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Contents

0. Introduction and Review of Previous Work

This paper is a contribution to the theory of viscometry of slow steady motions of a simple fluid and is presented as Part V of the work on slow motion and viscometric motion which formed the subject of the paper in four parts of JOSEPH (1974). The line of thought explored in all five parts, and in parts presently under preparation, is as follows: the constitutive relation for simple fluids is general enough to describe all motions of many fluids. The specification of the constitutive relation for a single fluid, the "name" of the fluid, is the general problem of viscometry. The very generality which is required to describe all of the possible responses of all simple fluids makes the solution of the general problem of viscometry difficult, to say the least. To circumvent this difficulty, we restrict our consideration to special motions on which the constitutive relation reduces to a manageable form; that is, a form for which it is possible to do the experiments to find the values of the quantities which name the fluid. We say that these manageable restrictions of the motion define restricted problems of viscometry (see Part I, Section 2). The two examples of restricted problems of viscometry under consideration are the viscometry of simple fluids in viscometric motions and the viscometry of simple fluids in slow steady motions. We are attempting to formulate a practically useful theory of viscometry for slow steady motions; this comes down to finding good relations from analysis to guide experiments to measure the Rivlin-Ericksen (RE, for short) constants which appear in the expressions (3.2) of Part I for the extra stress in slow steady flow.

The tilted trough viscometer, studied here, exploits the fact that the free surface on a viscoelastic fluid flowing down an open channel which is inclined to the horizontal will deform under the action of normal stresses induced by flow. The trough is a companion to the rotating rod viscometer (JOSEPH, BEAVERS & FOSDICK, 1974; BEAVERS & JOSEPH, 1975). Both the rod and the trough utilize the idea that the shape of the free surface depends on material parameters. By using the two instruments we can, at a minimum, determine the values of the two Rivlin-Ericksen constants of the 2^{nd} order approximation relating stress to deformation in the slow steady motion of simple fluid.

We propose to use the formulas derived here as the guiding theory for the experimental determination of the special combinations of RE constants which arise in the study of flow down a tilted trough. We calculate the solution in a series of powers of the tilt angle β and show that secondary motions do not appear until sixth order when the trough is infinitely deep or has a semicircular cross-section. For these troughs we give simple formulas relating the shape of the free surface to the RE constants of the first, second, third and fourth-order fluids.

WINEMAN & PIPKIN (1966) were the first to suggest that the deformation of the free surface on the liquid flowing in a trough could be used to obtain information about normal stress differences. They expand the stress into a series of RE tensors and expand the solutions into series of powers of $\sin\beta$ where β is the angle of tilt (our β equals their θ). They use the method of Stokes to treat the domain perturbation and they neglect surface tension. WINEMAN & PIPKIN show that the free surface on a liquid will bulge out if $2\alpha_1+\alpha_2<0$. This combination of parameters gives the limiting value of the second normal stress difference $\tilde{N}_2=2\alpha_1+\alpha_2=\lim_{\kappa\to 0} N_2(\kappa)/\kappa^2$, where κ is the rate of shearing. This bulging, observed in all non-Newtonian fluids so far studied, implies that $N_2(\kappa)<0$ when κ is small. Our results agree with those of WINEMAN & PIPKIN at the lowest order of approximation. At higher orders the results are not directly comparable because our expansion parameter is β and theirs is $\sin\beta$. Their second approximation corresponds to, but is not the same as, our analysis at $O(\beta^4)$. At this order, our results are not in agreement. They say, at the conclusion of their paper that "We have carried out all of the details of the solutions (in the second approximation) in the case of a trough of semi-circular cross-section, but since we did not find this exercise to be particularly edifying, we shall not report the results here. Our main conclusions are that the first approximation is particularly simple and the second approximation is excessively complicated." We have already mentioned that the analysis given in this paper shows that in semicircular troughs and infinitely deep troughs there is no secondary motion at $O(\beta^4)$. This superficially surprising result is actually in good accord with known results about the flow of simple fluids in pipes. If the cross-section of the pipe is not an annular ring or a limiting circle or parallel planes, a secondary motion, driven by normal stresses forced by the geometry of the walls, will appear at $O(\beta^4)$ (GIESEKUS, 1961; LANGLOIS & RIVLIN, 1963). This consideration led WINEMAN & PIPKIN to believe that secondary motions would exist at $O(\beta^4)$. In the annular round pipes, however, rectilinear flow is possible at all orders and the simultaneous presence of a free surface does not drive a secondary flow until $O(\beta^6)$, at least. The explicit solutions through $O(\beta^5)$ for these special cases are not excessively complicated; they are relatively compact and entirely elementary.

