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565

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FACTORIZATION THEOREMS FOR THE STABILITY OF BIFURCATING SOLUTIONS

by

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The theory of bifurcation at a simple complex eigenvalue, developed for ordinary differential equations by Hopf (1942) and extended to partial differential equations, like the Navier-Stokes equations, by Joseph and Sattinger (1972)*, using Hopf's methods, and by Iooss (1972), Yudovich (1971), and Marsden (1973), using other methods, is a local theory which is restricted to small values of ε, the amplitude of the bifurcating solution. In the local theory, bifurcating solutions which branch to the right (supercritical bifurcation) are stable and bifurcating solutions which branch to the left (subcritical bifurcation) are unstable. I am going to derive the form which this stability result must take when the restriction on the size of the amplitude of the bifurcating solution is removed.[†] Subject to conditions, we are going to replace Hopf's local statement of stability with a global statement of stability. The local statement, due to Hopf, is roughly: "Subcritical solutions branching at a simple eigenvalue are unstable; supercritical solutions are stable." The global statement is: "Solutions for which the response decreases with increasing amplitude are unstable; solutions for which the response increases with amplitude are stable." Expressed in physical terms, the global statement asserts that pipe flows for which the mass flux increases as the pressure gradient decreases are unstable or, for another example, convection for which the heat transported decreases as the temperature is increased is unstable.

The results to be given here trace the eigenvalues of the Frechet derivative of the nonlinear operator whose null space contains the bifurcating solution. The main result is a factorization theorem which shows among other things that the relevant eigenvalue vanishes at critical points of the bifurcation curve. When carried to small amplitudes we recover and extend Hopf's original stability results. We do not consider secondary bifurcations here; secondary bifurcations certainly alter the stability interpretation of the theorems but not the theorems.

The recovery of stability on subcritical branches which turn around is a physically important result which may have applications to observations of the mechanics of subcritical turbulence. I will discuss

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* This paper is designated in the sequel by the letters JS.

† Mathematically, the result takes form in the factorization theorems of Joseph (see Joseph & Nield, 1975). I wish to thank Professor Nield for his important contributions to the computations which at an early stage of the investigation led me to the factorization. The good suggestions of Professors P. Rabinowitz and M. Crandall about the local interpretation of the factorization are also most gratefully acknowledged.

these applications at the conclusion of this lecture.

We are now ready to state and prove our main result. Consider the following evolution problem on a Banach space:

$$\frac{dV}{dt} + L(\mu)V + N(\mu;V) = 0 \tag{1}$$

where μ is a real parameter,

$$L(\mu) = L_0 + \mu L_1 + \mu^2 L_2 + \ldots$$

is a linear operator, analytic in μ, and $N(\mu;V)$ is a nonlinear operator, analytic in V and μ, whose power series in V starts with terms of at least second degree. To simplify the computations, we take

$$L(\mu) = L_0 + \mu L_1 \tag{2}$$

and consider quadratic nonlinearities

$$N(\mu,\cdot) = N(\cdot,\cdot). \tag{3}$$

Without loss of generality we shall follow JS and assume that $V(t) \equiv 0$ loses stability when the eigenvalues of $\gamma(\mu) = \mathrm{Re}\gamma(\mu) + i\,\mathrm{Im}\gamma(\mu)$ of the spectral problem for $V \equiv 0$,

$$-\gamma\zeta + L_0\zeta + \mu L_1\zeta = 0, \tag{4}$$

cross the imaginary γ axis in conjugate pairs as μ passes through zero to the right,

$$\gamma(0) = i\omega_0, \qquad \bar{\gamma}(0) = -i\omega_0. \tag{5,6}$$

It is further assumed that $\gamma(\mu)$ is a simple isolated eigenvalue of L_0 and that the loss of stability is strict, $\mathrm{Re}\gamma_\mu(0) < 0$.

