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Stability of Bifurcating Time-Periodic and Steady Solutions of Arbitrary Amplitude

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The theory of bifurcation at a simple complex eigenvalue, developed for ordinary differential equations by HOPF (1942) and extended to partial differential equations, like the Navier-Stokes equations, by JOSEPH & SATTINGER (1972)*, is a local theory which is restricted to small values of ε, the amplitude of the bifurcating solution. In the local theory, bifurcating solutions which branch to the right (supercritical bifurcation) are stable and bifurcating solutions which branch to the left (subcritical bifurcation) are unstable. We are going to derive the form which this stability result must take when the restriction on the size of the amplitude of the bifurcating solution is removed.** Subject to conditions, we shall show that stability is associated with one sign, and instability with the other sign, of the slope of the bifurcating curve (see Fig. 1). The bifurcating curve in

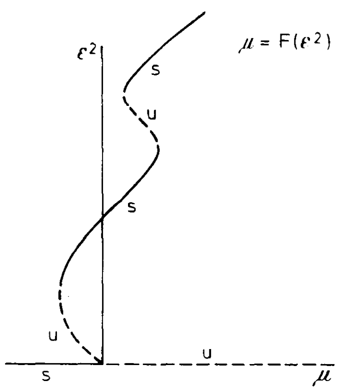

Fig. 1. Bifurcation curve for time-periodic solutions of (1). Stable solutions are indicated by solid lines; unstable solutions, by dotted lines.

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* This paper is designated in the sequel by the letters JS.

** The results and demonstrations given in this paper were formulated and proved by D. D. JOSEPH. The fundamental factorization formulas (24) were suggested to JOSEPH by computations with a pair of nonlinear ordinary differential equations initiated by JOSEPH and completed by D.A. NIELD. It is perhaps appropriate to note that our work flows in a natural way from the formulation of bifurcation theory which was developed by HOPF (1942) and extended and improved by JS. In particular, the improved construction of the Floquet exponents, which leads to the representation (19) of this paper, and the implicit function methods are so appropriate to the proofs and computations of this paper that we have merely to cite, or directly build onto, the earlier work. A very complete discussion of the functional analytic foundations of the theory of bifurcation at a simple complex eigenvalue may also be found in SATTINGER's (1972) monograph.

physical problems is properly regarded as a response curve generated by a response functional evaluated on solutions. For example, the response curve for pipe flow relates the pressure gradient $P'(\varepsilon)$ to the mass flux discrepancy ε. Here ε is the difference between the mass flux delivered by the time-periodic bifurcating flow and the mass flux delivered by laminar flow with the same pressure gradient. The physically unrealistic bifurcating solutions are those for which the mass flux discrepancy decreases as the pressure gradient is increased. Such physically unrealistic solutions certainly exist (see JOSEPH & CHEN, 1974; JOSEPH, 1974) and it is natural to expect that they are unstable, not only when ε is sufficiently small, but also for all values of ε for which $P'(\varepsilon)<0$.

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1. A Simple Example Motivating the Main Theorems

We are going to construct a mathematical theory of bifurcation in which the intuitive idea about the instability of unrealistic bifurcating solutions may be given a precise form. To clarify the mathematical nature of our effort, we first consider a simple nonlinear problem for which all the interesting results can be computed explicitly:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (\mu - F(x^2+y^2)) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

where F is an arbitrary analytic function with $F(0)=0$; one example is shown in the graph of Fig. 1. The conditional stability of the null solution of (1) is governed by the spectral problem of the linear theory of stability. This problem may be obtained by putting

$$\begin{pmatrix} x \\ y \end{pmatrix} = \delta \begin{pmatrix} x' \\ y' \end{pmatrix} e^{-\gamma t}$$

into (1). Then, after linearization, $\delta \to 0$, we find that

$$\begin{pmatrix} \gamma+\mu & -1 \\ 1 & \gamma+\mu \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0. \tag{2}$$

The eigenvalues of (2) are the complex conjugate roots

$$\gamma = -\mu \pm i.$$

The null solution is stable when $\mathrm{Re}(\gamma(\mu))>0$. For small values of μ, the null solution loses stability to a pair of imaginary eigenvalues $\gamma=\pm i$ as μ is increased from negative to positive values. The solutions of the spectral problem at criticality $(\mu=0, \mathrm{Re}(\gamma)=0)$ are all of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \text{const} \begin{pmatrix} \cos(t+\alpha) \\ \sin(t+\alpha) \end{pmatrix} \tag{3}$$

for any real number α. The time-periodic solutions of (1) with amplitude

$$\varepsilon^2 = x^2 + y^2 \tag{4}$$

which bifurcate from the null solution are given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varepsilon \begin{pmatrix} \cos(t+\alpha) \\ \sin(t+\alpha) \end{pmatrix}. \tag{5}$$

We shall say that a solution of the form (5) is in the α-class of bifurcating solutions. Substituting (4) and (5) into (1), we find that the bifurcating curve is given by

$$\mu = F(\varepsilon^2) \tag{6}$$

for all solutions in the α-class. The study of the stability of solutions in the α-class is most easily carried out using energy methods. Every solution of (1) satisfies the equation

$$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = [\mu - F(x^2+y^2)]\,(x^2+y^2).$$

To study the stability of bifurcating solutions (4), (5) and (6), we replace μ by $F(\varepsilon^2)$ and consider small disturbances

$$x^2 + y^2 - \varepsilon^2 = \delta\phi, \qquad \delta \to 0$$

of (4). We find that

$$\tfrac{1}{2}\dot{\phi} = -F'(\varepsilon^2)\phi\varepsilon^2.$$

Hence $\phi^2 \to 0$ if $F'(\varepsilon^2) > 0$ and ϕ^2 increases if $F'(\varepsilon^2) < 0$. If $\phi^2 \to 0$, then $x^2 + y^2 \to \varepsilon^2$. Apart from a possible change in phase, the α-class of bifurcating solutions is stable to small disturbances when $F'(\varepsilon^2) > 0$ and is unstable when $F'(\varepsilon^2) < 0$.

