (approximately) when

$$k_1^2 = 15.43.$$

For the (1, 2) fluid there are three disposable constants, k_1^2, k_2^2 and C_1. We fit the points at $\omega^2 = 25$, 100 and 400. There is a close fit for $I_{(1,2)}(\omega) = [\omega \colon 0 \leqq \omega^2 \leqq 450]$ when

$$k_1^2 = 14.50, \quad k_2^2 = 307.0, \quad C_1 = 0.9735.$$

It should be noted that k_1^2 does not change by much. We expect to find a similar extension of the interval of good fit for the higher members of the (N, M) sequence.

The existence of increasing intervals $I_{(N,M)}(\omega)$ over which we can fit the curves (and not just points) is one form of agreement between theory and experiment. This kind of agreement is already established for $I_{(1,1)}(\omega)$ and $I_{(1,1)}(\omega)$. Assuming for the discussion, that the intervals of agreement may be enlarged by matching up the (N, M) sequence, we come to the following concept: There is an increasing sequence of generalized (N, M) Maxwell fluids. Each fluid represents an approximation to a class of real simple fluids in small amplitude periodic motions over an increasing, but restricted, interval $I_{(N,M)}(\omega)$ of frequencies. The constants which are required to specify completely each fluid in the sequence may be determined by comparison with a universal function similar to that shown in Fig. 2. If the concept is good, it will be possible to predict other periodic motions of the now completely determined (N, M) fluids over similar ranges of frequency.

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