The operators $L(\mu)$, L_0, L_1 and $N(\mu;V)$ are defined in a precise way by JS and will not be discussed here. In the analysis it is sufficient to think of the simplest realizations of (1) - the systems of ordinary differential equations considered by Hopf (1942). For ordinary differential equations, V(t) is a vector, $L(\mu)$ is a matrix and $N(\mu;V)$ is the composition of matrices of functions of V and matrices independent of V. Our results hold for the general forms of L and N; the details of the computation in the demonstrations and the notations are more involved in the general case, but the results are the same. The extension of the results of this analysis to partial differential equations is immediate when $L(\mu)$ and $N(\mu;\cdot)$ satisfy the conditions stated by JS. For example, the results hold for nonlinear diffusion-reaction problems and for problems of fluid mechanics governed by the Navier-Stokes equations. Readers interested in this omitted aspect of the analysis may wish to consult Sattinger's monograph (1972).

To state the results, it is first necessary to specify the bifurcation problem and the spectral problem for the bifurcating solution. We introduce the scalar product

$$[a,b] = \frac{1}{2\pi}\int_0^{2\pi} a\cdot\overline{b}\ \ ds \tag{7}$$

for complex-valued vectors $a(s)$, $b(s)$ which are 2π periodic in $s = \omega t$. The angle brackets designate volume-averaged integrals; the averaging is over the spatial region on which the vectors $a(x,s)$ and $b(x,s)$ are defined. For ordinary differential equations, $\langle a\cdot b\rangle = a\cdot b$. Real-valued bifurcating time-periodic solutions $u(s;\varepsilon)$ of (1), with L and N given by (2) and (3), satisfy

$$Ju + N(u,u) = 0, \qquad 2\varepsilon^2 = [u\cdot u], \qquad u(s) = u(s+2\pi) \tag{8}$$

where

$$Ju = \omega(\varepsilon)\dot{u} + L_0 u + \mu(\varepsilon)L_1 u, \qquad \dot{u} \equiv \frac{du}{ds} \tag{9}$$

and

$$\begin{pmatrix} u(s;\varepsilon) \\ \omega(\varepsilon)-\omega_0 \\ \mu(\varepsilon) \end{pmatrix} = \sum_{\ell=1}^{\infty} \varepsilon^{\ell} \begin{pmatrix} u_\ell(s) \\ \omega_\ell \\ \mu_\ell \end{pmatrix} \tag{10}$$

are convergent power series in some complex neighborhood of $\varepsilon = 0$. The Taylor coefficients in (10) have the following properties:

$$\omega_0 = \mathrm{Im}\gamma(0),$$

$$\omega_{2\ell-1} = \mu_{2\ell-1} = 0 \qquad \ell \geqslant 1 \tag{11}$$

and

$$u_1(s) = z_1 + z_2$$

where

$$z_1 \doteq e^{-is}\zeta \text{ and } z_2 = \overline{z}_1 \tag{12}$$

and ζ is the eigenfunction of L_0 belonging to the simple eigenvalue $i\omega_0$. The amplitude of ζ is fixed by the requirement that

$$[u_1\cdot u_1] = 2\left\langle|\zeta|^2\right\rangle = 2. \tag{13}$$

The coefficients in the series (10) may be uniquely and sequentially determined from the boundary value problems which arise from (8) and (10). These problems are all in the form

$$J_0 u_\ell + f_\ell(s) = 0, \qquad J_0 = \omega_0\frac{d}{ds} + L_0 \tag{14}$$

where u_ℓ and 2π-periodic functions satisfying a normalizing condition arising from (8). The Fredholm alternative for these problems is proved in lemmas of section 7 in JS. The perturbation problems are uniquely solvable and have bounded inverses when

$$[f_\ell\cdot z_1^*] = [f_\ell\cdot z_2^*] = 0$$

where $J_0^* z_1^* = J_0^* z_2^* = 0$ are eigenvalue problems for the adjoint operator J_0^* (see JS),

$$z_1^* = e^{-is}\zeta^*, \qquad z_2^* = \overline{z}_1^* .$$

where

$$i\omega_0\zeta^* + L_0^*\zeta^* = 0 \tag{15}$$

and L_0^* is the adjoint operator for L_0. In the perturbation problem, $f_\ell(s)$ is real-valued and the one complex condition,

$$[f_\ell] \equiv [f_\ell \cdot z_1^*] = 0, \tag{16}$$

suffices for unique solvability. The amplitude of ζ^* is selected so that

Then

$$\left.\begin{aligned} [u_1] &= \left\langle \zeta\cdot\zeta^* \right\rangle = 1. \\ [\dot{u}_1] &= -i. \end{aligned}\right\} \tag{17}$$

The formula

$$-\gamma_\mu + [L_1 u_1] = 0, \qquad \gamma_\mu = \left.\frac{d\gamma}{d\mu}\right|_{\mu=0} \tag{18}$$

follows easily from (4), (15) and (17). The assumption that $V \equiv 0$ loses stability strictly as μ is increased past zero implies that $\mathrm{Re}\gamma_\mu < 0$.