2. The Factorization Theorem for the Stability of Time-Periodic Bifurcating Solutions

We are now ready to state and prove our main result. Consider the following evolution problem on a Banach space:

$$\frac{dV}{dt} + L(\mu)\,V + N(\mu; V) = 0 \tag{7}$$

where μ is a real parameter,

$$L(\mu) = L_0 + \mu L_1 + \mu^2 L_2 + \cdots$$

is a linear operator, analytic in μ, and $N(\mu; V)$ is a nonlinear operator, analytic in V and μ, whose power series in V starts with terms of at least second degree. To simplify the computations, we take

$$L(\mu) = L_0 + \mu L_1 \tag{8}$$

and consider quadratic nonlinearities

$$N(\mu, \cdot) = N(\cdot, \cdot). \tag{9}$$

Without loss of generality we shall follow JS and assume that $V(t)\equiv 0$ loses stability when the eigenvalues of $\gamma(\mu)=\operatorname{Re}\gamma(\mu)+i\operatorname{Im}\gamma(\mu)$ of the spectral problem for $V\equiv 0$,

$$-\gamma\zeta+L_0\zeta+\mu L_1\zeta=0, \tag{10}$$

cross the imaginary γ axis in conjugate pairs as μ passes through zero to the right,

$$\gamma(0)=i\omega_0, \quad \bar{\gamma}(0)=-i\omega_0. \tag{11}$$

It is further assumed that $\gamma(\mu)$ is a simple isolated eigenvalue of L_0 and that the loss of stability is strict, $\operatorname{Re}\gamma_\mu(0)<0$.

The operators $L(\mu)$, L_0, L_1 and $N(\mu, V)$ are defined in a precise way by JS* and will not be discussed here. In the analysis it is sufficient to think of the simplest realizations of (7) — the systems of ordinary differential equations considered by HOPF (1942). For ordinary differential equations, $V(t)$ is a vector, $L(\mu)$ is a matrix and $N(\mu; V)$ is the composition of matrices of functions of V and matrices independent of V. Our results hold for the general forms of L and N; the details of the computation in the demonstrations and the notations are more involved in the general case, but the results are the same. The extension of the results of this analysis to partial differential equations is immediate when $L(\mu)$ and $N(\mu; \cdot)$ satisfy the conditions stated by JS. For example, the results hold for nonlinear diffusion-reaction problems and for problems of fluid mechanics governed by the Navier-Stokes equations. Readers interested in this omitted aspect of the analysis may wish to consult SATTINGER's monograph (1972).

To state our results, it is first necessary to specify the bifurcation problem and the spectral problem for the bifurcating solution. We introduce the scalar product

$$[\boldsymbol{a}, \boldsymbol{b}]=\frac{1}{2\pi}\int_0^{2\pi}\langle \boldsymbol{a}\cdot\bar{\boldsymbol{b}}\rangle\, ds$$

for complex-valued vectors $\boldsymbol{a}(s)$, $\boldsymbol{b}(s)$ which are 2π periodic in $s=\omega t$. The angle brackets designate volume-averaged integrals; the averaging is over the spatial region on which the vectors $\boldsymbol{a}(\boldsymbol{x}, s)$ and $\boldsymbol{b}(\boldsymbol{x}, s)$ are defined. For ordinary differential equations, $\langle \boldsymbol{a}\cdot\boldsymbol{b}\rangle=\boldsymbol{a}\cdot\boldsymbol{b}$. Real-valued bifurcating time-periodic solutions $\boldsymbol{u}(s;\varepsilon)$ of (7), with L and N given by (8) and (9), satisfy

$$J\boldsymbol{u}+N(\boldsymbol{u},\boldsymbol{u})=0, \quad 2\varepsilon^2=[\boldsymbol{u}\cdot\boldsymbol{u}], \quad \boldsymbol{u}(s)=\boldsymbol{u}(s+2\pi) \tag{12}$$

where

$$J\boldsymbol{u}=\omega(\varepsilon)\,\dot{\boldsymbol{u}}+L_0\boldsymbol{u}+\mu(\varepsilon)\,L_1\boldsymbol{u}, \quad \dot{\boldsymbol{u}}\equiv\frac{d\boldsymbol{u}}{ds}$$

and

$$\begin{pmatrix}\boldsymbol{u}(s;\varepsilon)\\ \omega(\varepsilon)-\omega_0\\ \mu(\varepsilon)\end{pmatrix}=\sum_{l=1}^{\infty}\varepsilon^l\begin{pmatrix}\boldsymbol{u}_l(s)\\ \omega_l\\ \mu_l\end{pmatrix} \tag{13}$$

* The operator $N(\mu; V)$ is introduced in JS under Eq. (9.1). The operator $\mathscr{L}$ defined under (2.3) in JS corresponds to $L(\mu)$ as used here and in sections 7 and 8 of JS.