The other studies of trough flow (TANNER 1970; KUO & TANNER 1972, 1974; PIPKIN & TANNER, 1973) use a different procedure which relies on what we shall call the "cut away" argument. In the case of flow down a pipe, this is equivalent to imagining the top half of the fluid has been cut away without changing the now unequilibrated stresses which were present in the developed flow before the cut away. The free surface on the cut away plane is then determined from the over-burden which must develop to balance the unequilibrated stresses. PIPKIN & TANNER (1973) say that this procedure is accurate to within the first order in the height. There is some ambiguity in this statement. The height is a dependent variable; it is not a given quantity and it must be determined as part of the solution. We interpret the "first order in the height" to mean $O(\beta^2)$. Then the statement of PIPKIN and TANNER is correct. However, if this is the correct interpretation, then we think that the use of N_2 in their equations is misleading unless the further restriction $N_2=(2\alpha_1+\alpha_2)\kappa^2$ is specified, and it is not specified. In fact, we think that TANNER (1970), KUO (1973) and KUO & TANNER (1974) claim more for the cut away argument than can be supported by analysis (see the discussions at the end of Section 5). They use the cut away argument to derive an equation which they say gives the second normal stress *function*. Our analysis shows that it is possible to determine the second normal stress function but only through terms of $O(\beta^4)$ and then only when the trough is infinitely deep within plane walls or has a semicircular cross-section. The second restriction is stated clearly by KUO (1973) and KUO & TANNER (1974) but, unfortunately, they have given the order of terms neglected incorrectly and have omitted terms of same order (β^4) as others retained (see Section 5). Their analysis is valid through terms of $O(\beta^2)$; our analysis is valid through terms of $O(\beta^4)$.

The analysis to follow is purely formal and mostly elementary. We should like to call attention, however, to the theorem about Rivlin-Ericksen tensors which is proved in Appendix 1; this theorem is used in our computations but has a more general application.

1. Mathematical Formulation

We consider the motion of an incompressible simple fluid down an open channel tilted at an angle β with the vertical. Troughs of rectangular cross-section are probably of greatest practical value for free surface viscometry. It will be convenient in Sections 1, 2 and 3 to develop the analysis for this convenient geometry (see Fig. 1). The analysis of Sections 1 and 2 applies, with only slight modifications, to flow down troughs of arbitrary cross-section, and, in Section 5, we give brief consideration to the problem of flow down troughs of semicircular cross-section.

The notation of our analysis is defined in Fig. 1. The fluid occupies the region $\mathscr{V}$ with coordinates (x, y, z) and corresponding velocity components (u, v, w). The wetted perimeter of the surface of the container $\mathscr{C}$ is $\mathscr{S}$. The free surface is

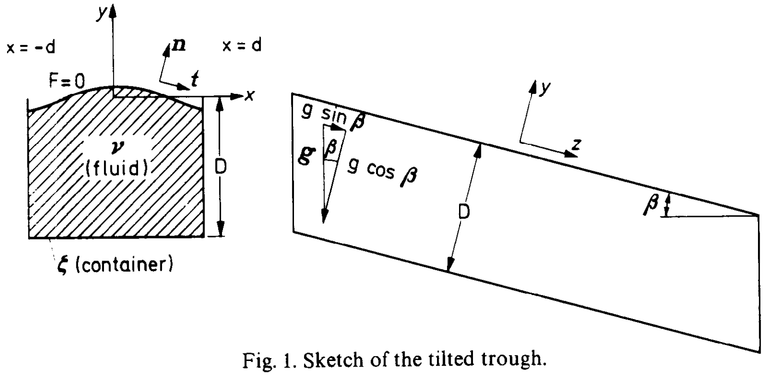

Fig. 1. Sketch of the tilted trough.