The spectral problem for the conditional stability of (10) is obtained by introducing disturbances of the form

$$\begin{aligned} V &= \delta e^{-\sigma t}\Gamma + u(s;\varepsilon), \\ \Gamma &= \alpha(\varepsilon)\dot{u}(s;\varepsilon) + \gamma(s;\varepsilon) \end{aligned} \tag{19}$$

into (1) followed by linearization, $\delta \to 0$. The function Γ or, equivalently, the function γ, may be normalized by any convenient convention. We find that

$$\tau\dot{u} - \sigma\gamma + \mathcal{J}\gamma = 0 \tag{20}$$

where

$$\mathcal{J}(\cdot) = J(\cdot) + N(u,\cdot) + N(\cdot,u)$$

and

$$\tau = -\sigma\alpha.$$

According to Floquet theory, solutions of $\gamma(s)$ of (20) must be 2π-periodic functions of s. Moreover (see JS, section 5),

$$\begin{pmatrix} \gamma(s;\varepsilon)-u_1(s) \\ \tau(\varepsilon) \\ \sigma(\varepsilon) \end{pmatrix} = 2\begin{pmatrix} u_2(s) \\ \omega_2-\mu_2\mathrm{Im}\gamma_\mu \\ 0 \end{pmatrix}\varepsilon + \sum_{\ell=2}\varepsilon^\ell \begin{pmatrix} \gamma_\ell(s) \\ \tau_\ell \\ \sigma_\ell \end{pmatrix} \tag{21}$$

where τ_ℓ and σ_ℓ are real; and

$$i\tau_1 + \sigma_2 = 2(i\omega_2 - \mu_2\gamma_\mu). \tag{22}$$

The equation

$$\sigma_2 = -2\mu_2\mathrm{Re}\gamma_\mu, \qquad \mathrm{Re}\gamma_\mu < 0 \tag{23}$$

shows that subcritical solutions ($\mu_2 < 0$) are unstable. The series (21) has a finite, but possibly small, radius of convergence. The proof of convergence follows a slightly different path which allows the use of the implicit function theorem (see JS).

We are now ready to state and prove an extension of Hopf's theorem. The extension takes form as a factorization theorem. The factorization holds globally provided only that the quantities mentioned in the theorem are continuous functions of ε. No matter what the regularity properties of the solution may be for large values of ε they are regular analytic functions in some circle at the origin of the complex ε plane.

Suppose $\underset{\sim}{u}(\underset{\sim}{x},s;\varepsilon)$, $\omega(\varepsilon)$ and $\mu(\varepsilon)$ are real analytic functions on an open interval I_1 containing the point $\varepsilon = 0$. Then,

$$\underset{\sim}{\phi}(\underset{\sim}{x},s;\varepsilon) = \underset{\sim}{u}_\varepsilon(\underset{\sim}{x},s;\varepsilon) + \mu_\varepsilon(\varepsilon)\hat{\underset{\sim}{\phi}}(\underset{\sim}{x},s;\varepsilon),$$

$$\tau(\varepsilon) = \omega_\varepsilon(\varepsilon) + \mu_\varepsilon(\varepsilon)\hat{\tau}(\varepsilon) \tag{24}$$

and

$$\gamma(\varepsilon) = \mu_\varepsilon(\varepsilon)\hat{\gamma}(\varepsilon)$$

where $\hat{\underset{\sim}{\phi}}(\underset{\sim}{x},s;\varepsilon)$, $\hat{\tau}(\varepsilon)$ and $\hat{\gamma}(\varepsilon)$ are real analytic functions on an interval $I_2 < I_1$ containing the point $\varepsilon = 0$. Moreover, $\hat{\tau}(\varepsilon)$ and $\hat{\gamma}(\varepsilon)/\varepsilon$ are even functions of ε and such that

$$\hat{\gamma}_1 = -\text{re}\ \sigma_\mu, \qquad \tau_0 = -\text{im}\ \sigma_\mu.$$