are convergent power series in some complex neighborhood of $\varepsilon=0$. The Taylor coefficients in (13) have the following properties:

$$\omega_0=\operatorname{Im}\gamma(0),$$

$$\omega_{2l-1}=\mu_{2l-1}=0 \qquad \forall l\geqq 1$$

and

$$\boldsymbol{u}_1(s)=z_1+z_2$$

where

$$z_1=e^{-is}\zeta \quad \text{and} \quad z_2=\bar{z}_1$$

and ζ is the eigenfunction of L_0 belonging to the simple eigenvalue $i\omega_0$. The amplitude of ζ is fixed by the requirement that

$$[\boldsymbol{u}_1\cdot\boldsymbol{u}_1]=2\langle|\zeta|^2\rangle=2.$$

The coefficients in the series (13) may be uniquely and sequentially determined from the boundary value problems which arise from (12) and (13). These problems are all in the form

$$J_0\,\boldsymbol{u}_l+f_l(s)=0, \qquad J_0=\omega_0\frac{d}{ds}+L_0 \tag{14}$$

where $\boldsymbol{u}_l$ are 2π-periodic functions satisfying a normalizing condition arising from (12). The Fredholm alternative for these problems is proved in lemmas of section 7 in JS. The perturbation problems are uniquely solvable and have bounded inverses when

$$[f_l\cdot z_1^*]=[f_l\cdot z_2^*]=0$$

where $J_0^*z_1^*=J_0^*z_2^*=0$ are eigenvalue problems for the adjoint operator J_0^* (see JS),

$$z_1^*=e^{-is}\zeta^*, \qquad z_2^*=\bar{z}_1^*$$

where

$$i\omega_0\zeta^*+L_0^*\zeta^*=0 \tag{15}$$

and L_0^* is the adjoint operator for L_0.* In the perturbation problem, $f_l(s)$ is real-valued and the one complex condition,

$$[f_l]\equiv[f_l\cdot z_1^*]=0, \tag{16}$$

suffices for unique solvability. The amplitude of ζ^* is selected so that

Then

$$\left.\begin{aligned}&[\boldsymbol{u}_1]=\langle\zeta\cdot\zeta^*\rangle=1.\\ &[\dot{\boldsymbol{u}}_1]=-i.\end{aligned}\right\} \tag{17}$$

The formula

$$-\gamma_\mu+[L_1\,\boldsymbol{u}_1]=0, \qquad \gamma_\mu=\left.\frac{d\gamma}{d\mu}\right|_{\mu=0} \tag{18}$$

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* The operator L_0^* is defined by the equation $\langle L_0\boldsymbol{a}\cdot\bar{\boldsymbol{b}}\rangle=\langle\boldsymbol{a}\cdot L_0^*\bar{\boldsymbol{b}}\rangle$ for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

follows easily from (10), (15) and (17). The assumption that $V\equiv 0$ loses stability strictly as μ is increased past zero implies that $\mathrm{Re}\,\gamma_\mu<0$.

The spectral problem for the conditional stability of (13) is obtained by introducing disturbances of the form

$$\begin{aligned} V&=\delta e^{-\sigma t}\boldsymbol{\Gamma}+\boldsymbol{u}(s;\varepsilon),\\ \boldsymbol{\Gamma}&=\alpha(\varepsilon)\,\dot{\boldsymbol{u}}(s;\varepsilon)+\gamma(s;\varepsilon)\end{aligned}\tag{19}$$

into (7) followed by linearization, $\delta\to 0$.* The function $\boldsymbol{\Gamma}$ or, equivalently, the function γ, may be normalized by any convenient convention. We find that

$$\tau\dot{\boldsymbol{u}}-\sigma\gamma+\mathscr{J}\gamma=0\tag{20}$$

where

$$\mathscr{J}(\cdot)=J(\cdot)+N(\boldsymbol{u},\cdot)+N(\cdot,\boldsymbol{u})$$

and

$$\tau=-\sigma\alpha.$$

According to Floquet theory, solutions of $\gamma(s)$ of (20) must be 2π-periodic functions of s. Moreover (see JS**, section 5),

$$\begin{pmatrix}\gamma(s;\varepsilon)-\boldsymbol{u}_1(s)\\ \tau(\varepsilon)\\ \sigma(\varepsilon)\end{pmatrix}=2\begin{pmatrix}\boldsymbol{u}_2(s)\\ \omega_2-\mu_2\,\mathrm{Im}\,\gamma_\mu\\ 0\end{pmatrix}\varepsilon+\sum_{l=2}\varepsilon^l\begin{pmatrix}\gamma_l(s)\\ \tau_l\\ \sigma_l\end{pmatrix}\tag{21}$$

where τ_l and σ_l are real; and

$$i\tau_1+\sigma_2=2(i\omega_2-\mu_2\gamma_\mu).\tag{22}$$

The equation

$$\sigma_2=-2\mu_2\,\mathrm{Re}\,\gamma_\mu,\qquad \mathrm{Re}\,\gamma_\mu<0\tag{23}$$

shows that subcritical solutions ($\mu_2<0$) are unstable. The series (21) has a finite, but possibly small, radius of convergence. The proof of convergence follows a slightly different path which allows the use of the implicit function theorem (see JS).

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* The representation (19) arises from a consideration of the stability problem when $\varepsilon=0$. The function $\dot{\boldsymbol{u}}(s;\varepsilon)$ is always an eigenfunction of the operator $\mathscr{J}$ (defined under (20)) belonging to $\sigma=0$. The function γ is also an eigenfunction of $\mathscr{J}$ when $\varepsilon=0$. Hence at $\varepsilon=0$, the eigenvalue $\sigma=0$ has a multiplicity two and any linear combination of the two eigenfunctions have the same eigenvalue $\sigma=0$. Not all linear combinations perturb when ε is varied away from zero. The correct combinations of eigenfunctions is determined by the perturbation; it comes down to a determination of the function $\alpha(\varepsilon)$, and the combination $\alpha(\varepsilon)\dot{\boldsymbol{u}}$ may be regarded as giving the contribution to $\boldsymbol{\Gamma}$ which lies on the null space of the operator $\mathscr{J}$. This contribution varies with ε.