(x, y, z)	(coordinates)
(u, v, w)	(velocity components)
$\mathscr{S} = \mathscr{C} \cap \mathscr{V}$	(wet container)
$\mathscr{F} = y - h(x) = 0$	(free surface)
$\partial\mathscr{S} = \mathscr{F} \cap \mathscr{C}$	(contact line)

$\mathscr{F} = y - h(x; \beta) = 0$ and the contact line is the trace of the free surface $\mathscr{F} = 0$ on the container wall. We are going to assume that the air exerts no tangential tractions on the moving fluid and that the fluid satisfies the classical surface tension equation of Laplace and Young with constant surface tension σ. We shall also assume that the contact line is fixed or that the fluid and solid meet at an angle of $\pi/2$. The first of these conditions expresses the natural affinity which liquids have for sharp corners, the second condition can be obtained by treating the solid surface with a non-wetting agent, like "scotchgard".

The flow down the trough is driven by gravity and gravity enters the force balance shaping the free surface. We may separate these two effects by decomposing gravity into a specific driving force and a potential:

$$\boldsymbol{g} = g \sin\beta\, \boldsymbol{e}_z - g \cos\beta\, \nabla y.$$

The equations of motion may then be written as

$$\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \Phi = \nabla \cdot \boldsymbol{S} + \rho g \sin\beta\, \boldsymbol{e}_z, \tag{1.1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \tag{1.1b}$$

where

$$\Phi = p + \rho g y \cos\beta$$

is the head, $-p\,\mathbf{1}$ is the isotropic part of the stress and $\boldsymbol{S}$ is the extra stress. The extra stress relates stress to deformation while p arises as a kinematic constraint of incompressibility and is to be determined as part of the solution. The fluid satisfies a no-slip condition at solid boundaries:

$$\boldsymbol{u}|_{\mathscr{S}} = 0. \tag{1.1c}$$

On the free surface, $y=h(x)$, the normal component of the velocity and the components of the shear stress vanish; the jump in the normal stress is balanced by surface tension. The free surface conditions may be written in component form:

$$v-h_{,x}u=0, \tag{1.1d}$$

$$h_{,x}(S_{yy}-S_{xx})+(1-h_{,x}^2)\,S_{xy}=0, \tag{1.1e}$$

$$S_{yz}-h_{,x}S_{xz}=0, \tag{1.1f}$$

$$S_{yy}-h_{,x}S_{xy}-\Phi=\sigma\frac{h_{,xx}}{(1+h_{,x}^2)^{3/2}}-\rho g h\cos\beta, \tag{1.1g}$$

on $y=h$. In the rest state, $\beta=0$, we assume that the free surface is everywhere flat, $h(x)\equiv 0$. This assumption is equivalent to an assumption about the nature of the contact between the free surface and solid wall. In the following fluid, the shape of the free surface depends on the forces due to motion and on the total volume of fluid in the channel. The total volume in a developed flow can be specified as a volume per unit length. The specification of the volume is a separate kinematic condition and it, or an equivalent condition, must be prescribed to guarantee uniqueness. To emphasize this point, consider the rectangular trough of Fig. 1 when $\beta=0$ and there is no motion. The trough is now filled with liquid so that $y=h(x)\equiv 0$. If we now add or remove some liquid, the fluid will continue to grip the sharp corner but the free surface will not remain flat. To guarantee flatness when $\beta=0$, we properly specify the volume of fluid in the trough. For essentially the same reason, we must specify the volume when $\beta\neq 0$. Without loss of generality, we require that this condition take form as a prescription of the mean height

$$Q(\beta)=\frac{1}{2d}\int_{-d}^{d}h(x;\beta)\,dx. \tag{1.1h}$$

We require that

$$Q(0)=0$$

and

$$Q(\beta)=Q(-\beta).$$

Otherwise Q is an arbitrary analytic function which may be selected to match experimental conditions. We are assuming that the solution corresponding to a given prescription of the mean height is unique; the formal solutions which we compute are unique. The condition that $h\equiv 0$ when $\beta=0$ is essential in our analysis because it allows us to use a convenient reference domain, under a flat surface, in the domain perturbation. We assume contact line conditions

$$h(\pm d,\beta)=0,$$

or contact angle conditions

$$h'(\pm d;\beta)=0,$$

which lead to the desired reference configuration $h(x;0)\equiv 0$.