The representation for $\sigma(\varepsilon)$ shows that $\sigma(\varepsilon)$ has all of the zeros of μ_ε. Unfortunately, we cannot assert that the function $\hat{\sigma}(\varepsilon)$ is of one sign when ε is large. When ε is small the sign of $\hat{\sigma}$ is known and the representation (24) leads to an extension of Hopf's stability theorem (see Theorem 2). It is of particular interest to determine the sign of $\hat{\sigma}(\varepsilon)$ at critical points $\tilde{\varepsilon}$, defined by

$$\mu^{\langle 1 \rangle} = \left.\frac{d\mu}{d\varepsilon}\right|_{\varepsilon=\tilde{\varepsilon}} \equiv \mu_\varepsilon(\tilde{\varepsilon}) = 0 \tag{25}$$

of the bifurcating curve.

Proof of Theorem 1. We first introduce the representation (24) into (20) and find that

$$(\omega_\varepsilon+\mu_\varepsilon\hat{\tau})\dot{u} + \mathcal{J}u_\varepsilon + \mu_\varepsilon\mathcal{J}\hat{\gamma} - \mu_\varepsilon\hat{\sigma}(u_\varepsilon+\mu_\varepsilon\hat{\gamma}) = 0 \tag{26}$$

Differentiating (8) with respect to ε, we find that

$$\mathcal{J}u_\varepsilon + \omega_\varepsilon\dot{u} + \mu_\varepsilon L_1 u = 0. \tag{27}$$

When equation (27) is subtracted from (26), $\mathcal{J}u_\varepsilon + \omega_\varepsilon\dot{u}$ cancels and μ_ε may be factored from each of the remaining terms. It follows that

$$\mathcal{J}\hat{\gamma} + \hat{\tau}\dot{u} - L_1 u - \hat{\sigma}(u_\varepsilon+\mu_\varepsilon\hat{\gamma}) = 0. \tag{28}$$

The factorization (28) does not require analyticity in ε. To establish (24) under better hypotheses which replace analyticity with continuity, it would be necessary to prove the existence of continuous (in ε) functions $\hat{\underset{\sim}{\gamma}}(s;\varepsilon) = \hat{\gamma}(s+2\pi;\varepsilon)$, $\hat{\tau}(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$ solving (28). I was unable to construct such an existence theorem. However, (28) does have a unique solution which is analytic in ε and has a power series converging in some circle centered at the origin of the complex ε plane. To prove this, we first assume that these functions have the following representations:

$$\begin{bmatrix} \hat{\gamma}(s;\varepsilon) \\ \hat{\tau}(\varepsilon)-\hat{\tau}_0 \\ \hat{\sigma}(\varepsilon) \end{bmatrix} = \sum_{\ell=1} \varepsilon^{\ell} \begin{bmatrix} \hat{\gamma}_{\ell}(s) \\ \hat{\tau}_{\ell} \\ \hat{\sigma}_{\ell} \end{bmatrix} \tag{29}$$

Inserting (29) and (10) into (28), we find that

$$J_0\hat{\gamma}_1 - L_1u_1 + \hat{\tau}_0\dot{u}_1 - \hat{\sigma}_1u_1 = 0, \tag{30}$$

$$J_0\hat{\gamma}_2 - L_1u_2 + \sum_{n+m=2}[\hat{\tau}_n\dot{u}_m-(m+1)u_{m+1}\hat{\sigma}_n+\mathcal{J}_n\hat{\gamma}_m] = 0, \tag{31}$$

and

$$\begin{aligned} J_0\hat{\gamma}_{\ell} - L_1u_{\ell} &+ \sum_{n+m=\ell}[\hat{\tau}_n\dot{u}_m-(m+1)u_{m+1}\hat{\sigma}_n+\mathcal{J}_n\hat{\gamma}_m] \\ &- \sum_{n+m+r=\ell}(n+1)\mu_{n+1}\hat{\sigma}_m\hat{\gamma}_r = 0, \qquad \ell \geqslant 3 \end{aligned} \tag{32}$$

where

$$\mathcal{J}_n(0) = \omega_n\frac{d}{ds}(\cdot) + \mu_nL_1(\cdot) + N(\cdot,u_n) + N(u_n,\cdot).$$