** Readers who wish to compare our formulas with those given in JS should take note of the following differences in notation: Our symbols $[\mu, \mu_n, \boldsymbol{u}, \boldsymbol{u}_n, \sigma, \varepsilon\tau, \varepsilon\alpha, \mathrm{Re}\,\gamma_\mu]$ correspond to the symbols $[\lambda_0-\lambda, -\lambda_n, \varepsilon\hat{\boldsymbol{u}}, \boldsymbol{u}_{n-1}, \sigma\omega, -\sigma\omega a, a, -\xi']$ used in sections 1–5 of JS. The other symbols are the same. To correct misprints which appear in JS: replace ξ with γ in (3.7)–(4.9); ω_m with $\omega_m-\lambda_m Im\gamma'$ in (4.10), (4.14b) and (5.12); σ with $\sigma\omega_0$ in (5.4)–(5.6); the last equation on p. 91 with

$$-\sigma_2\omega_0(1-ia_0)-i\omega_2+\gamma'\lambda_2+3[\boldsymbol{F}_2]_1=0;$$

$\varepsilon=0$ with $\varepsilon\neq 0$ before (5.6); $\boldsymbol{u}$ with $\hat{\boldsymbol{u}}$ in (3.16), (3.20) and (5.12); $-\bar{\gamma}$ with $\bar{\gamma}$ in (3.2); a_0 with $-a_0$ in (5.9), s with δ in (3.15), and λ_2 with $-\lambda_2$ in (4.14c); $\gamma_1=\boldsymbol{u}_1$ with $\gamma_1=2\boldsymbol{u}_1$ after (5.11), (4.11) by (4.13) before (5.11).

We are now ready to state and prove the factorization theorem:

Theorem 1. *Suppose $\boldsymbol{u}(s;\varepsilon)$, $\omega(\varepsilon)$ and $\mu(\varepsilon)$ are real analytic functions on an open interval I_1 containing the point $\varepsilon=0$. Then*

$$\left.\begin{aligned} \gamma(s;\varepsilon)&=\boldsymbol{u}_\varepsilon(s;\varepsilon)+\mu_\varepsilon(\varepsilon)\,\hat{\gamma}(s;\varepsilon),\\ \tau(\varepsilon)&=\omega_\varepsilon(\varepsilon)+\mu_\varepsilon(\varepsilon)\,\hat{\tau}(\varepsilon)\\ \textit{and}\qquad&\\ \sigma(\varepsilon)&=\mu_\varepsilon(\varepsilon)\,\hat{\sigma}(\varepsilon)\end{aligned}\right\}\tag{24}$$

where $\hat{\gamma}(s;\varepsilon)$, $\hat{\tau}(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$ are real analytic functions on an open interval $I_2\subseteq I_1$.

This theorem is a result of analytic continuation. We prove that the solutions of (20) have the representations (24) within their circle of convergence in the complex ε plane; the representations may then be extended on the real ε axis beyond this circle provided only that they are locally analytic in a small neighborhood of each point ε in the interval I_2.*

The representation for $\sigma(\varepsilon)$ shows that $\sigma(\varepsilon)$ has all of the zeros of μ_ε (see Fig. 1). Unfortunately, we cannot assert that the function $\hat{\sigma}(\varepsilon)$ is of one sign when ε is large. When ε is small the sign of $\hat{\sigma}$ is known and the representation (24) leads to an extension of HOPF's stability theorem (see Theorem 2). It is of particular interest to determine the sign of $\hat{\sigma}(\varepsilon)$ at critical points $\tilde{\varepsilon}$, defined by

$$\mu^{\langle 1\rangle}=\left.\frac{d\mu}{d\varepsilon}\right|_{\varepsilon=\tilde{\varepsilon}}=\mu_\varepsilon(\tilde{\varepsilon})=0 \tag{25}$$

of the bifurcating curve. We shall study the stability of the bifurcating solution near critical points in section 3.

Proof of Theorem 1. We first introduce the representation (24) into (20) and find that

$$(\omega_\varepsilon+\mu_\varepsilon\hat{\tau})\,\dot{\boldsymbol{u}}+\mathscr{J}\boldsymbol{u}_\varepsilon+\mu_\varepsilon\mathscr{J}\hat{\gamma}-\mu_\varepsilon\,\hat{\sigma}(\boldsymbol{u}_\varepsilon+\mu_\varepsilon\hat{\gamma})=0. \tag{26}$$

Differentiating (12) with respect to ε, we find that

$$\mathscr{J}\boldsymbol{u}_\varepsilon+\omega_\varepsilon\dot{\boldsymbol{u}}+\mu_\varepsilon L_1\,\boldsymbol{u}=0. \tag{27}$$

When equation (27) is subtracted from (26), $\mathscr{J}\boldsymbol{u}_\varepsilon+\omega_\varepsilon\dot{\boldsymbol{u}}$ cancels and μ_ε may be factored from each of the remaining terms. It follows that

$$\mathscr{J}\hat{\gamma}+\hat{\tau}\,\dot{\boldsymbol{u}}-L_1\,\boldsymbol{u}-\hat{\sigma}(\boldsymbol{u}_\varepsilon+\mu_\varepsilon\hat{\gamma})=0. \tag{28}$$

It remains to show that problem (28) may be solved for analytic functions $\hat{\gamma}(s;\varepsilon)=\hat{\gamma}(s+2\pi,\varepsilon)$, $\hat{\tau}(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$. We first assume that these functions have the following representations:

$$\begin{pmatrix}\hat{\gamma}(s;\varepsilon)\\ \hat{\tau}(\varepsilon)-\hat{\tau}_0\\ \hat{\sigma}(\varepsilon)\end{pmatrix}=\sum_{l=1}\varepsilon^l\begin{pmatrix}\hat{\gamma}_l(s)\\ \hat{\tau}_l\\ \hat{\sigma}_l\end{pmatrix}. \tag{29}$$

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* We originally found the fundamental factorization (24) by working out and summing all the terms in the series (21) for a special case. The better proof, (26)–(28), of the factorization does not require analyticity. It is enough that the functions of ε defined by (27), (28) exist and are continuous. Analyticity in I_2 follows from analyticity of the operators in (7).