To solve problem (1.1), we use the Lagrangian theory of domain perturbations. The most recent and most complete statement of this theory appears in the (1975) paper of JOSEPH & STURGES. We first define a scaling transformation

$$\begin{aligned} x &= x_0, \\ z &= z_0, \\ y &= y_0 \frac{D+h}{D} + h, \end{aligned} \tag{1.2}$$

which is a one-to-one mapping of the deformed domain

$$\mathscr{V}_\beta = [x, y, z: \ -d \leqq x \leqq d, \ -D \leqq y \leqq h(x;\beta), \ -\infty < z < \infty]$$

into the refernce domain

$$\mathscr{V}_0 = [x_0, y_0, z_0: \ -d \leqq x_0 \leqq d; \ -D \leqq y_0 \leqq 0, \ -\infty < z < \infty]$$

of the rest state. This mapping is analytic in β when $h(x;\beta)$ is analytic in β. The solutions of (1.1) are mapped in $\mathscr{V}_0$ and expanded there in a series of powers of β:

$$w(x, y; \beta) = w^{[1]}(x_0, y_0)\,\beta + w^{[3]}(x_0, y_0)\,\beta^3/3! + w^{[5]}(x_0, y_0)\,\beta^5/5! + \cdots \tag{1.3a}$$

$$\begin{pmatrix} \psi(x, y; \beta) \\ \Phi(x, y; \beta) \\ h(x; \beta) \end{pmatrix} = \begin{pmatrix} \psi^{[2]}(x_0, y_0) \\ \Phi^{[2]}(x_0, y_0) \\ h^{[2]}(x_0) \end{pmatrix} \frac{\beta^2}{2} + \begin{pmatrix} \psi^{[4]}(x_0, y_0) \\ \Phi^{[4]}(x_0, y_0) \\ h^{[4]}(x_0) \end{pmatrix} \frac{\beta^4}{4!} + \cdots \tag{1.3b}$$

where $\psi(x, y; \beta)$ is the stream function for the secondary motion and

$$(\cdot)^{[n]} = \left(\frac{\partial}{\partial \beta} + \frac{dy}{d\beta} \frac{\partial}{\partial y} \right)^n (\cdot) \Big|_{\beta=0}$$

is a substantial derivative following the mapping evaluated in the reference domain $\mathscr{V}_0$.

To express the solution in the deformed domain, we invert the mapping:

$$\begin{aligned} w(x, y; \beta) &= \sum_{n=1}^{\infty} \frac{\beta^{2n-1}}{(2n-1)!} w^{[2n-1]} \left(x, \frac{(y-h)\,D}{D+h} \right) \\ &= \sum_{n=1}^{\infty} \frac{\beta^{2n-1}}{(2n-1)!} w^{\langle 2n-1 \rangle}(x, y), \end{aligned} \tag{1.4a}$$

$$\begin{pmatrix} \psi(x, y; \beta) \\ \Phi(x, y; \beta) \\ h(x; \beta) \end{pmatrix} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{(2n)!} \begin{pmatrix} \psi^{[2n]}\left(x, \dfrac{(y-h)\,D}{D+h} \right) \\ \Phi^{[2n]}\left(x, \dfrac{(y-h)\,D}{D+h} \right) \\ h^{[2n]}(x) \end{pmatrix} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{(2n)!} \begin{pmatrix} \psi^{\langle 2n \rangle}(x, y) \\ \Phi^{\langle 2n \rangle}(x, y) \\ h^{\langle 2n \rangle}(x) \end{pmatrix}. \tag{1.4b}$$

The last equality in (1.4a, b) is not obvious and the equality of the series does not imply term by term equality of the Taylor coefficients (JOSEPH, 1973). The functions with angle bracket superscripts are partial derivatives holding y fixed and evaluated at $\beta=0$; *e.g.*,

$$w^{\langle n\rangle}(x_0, y_0)=\left.\frac{\partial^n w(x, y; \beta)}{\partial \beta^n}\right|_{\beta=0}.$$

The functions $w^{\langle n\rangle}(x, y)$, $(x, y)\in\mathscr{V}_\beta$ can be interpreted as the analytic extension by declaration of the functions $w^{\langle n\rangle}(x_0, y_0)$, $(x_0, y_0)\in\mathscr{V}_0$. The substantial derivatives $w^{[n]}(x_0, y_0)$ may be computed when the $w^{\langle n\rangle}(x_0, y_0)$ and the coefficients for height of rise $h^{\langle n\rangle}(x_0)$ are known.