We next invoke the solvability condition (16) and use (17) to find that

$$\gamma_{\mu} + i\hat{\tau}_0 + \hat{\sigma}_1 = 0, \tag{33}$$

$$[L_1u_2] + i\hat{\tau}_1 + \hat{\sigma}_2 - \hat{\tau}_0[\dot{u}_2] + 2\hat{\sigma}_1[u_2] + [\mathcal{J}_1\hat{\gamma}_1] = 0 \tag{34}$$

and

$$\begin{aligned} [L_1u_{\ell}] + i\hat{\tau}_{\ell-1} + \hat{\sigma}_{\ell} &- \sum_{\substack{n+m=\ell \\ n\neq\ell}}\{\hat{\tau}_{n-1}[\dot{u}_{m+1}] - (m+1)\hat{\sigma}_n[u_{m+1}] \\ &+ [\mathcal{J}_n\hat{\gamma}_m]\} - \sum_{n+m+r=\ell}(n+1)\mu_{n+1}\hat{\sigma}_m[\hat{\gamma}_r], \qquad \ell \geqslant 3 \end{aligned}$$

The operator J_0 in equations (30), (31) and (32) has a bounded inverse on the complement of the null space of the operator J_0. When normalized in any convenient way, the solutions $\hat{\gamma}_{\ell}$ are unique. The normalized coefficients $\hat{\gamma}_{\ell}$, $\hat{\tau}_{\ell}$ and $\hat{\sigma}_{\ell}$ may be determined uniquely and sequentially and the series (29) converge when ε is sufficiently small. The proof of convergence copies the proof given in section 7 and 8 of JS. The functions $\hat{\gamma}(\varepsilon)$, $\hat{\tau}(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$ may then be extended as real analytic functions defined on the interval I_2 of analyticity. In general, I_2 could not extend beyond the interval I_1 of analyticity of the operators defined in (20). This completes the proof of theorem 1.

When $\mu_2 \neq 0$, representation $\sigma = \mu_{\varepsilon}\hat{\sigma}$ leads to equation (23). This equation was derived by Hopf and used by him to prove the instability of subcritical solutions and the stability of supercritical solutions. The following local extension of Hopf's theorem may now be proved.

Theorem 2. When ε is small,

$$\sigma(\varepsilon) = -\mu_\varepsilon \varepsilon \mathrm{Re}\gamma_\mu + \mu_\varepsilon O(\varepsilon^3) \tag{35}$$

Subcritical solutions are unstable and supercritical solutions are stable.

Stability, in theorem 2 and throughout this paper, is in the sense of linearized theory. The result stated in theorem 2 does not require that $\mu_2 \neq 0$. To prove (35) we first note, following the computation (5.10) of JS, that $i\hat{\tau}_1 + \hat{\sigma}_2 = 0$. Hence, $\hat{\sigma}_2 = 0$ and (35) follows directly from (24) and (33).

The stability of steady bifurcating solutions can also be studied by the factorization method. The analysis follows along the lines laid out in the analysis of the stability of time-periodic bifurcating solutions. The following theorem holds:

Theorem 3 (Steady bifurcation). Suppose the $u(\varepsilon)$ and $\mu(\varepsilon)$ are real analytic functions on an open interval I_1 containing the point $\varepsilon = 0$. Then

$$\gamma(\varepsilon) = u_\varepsilon + \mu_\varepsilon \hat{\gamma}(\varepsilon)$$

and

$$\sigma(\varepsilon) = \mu_\varepsilon \hat{\sigma}(\varepsilon) \tag{36}$$

where $\hat{\gamma}(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$ are real analytic functions on an open interval I_2 containing the point $\varepsilon = 0$ and

$$\hat{\sigma}_1 = -\sigma_\mu + O(\varepsilon). \tag{37}$$

If $\mu_1 \neq 0$, then the bifurcation is two-sided and the subcritical branch is unstable.