Inserting (29) and (13) into (28), we find that

$$J_0 \hat{\gamma}_1 - L_1 \boldsymbol{u}_1 + \hat{\tau}_0 \dot{\boldsymbol{u}}_1 - \hat{\sigma}_1 \boldsymbol{u}_1 = 0, \tag{30}$$

$$J_0 \hat{\gamma}_2 - L_1 \boldsymbol{u}_2 + \sum_{n+m=2} [\hat{\tau}_n \dot{\boldsymbol{u}}_m - (m+1) \boldsymbol{u}_{m+1} \hat{\sigma}_n + \mathscr{J}_n \hat{\gamma}_m] = 0, \tag{31}$$

and

$$\begin{aligned} J_0 \hat{\gamma}_l - L_1 \boldsymbol{u}_l + \sum_{n+m=l} [\hat{\tau}_n \dot{\boldsymbol{u}}_m - (m+1) \boldsymbol{u}_{m+1} \hat{\sigma}_n + \mathscr{J}_n \hat{\gamma}_m] \\ - \sum_{n+m+r=l} (n+1) \mu_{n+1} \hat{\sigma}_m \hat{\gamma}_r = 0, \qquad l \geqq 3 \end{aligned} \tag{32}$$

where

$$\mathscr{J}_n(0) = \omega_n \frac{d}{ds}(\cdot) + \mu_n L_1(\cdot) + N(\cdot, \boldsymbol{u}_n) + N(\boldsymbol{u}_n, \cdot).$$

We next invoke the solvability condition (16) and use (17) to find that

$$\gamma_\mu + i\hat{\tau}_0 + \hat{\sigma}_1 = 0, \tag{33}$$

$$[L_1 \boldsymbol{u}_2] + i\hat{\tau}_1 + \hat{\sigma}_2 - \hat{\tau}_0 [\dot{\boldsymbol{u}}_2] + 2\hat{\sigma}_1 [\boldsymbol{u}_2] + [\mathscr{J}_1 \hat{\gamma}_1] = 0 \tag{34}$$

and

$$\begin{aligned} [L_1 \boldsymbol{u}_l] + i\hat{\tau}_{l-1} + \hat{\sigma}_l - \sum_{\substack{n+m=l \\ n \neq l}} \{\hat{\tau}_{n-1} [\dot{\boldsymbol{u}}_{m+1}] - (m+1) \hat{\sigma}_n [\boldsymbol{u}_{m+1}] \\ + [\mathscr{J}_n \hat{\gamma}_m]\} - \sum_{n+m+r=l} (n+1) \mu_{n+1} \hat{\sigma}_m [\hat{\gamma}_r] = 0, \qquad l \geqq 3. \end{aligned}$$

The operator J_0 in equations (30), (31) and (32) has a bounded inverse on the complement of the null space of the operator J_0. When normalized in any convenient way, the solutions $\hat{\gamma}_l$ are unique. The normalized coeffients $\hat{\gamma}_l$, $\hat{\tau}_l$ and $\hat{\sigma}_l$ may be determined uniquely and sequentially and the series (29) converge when ε is sufficiently small. The proof of convergence copies the proof given in section 7 and 8 of JS. The functions $\hat{\gamma}(\varepsilon)$, $\hat{\tau}(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$ may then be extended as real analytic functions defined on the interval I_2 of analyticity. In general, I_2 could not extend beyond the interval I_1 of analyticity of the operators defined in (20). This completes the proof of theorem 1.

When $\mu_2 \neq 0$ the representation $\sigma = \mu_\varepsilon \hat{\sigma}$ leads to equation (23). This equation was derived by HOPF and used by him to prove the instability of subcritical solutions and the stability of supercritical solutions. The following extension of HOPF's theorem may now be proved:

Theorem 2. *When ε is small,*

$$\sigma(\varepsilon) = -\mu_\varepsilon \varepsilon \operatorname{Re} \gamma_\mu + \mu_\varepsilon O(\varepsilon^3) \tag{35}$$

Subcritical solutions are unstable and supercritical solutions are stable.

Stability, in theorem 2 and throughout this paper, is in the sense of the linearized theory. The result stated in theorem 2 does not require that $\mu_2 \neq 0$. To prove (35) we first note, following the computation (5.10) of JS, that $i\hat{\tau}_1 + \hat{\sigma}_2 = 0$. Hence, $\hat{\sigma}_2 = 0$ and (35) follows directly from (24) and (33).