In the perturbation analysis which follows, we compute $w^{\langle n\rangle}(x_0, y_0)$, $\psi^{\langle n\rangle}(x_0, y_0)$, $\Phi^{\langle n\rangle}(x_0, y_0)$ and $h^{\langle n\rangle}(x_0)$. From this point on, we shall work exclusively in the reference domain $\mathscr{V}_0$ and we simplify the notation by dropping the subscript zero; that is, in the equations which follow, the coordinates $(x, y)\in\mathscr{V}_0$. Since w is independent of z, and u and v are obtained from a stream function, the equation $\operatorname{div}\boldsymbol{u}=\operatorname{div}[\boldsymbol{e}_z\wedge\nabla\psi]$ is automatically satisfied. We shall not always introduce ψ explicitly in our equations because we never do come to an actual computation of secondary flows and we want to avoid unnecessary computations. We are now ready to consider the perturbation equations for $u^{\langle n\rangle}$, $v^{\langle n\rangle}$, $w^{\langle n\rangle}$, $\Phi^{\langle n\rangle}$ and $h^{\langle n\rangle}$.

2. Series Solution

The perturbation equations for the coefficients in the expansions (1.4) are obtained by the methods prescribed in the Lagrangian theory (see JOSEPH & STURGES, 1975). When n is odd,

$$\begin{gathered}\rho a_z^{\langle n\rangle}=S_{xz,x}^{\langle n\rangle}+S_{yz,y}^{\langle n\rangle}+S_{zz,z}^{\langle n\rangle}+\rho g \quad \text{in } \mathscr{V}_0,\\ S_{yz}^{[n]}-(h_{,x}S_{xz})^{[n]}=0 \quad \text{on } y=0,\\ w^{\langle n\rangle}|_{\mathscr{S}}=0,\end{gathered} \tag{2.1}$$

where $\boldsymbol{a}=\boldsymbol{u}\cdot\nabla\boldsymbol{u}$. When n is even,

$$\left.\begin{aligned}\rho a_x^{\langle n\rangle}&=S_{xx,x}^{\langle n\rangle}+S_{xy,y}^{\langle n\rangle}+S_{xz,z}^{\langle n\rangle}-\Phi_{,x}^{\langle n\rangle},\\ \rho a_y^{\langle n\rangle}&=S_{xy,x}^{\langle n\rangle}+S_{yy,y}^{\langle n\rangle}+S_{yz,z}^{\langle n\rangle}-\Phi_{,y}^{\langle n\rangle},\end{aligned}\right\} \quad \text{in } \mathscr{V}_0 \tag{2.2a}$$

$$u^{\langle n\rangle}|_{\mathscr{S}}=v^{\langle n\rangle}|_{\mathscr{S}}=0, \tag{2.2b}$$

$$\left.\begin{aligned}&v^{[n]}-(h_{,x}u)^{[n]}=0,\\ &\{h_{,x}(S_{yy}-S_{xx})\}^{[n]}+S_{xy}^{[n]}-(h_{,x}^2 S_{xy})^{[n]}=0,\\ &S_{yy}^{[n]}-(h_{,x}S_{xy})^{[n]}-\Phi^{[n]}=\sigma\left[\frac{h_{,xx}}{(1+h_{,x}^2)^{\frac{3}{2}}}\right]^{[n]}-\rho g[h\cos\beta]^{[n]},\end{aligned}\right\} \quad \text{on } y=0 \tag{2.2c}$$

$$Q^{\langle n\rangle}=\frac{1}{2d}\int_{-d}^{d} h^{\langle n\rangle}(x)\,dx \tag{2.2d}$$

and

$$h^{\langle n\rangle}(\pm d)=0 \tag{2.2e$_1$}$$

or

$$h_{,x}^{\langle n\rangle}(\pm d)=0. \tag{2.2e$_2$}$$

The substantial derivatives $(\cdot)^{[n]}$ are related to partial derivatives $(\cdot)^{\langle n\rangle}$ by the chain rule; for example, since $w^{\langle 0\rangle}$, $\Phi^{\langle 0\rangle}$, $\psi^{\langle 0\rangle}$, $h^{\langle 0\rangle}$ and $h^{\langle 2n+1\rangle}$ are all zero,