If $\hat{\sigma}(\varepsilon)$ does not change sign, then $\sigma(\varepsilon)$ is negative when $\mu_\varepsilon > 0$ and is positive when $\mu_\varepsilon < 0$ (see Figs. 1 and 2). In the simplest situations, those in which $\hat{\sigma}$ controls stability and does not change sign, the unstable bifurcating solution regains stability as it turns around a critical point of the bifurcation curve. For partial differential equations and ordinary differential equations in R_n with $n > 2$ it is possible to have secondary instability and repeated branching. $\hat{\sigma}(\varepsilon)$ may exist but fail to control stability. For this reason, it is not possible to give a generally valid interpretation of the physical implications of the fundamental factorization. In examples of steady bifurcation in which it has been possible to construct global representations of the subcritical branch, $\hat{\sigma}(\varepsilon) \neq 0$ and stability is associated uniquely with the sign of μ_ε. In such cases we get snap-through instabilities. Computed global representations of subcritical bifur-

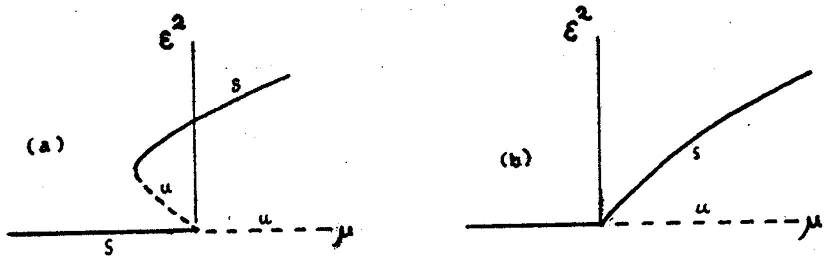

Fig. 1: Time-periodic bifurcation at a simple eigenvalue is one-sided. The time periodic solution bifurcates subcritically in (a) and supercritically in (b). Assuming $\hat{\sigma}$ is of one sign and controls stability, branches for which μ decreases as ε^2 increases are unstable.

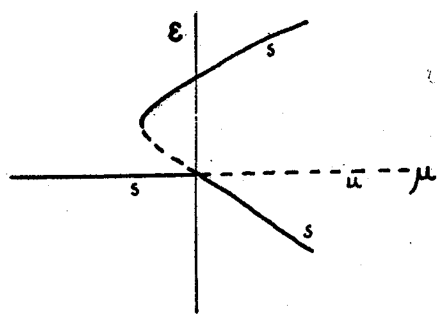

Fig. 2: Steady bifurcation at simple eigenvalue is usually two-sided. See caption for Fig. 1.

cation of time-periodic solutions are rare but, again, in one example, the numerical study of bifurcating time-periodic Poiseuille flow by Zahn, Toomre, Spiegel and Gough, (1973), $\hat{\sigma}(\varepsilon) \neq 0$ and we have restabilization of the subcritical branch and snap-through instabilities. It is necessary to add that, though the computations of Zahn, et al. proceed from a severely truncated version of the Navier-Stokes equation, the factorization (24) applies equally to the full equations and to the truncated version. Zahn, et al. consider traveling wave solutions of their equations; more general disturbances could possibly lead to instability and repeated branching on the conditionally stable upper branch of the bifurcation curve beyond the critical point. Assuming for the sake of the argument, that $\hat{\sigma}(\varepsilon)$ controls stability and that stability is associated uniquely with the sign of μ_ε, we are again led to a bifurcation picture for snap-through instability; at subcritical values of μ, $\mu_G < \mu < 0$, there are two conditionally stable solutions: laminar Poiseuille flow and time-periodic bifurcating Poiseuille flow on the stable subcritical upper branch of $\mu(\varepsilon)$, where $\mu_\varepsilon > 0$. The analysis applies to spatially periodic disturbances in infinitely long pipes and comparisons with experiments in pipes of finite length are at best suggestive. In finite pipes, where $\mu \in (\mu_G, 0)$, there also seem to be two "stable" solutions, one of which is laminar (Wygnanski and Champagne, 1973; Wygnanski, Sokolov, Friedman, 1975). The flow is spatially segregated into distinct patches of traveling packets of laminar and turbulent flow (turbulent "puffs" when μ is near μ_G, and "slugs" at higher values of μ). The transition from laminar to turbulent flow at a fixed place occurs suddenly as a puff or slug sweeps over the place, and the reverse transition occurs just as suddenly when it leaves the place. These observations suggest a sort of cycling in "phase space" between two distinct relatively stable but weakly attracting solutions.

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