3. Expansions Relative to Critical Points of the Bifurcation Curve

Since $\boldsymbol{u}$, $\hat{\gamma}$, ω, $\hat{\tau}$, μ and $\hat{\sigma}$ are real analytic functions of ε when $\varepsilon \in I_2$, they may be expanded relative to any critical point of the bifurcation curve which is on the interval I_2 of analyticity. Relative to such points, we assume series solutions in the form

$$\begin{pmatrix} \boldsymbol{u}(s;\varepsilon) - \boldsymbol{u}^{\langle 0\rangle}(s) \\ \omega(\varepsilon) - \omega^{\langle 0\rangle} \\ \mu(\varepsilon) - \mu^{\langle 0\rangle} \\ \hat{\gamma}(s;\varepsilon) - \hat{\gamma}^{\langle 0\rangle}(s) \\ \hat{\tau}(\varepsilon) - \hat{\tau}^{\langle 0\rangle} \\ \hat{\sigma}(\varepsilon) - \hat{\sigma}^{\langle 0\rangle} \end{pmatrix} = \sum_{l=1} (\varepsilon - \tilde{\varepsilon})^l \begin{pmatrix} \boldsymbol{u}^{\langle l\rangle}(s) \\ \omega^{\langle l\rangle} \\ \mu^{\langle l\rangle} \\ \hat{\gamma}^{\langle l\rangle}(s) \\ \hat{\tau}^{\langle l\rangle} \\ \hat{\sigma}^{\langle l\rangle} \end{pmatrix}. \tag{36}$$

Inserting the series (36) into (28), we find that

$$\mathscr{J}^{\langle 0\rangle}\hat{\gamma}^{\langle 0\rangle} - L_1\boldsymbol{u}^{\langle 0\rangle} + \hat{\tau}^{\langle 0\rangle}\dot{\boldsymbol{u}}^{\langle 0\rangle} - \hat{\sigma}^{\langle 0\rangle}\boldsymbol{u}^{\langle 1\rangle} = 0, \tag{37}$$

$$\begin{aligned} \mathscr{J}^{\langle 0\rangle}\hat{\gamma}^{\langle l\rangle} - L_1\boldsymbol{u}^{\langle l\rangle} + \sum_{n+m=l} \{\hat{\tau}^{\langle n\rangle}\dot{\boldsymbol{u}}^{\langle m\rangle} - (m+1)\boldsymbol{u}^{\langle m+1\rangle}\hat{\sigma}^{\langle n\rangle} \\ + \mathscr{J}^{\langle n\rangle}\hat{\gamma}^{\langle m\rangle}\} - \sum_{n+m+r=l} (n+1)\mu^{\langle n+1\rangle}\hat{\sigma}^{\langle m\rangle}\hat{\gamma}^{\langle r\rangle} = 0, \qquad l \geqq 1 \end{aligned} \tag{38}$$

where

$$\mathscr{J}^{\langle 0\rangle}(\cdot) = \omega^{\langle 0\rangle}\frac{d}{ds}(\cdot) + L_0(\cdot) + \mu^{\langle 0\rangle}L_1(\cdot) + N(\cdot, \boldsymbol{u}^{\langle 0\rangle}) + N(\boldsymbol{u}^{\langle 0\rangle}, \cdot)$$

and

$$\mathscr{J}^{\langle n\rangle}(\cdot) = \omega^{\langle n\rangle}\frac{d}{ds}(\cdot) + \mu^{\langle n\rangle}L_1(\cdot) + N(\cdot, \boldsymbol{u}^{\langle n\rangle}) + N(\boldsymbol{u}^{\langle n\rangle}, \cdot). \tag{39}$$

We now confront the problem of inverting (37) and (38). To do this we need to know certain properties of the spectrum of the operator $\mathscr{J}^{\langle 0\rangle}$. Since

$$\mathscr{J}^{\langle 0\rangle}\dot{\boldsymbol{u}}^{\langle 0\rangle} = 0, \tag{40}$$

zero is an eigenvalue of $\mathscr{J}^{\langle 0\rangle}$. The formal adjoint of this operator may be defined using the scalar product for 2π-periodic functions of s. Thus, $\mathscr{J}_*^{\langle 0\rangle}$ is called the adjoint of the operator $\mathscr{J}^{\langle 0\rangle}$ if

$$[\mathscr{J}^{\langle 0\rangle}\boldsymbol{a}\cdot\boldsymbol{b}] = [\boldsymbol{a}\cdot\mathscr{J}_*^{\langle 0\rangle}\boldsymbol{b}].$$

We assume that the eigenvalues of $\mathscr{J}_*^{\langle 0\rangle}$ and $\mathscr{J}^{\langle 0\rangle}$ coincide. Hence, zero is an eigenvalue of $\mathscr{J}_*^{\langle 0\rangle}$, with at least one eigenfunction*, $\boldsymbol{\theta}$ say, so that

$$\mathscr{J}_*^{\langle 0\rangle}\boldsymbol{\theta} = 0.$$

To invert problems in the form

$$\mathscr{J}^{\langle 0\rangle}\boldsymbol{u}(s) = \boldsymbol{f}(s), \qquad \boldsymbol{u}(s+2\pi) = \boldsymbol{u}(s) \tag{41}$$

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* At a critical point the two eigenvalue branches, $\sigma(\varepsilon) \neq 0$ with eigenfunction $\boldsymbol{\Gamma}$ (see 19) and $\sigma = 0$ with eigenfunction $\dot{\boldsymbol{u}}$, coalesce.

where $f(s)$ is a 2π-periodic function of s, it is necessary that

$$[f \cdot \theta] = 0. \tag{42}$$

We next note that at a critical point, $\mu_\varepsilon = \mu^{\langle 1 \rangle} = 0$, and when $\varepsilon = \tilde{\varepsilon}$, (27) may be written as

$$\mathcal{J}^{\langle 0 \rangle} \boldsymbol{u}^{\langle 1 \rangle} + \omega^{\langle 1 \rangle} \dot{\boldsymbol{u}}^{\langle 0 \rangle} = 0. \tag{43}$$

Applying (42) to (43), we find that

$$\omega^{\langle 1 \rangle} [\dot{\boldsymbol{u}}^{\langle 0 \rangle} \cdot \theta] = 0. \tag{44}$$

In general, $\omega^{\langle 1 \rangle} \neq 0$.