$$\begin{aligned}
&(\cdot)^{[2]}=(\cdot)^{\langle 2\rangle},\\
&(\cdot)^{[3]}=(\cdot)^{\langle 3\rangle}+3h^{\langle 2\rangle}(\cdot)^{\langle 1\rangle}_{,y},\\
&(\cdot)^{[4]}=(\cdot)^{\langle 4\rangle}+6h^{\langle 2\rangle}(\cdot)^{\langle 2\rangle}_{,y},\\
&(\cdot)^{[5]}=(\cdot)^{\langle 5\rangle}+5h^{\langle 4\rangle}(\cdot)^{\langle 1\rangle}_{,y}+10h^{\langle 2\rangle}(\cdot)^{\langle 3\rangle}_{,y}+15h^{\langle 2\rangle^2}(\cdot)^{\langle 1\rangle}_{,yy},\\
&(\cdot)^{[6]}=(\cdot)^{\langle 6\rangle}+15h^{\langle 4\rangle}(\cdot)^{\langle 2\rangle}_{,y}+15h^{\langle 2\rangle}(\cdot)^{\langle 4\rangle}_{,y}+45h^{\langle 2\rangle^2}(\cdot)^{\langle 2\rangle}_{,yy}.
\end{aligned} \tag{2.3}$$

Derivatives of stresses are given by

$$\begin{aligned}
S^{\langle n\rangle}&=S_1^{\langle n\rangle}+S_2^{\langle n\rangle}+\cdots+S_n^{\langle n\rangle},\\
\operatorname{div} S^{\langle n\rangle}&=\mu\nabla^2 u^{\langle n\rangle}+\nabla\cdot\{S_2^{\langle n\rangle}+\cdots+S_n^{\langle n\rangle}\}.
\end{aligned}$$

The stress tensors $S_l^{\langle n\rangle}$ are related to the functions $u^{\langle l\rangle}$ through the expressions involving the Rivlin-Ericksen tensors A_l (see (3.6) of JOSEPH, 1974). This elimination of the $S_l^{\langle n\rangle}$ is carried out below with help of the formulas in Appendices 1 and 2 for $n=1, 2, 3, 4$. To simplify the writing of formulas, we introduce the following notations:

$$\begin{aligned}
\tilde{N}_2&=2\alpha_1+\alpha_2,\\
\tilde{\beta}&=\beta_2+\beta_3,\\
\tilde{\gamma}&=\gamma_3+\gamma_4+\gamma_5+\tfrac{1}{2}\gamma_6,\\
V&=\frac{\rho g d^2}{\mu}.
\end{aligned}$$

$$\begin{aligned}
n=1:\quad &S^{\langle 1\rangle}=\mu A_1^{\langle 1\rangle},\\
&A_1^{\langle 1\rangle}=(e_x e_z+e_z e_x)\,w^{\langle 1\rangle}_{,x}+(e_y e_z+e_z e_y)\,w^{\langle 1\rangle}_{,y},
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\nabla^2 w^{\langle 1\rangle}+\frac{\rho g}{\mu}&=0,\\
w^{\langle 1\rangle}|_{\mathscr{S}}&=0,\\
w^{\langle 1\rangle}_{,y}|_{y=0}&=0.
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
n=2:\quad S^{\langle 2\rangle}&=\mu A_1^{\langle 2\rangle}+\alpha_1 A_2^{\langle 2\rangle}+\alpha_2(A_1^2)^{\langle 2\rangle}=\mu A_1^{\langle 2\rangle}+2\alpha_2 e_z e_z\,|\nabla w^{\langle 1\rangle}|^2\\
&\quad+2\tilde{N}_2\,[e_x e_x w^{\langle 1\rangle^2}_{,x}+e_y e_y w^{\langle 1\rangle^2}_{,y}+(e_x e_y+e_y e_x)\,w^{\langle 1\rangle}_{,x}\,w^{\langle 1\rangle}_{,y}].
\end{aligned} \tag{2.6}$$