Turning now to (37) and (38), we find, using (42) and (44) that

$$[L_1 \boldsymbol{u}^{\langle 0 \rangle} \cdot \theta] + \hat{\sigma}^{\langle 0 \rangle} [\boldsymbol{u}^{\langle 1 \rangle} \cdot \theta] = 0 \tag{45}$$

and

$$\begin{aligned} &[L_1 \boldsymbol{u}^{\langle l \rangle} \cdot \theta] + \hat{\sigma}^{\langle l \rangle} [\boldsymbol{u}^{\langle 1 \rangle} \cdot \theta] - [\mathcal{J}^{\langle l \rangle} \hat{\gamma}^{\langle 0 \rangle} \cdot \theta] \\ &\quad - \sum_{\substack{n+m=l \\ n \neq l}} \{\hat{\tau}^{\langle n \rangle} [\dot{\boldsymbol{u}}^{\langle m \rangle} \cdot \theta] - (m+1)\, \hat{\sigma}^{\langle n \rangle} [\boldsymbol{u}^{\langle m+1 \rangle} \cdot \theta] \\ &\quad + [\mathcal{J}^{\langle n \rangle} \hat{\gamma}^{\langle m \rangle} \cdot \theta]\} + \sum_{n+m+r=l} (n+1)\, \mu^{\langle n+1 \rangle} \hat{\sigma}^{\langle m \rangle} [\hat{\gamma}^{\langle r \rangle} \cdot \theta] = 0. \end{aligned} \tag{46}$$

Equations (45) and (24) lead to the representation

$$\sigma(\varepsilon) = -2\mu^{\langle 2 \rangle} \frac{[L_1 \boldsymbol{u}^{\langle 0 \rangle} \cdot \theta]}{[\boldsymbol{u}^{\langle 1 \rangle} \cdot \theta]} (\varepsilon - \tilde{\varepsilon}) + O(\varepsilon - \tilde{\varepsilon})^2. \tag{47}$$

Equation (45) gives the value of the analytic function $\hat{\sigma}(\varepsilon)$ at every critical point $\varepsilon = \tilde{\varepsilon}$. Of course, the functions $\boldsymbol{u}^{\langle 0 \rangle}$, $\boldsymbol{u}^{\langle 1 \rangle}$ and θ change with ε so that the values $\hat{\sigma}(\tilde{\varepsilon})$ change from critical point to critical point.

Unfortunately, we do not know enough about the structure of this problem to complete the construction of the series (36). The given construction, which is embodied in (45) and (46), does not lead to a unique determination of the coefficients $\hat{\tau}^{\langle n \rangle}$. These difficulties do not arise when the bifurcating solution is steady.

4. Analytic Continuation Theorem for the Stability of Steady Bifurcating Solutions. Convergent Expansions Near Critical Points

When the bifurcating solution is steady, the spectral problem for the stability of bifurcating solutions has a simpler structure and it is possible to prove more. In this case $\dot{\boldsymbol{u}}$, $\dot{\gamma}$, ω, α and τ are all zero, $J = L_0 + \mu(\varepsilon) L_1$, $J_0 = L_0$, $L_0 \boldsymbol{\zeta} = 0$, $\boldsymbol{\zeta} = \boldsymbol{u}_1$ and $\gamma(\mu)$ are real, $[\boldsymbol{a} \cdot \boldsymbol{b}] = \langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle$, $\langle \boldsymbol{u} \cdot \boldsymbol{u} \rangle = \varepsilon^2$, $\langle \boldsymbol{u}_1 \cdot \boldsymbol{u}_1 \rangle = 1$ and $[\boldsymbol{a}] = \langle \boldsymbol{a} \cdot \boldsymbol{\zeta}^* \rangle$ where $L_0^* \boldsymbol{\zeta}^* = 0$ and $L_0^* = J_0^*$ is the adjoint to L_0. In general $\mu_1 \gamma_\mu + \langle N[\boldsymbol{u}_1 \cdot \boldsymbol{u}_1] \cdot \boldsymbol{\zeta}^* \rangle = 0$ so that $\mu_1 \neq 0$ except in the special cases for which $\langle N[\boldsymbol{u}_1 \cdot \boldsymbol{u}_1] \cdot \boldsymbol{\zeta}^* \rangle = 0$. When $\mu_1 \neq 0$ the bifurcation of steady solutions is two-sided and one branch of the bifurcating solution is subcritical. Analysis of the spectral problem for steady bifurcating solutions shows that

$$\sigma(\varepsilon) = \sigma_1 \varepsilon + O(\varepsilon^2) \tag{48}$$

and

$$\sigma_1 = -\mu_1 \gamma_\mu \tag{49}$$

so that steady subcritical bifurcating solutions at a simple eigenvalue (zero of L_0) are unstable and supercritical solutions are stable when ε is small.* These results are, by now, all well known and the techniques which are used to establish them also serve to establish the following theorem.

Theorem 3 (Steady bifurcation). *Suppose the $\boldsymbol{u}(\varepsilon)$ and $\mu(\varepsilon)$ are real analytic functions on an open interval I_1 containing the point $\varepsilon = 0$. Then*

$$\gamma(\varepsilon) = \boldsymbol{u}_\varepsilon + \mu_\varepsilon \hat{\gamma}(\varepsilon)$$

and (50)

$$\sigma(\varepsilon) = \mu_\varepsilon \hat{\sigma}(\varepsilon)$$

where $\hat{\gamma}(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$ are real analytic functions on an open interval I_2 containing the point $\varepsilon = 0$. Moreover, if zero is a simple eigenvalue of $\mathscr{J}^{\langle 0 \rangle}$, then the eigenfunction $\hat{\gamma}(\varepsilon)$ and eigenvalue $\hat{\sigma}(\varepsilon)$ of the spectral problem for the steady bifurcating solutions may be developed in a convergent series of powers of $\varepsilon - \tilde{\varepsilon}$ near each critical point $\tilde{\varepsilon}$ and

$$\hat{\sigma}(\tilde{\varepsilon}) = \hat{\sigma}^{\langle 0 \rangle} = -\langle L_1 \boldsymbol{u}^{\langle 0 \rangle} \cdot \boldsymbol{\theta} \rangle / \langle \boldsymbol{u}^{\langle 1 \rangle} \cdot \boldsymbol{\theta} \rangle. \tag{51}$$

The reduction of the spectral problem which leads to (50) follows along the path laid out in the proof of (24). The calculation of σ near critical points is the one given in section 3 but, of course, $\hat{\omega}^{\langle l \rangle}$ and $\hat{\tau}^{\langle l \rangle}$ are now zero. Since $\dot{\boldsymbol{u}}^{\langle 0 \rangle} \equiv 0$, (40) is satisfied identically but equation (43) now shows that zero is in the spectrum of $\mathscr{J}^{\langle 0 \rangle}$. The vector $\boldsymbol{\theta}$ is the adjoint eigenfunction for $\mathscr{J}^{\langle 0 \rangle}$ relative to $\langle \cdot, \cdot \rangle$. Under the broad mathematical assumptions specified in theorem 3.2 of Crandall & Rabinowitz (1973), the formal adjoint $\mathscr{J}_*^{\langle 0 \rangle}$ is the real adjoint and the bifurcating solution may be developed into a convergent series of powers of $\varepsilon - \tilde{\varepsilon}$. The convergence of the series for $\hat{\gamma}(\varepsilon)$ and $\hat{\sigma}(\varepsilon)$ also follows from the same assumptions. Related results which hold in both the analytic and non-analytic case have been proved by Crandall & Rabinowitz (1973).

Theorem 3 suffers from the same basic defect as Theorem 1: the sign of the function $\hat{\sigma}(\varepsilon)$ is left undetermined when ε is not small.

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* The instability of subcritical and stability of supercritical bifurcating solutions was proved, using topological methods, by Sattinger (1971). The relation (49) was proved, using analytic perturbation theory, by Joseph (1971). A similar result for the non-analytic case was proved by Crandall & Rabinowitz (1973).

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Note added in proof: In the simplest situations, those in which $\hat{\sigma}$ controls stability and does not change sign, the unstable bifurcating solution regains stability as it turns around a critical point of the bifurcation curve. For partial differential equations and ordinary differential equations in R_n with $n>2$ it is possible to have secondary instability and repeated branching. $\hat{\sigma}(\varepsilon)$ may exist but fail to control stability. For this reason, it is not possible to give a generally valid interpretation of the physical implications of the fundamental factorization. In examples of steady bifurcation in which it has been possible to construct global representations of the subcritical branch, $\hat{\sigma}(\varepsilon)\neq 0$ and stability is associated uniquely with the sign of μ_ε. In such cases we get snap-through instabilities. Computed global representations of subcritical bifurcation of time-periodic solutions are rare but, again, in one example, the numerical study of bifurcating time-periodic Poiseuille flow by ZAHN, TOOMRE, SPIEGEL & GOUGH (J. Fluid Mech. **64**, 319–345, 1973), $\hat{\sigma}(\varepsilon)\neq 0$ and we have restabilization of the subcritical branch and snap-through instabilities. It is necessary to add that, though the computations of ZAHN *et al.* proceed from a severely truncated version of the Navier-Stokes equation, the factorization (24) applies equally to the full equations and to the truncated version. ZAHN *et al.* consider traveling wave solutions of their equations; more general disturbances could possibly lead to instability and repeated branching on the conditionally stable upper branch of the bifurcation curve beyond the critical point. Assuming for the sake of the argument that $\hat{\sigma}(\varepsilon)$ controls stability and that stability is associated uniquely with the sign of μ_ε, we are again led to a bifurcation picture for snap-through instability; at subcritical values of μ, $\mu_G \leqq \mu < 0$, there are two conditionally stable solutions: laminar Poiseuille flow and time-periodic bifurcating Poiseuille flow on the stable subcritical upper branch of $\mu(\varepsilon)$, where $\mu_\varepsilon > 0$. The analysis applies to spatially periodic disturbances in infinitely long pipes and comparisons with experiments in pipes of finite length are at best suggestive. In finite pipes, where $\mu\in[\mu_G, 0)$, there also seem to be two "stable" solutions, one of which is laminar (WYGNANSKI & CHAMPAGNE, J. Fluid Mech. **59**, 281–335, 1973; WYGNANSKI, SOKOLOV & FRIEDMAN, J. Fluid Mech. **69**, 283–304, 1975). The flow is spatially segregated into distinct patches of traveling packets of laminar and turbulent flow (turbulent "puffs" when μ is near μ_G, and "slugs" at higher values of μ). The transition from laminar to turbulent flow at a fixed place occurs suddenly as a puff or slug sweeps over the place, and the reverse transition occurs just as suddenly when it leaves the place.