On $y=0$, $w^{\langle 1\rangle}_{,y}=0$. Hence

$$S^{\langle 2\rangle}_{xy}=\mu(u^{\langle 2\rangle}_{,y}+v^{\langle 2\rangle}_{,x})$$

on $y=0$. Moreover, on $y=0$,

$$v^{\langle 2\rangle}=0.$$

Equations (2.2a) may be written

$$\mu \nabla^2 \boldsymbol{u}^{\langle 2\rangle} + \nabla[\tilde{N}_2(|\nabla w^{\langle 1\rangle}|^2 + 2w^{\langle 1\rangle}\nabla^2 w^{\langle 1\rangle}) - \Phi^{\langle 2\rangle}] = 0 \quad \text{in } \mathscr{V}_0$$

and

$$\boldsymbol{u}^{\langle 2\rangle}|_{\mathscr{S}} = 0.$$

Hence

$$\boldsymbol{u}^{\langle 2\rangle} \equiv 0 \quad \text{in } \mathscr{V}_0 \tag{2.7}$$

and

$$\Phi^{\langle 2\rangle} = \tilde{N}_2(|\nabla w^{\langle 1\rangle}|^2 + 2w^{\langle 1\rangle}\nabla^2 w^{\langle 1\rangle}) + \text{Const.} \tag{2.8}$$

The free surface is determined by the differential equation arising from the third of equations (2.2c):

$$\sigma h^{\langle 2\rangle}_{,xx} - \rho g h^{\langle 2\rangle} = 2\tilde{N}_2 w^{\langle 1\rangle 2}_{,y} - \Phi^{\langle 2\rangle} = -\Phi^{\langle 2\rangle}|_{y=0}. \tag{2.9}$$

This equation is to be solved subject to the boundary conditions

$$h^{\langle 2\rangle}(\pm d) = 0 \tag{2.10$_1$}$$

or

$$h^{\langle 2\rangle}_{,x}(\pm d) = 0, \tag{2.10$_2$}$$

and the prescription of the mean height

$$Q^{\langle 2\rangle} = \frac{1}{2d}\int_{-d}^{d} h^{\langle 2\rangle}(x)\,dx. \tag{2.11}$$

$$\begin{aligned} n=3:\quad \boldsymbol{S}^{\langle 3\rangle} &= \mu \boldsymbol{A}_1^{\langle 3\rangle} + \alpha_1 \boldsymbol{A}_2^{\langle 3\rangle} + \alpha_2 (\boldsymbol{A}_1^2)^{\langle 3\rangle} + \beta_1 \boldsymbol{A}_3^{\langle 3\rangle} \\ &\quad + \beta_2[\boldsymbol{A}_2\cdot\boldsymbol{A}_1 + \boldsymbol{A}_1\cdot\boldsymbol{A}_2]^{\langle 3\rangle} + \beta_3[\boldsymbol{A}_1 \operatorname{tr}\boldsymbol{A}_2]^{\langle 3\rangle} \\ &= (\boldsymbol{e}_x\boldsymbol{e}_z + \boldsymbol{e}_z\boldsymbol{e}_x)[\mu w^{\langle 3\rangle}_{,x} + 12\tilde{\beta} w^{\langle 1\rangle}_{,x}|\nabla w^{\langle 1\rangle}|^2] + (\boldsymbol{e}_y\boldsymbol{e}_z + \boldsymbol{e}_z\boldsymbol{e}_y) \\ &\quad \cdot [\mu w^{\langle 3\rangle}_{,y} + 12\tilde{\beta} w^{\langle 1\rangle}_{,y}|\nabla w^{\langle 1\rangle}|^2]. \end{aligned} \tag{2.12}$$

Evaluating (2.1) when $n=3$, we get

$$\begin{aligned} &\nabla^2 w^{\langle 3\rangle} + \frac{12\tilde{\beta}}{\mu}(3w^{\langle 1\rangle 2}_{,x} w^{\langle 1\rangle}_{,xx} + 3w^{\langle 1\rangle 2}_{,y} w^{\langle 1\rangle}_{,yy} + w^{\langle 1\rangle 2}_{,x} w^{\langle 1\rangle}_{,yy} \\ &\qquad + w^{\langle 1\rangle 2}_{,y} w^{\langle 1\rangle}_{,xx} + 4 w^{\langle 1\rangle}_{,x} w^{\langle 1\rangle}_{,y} w^{\langle 1\rangle}_{,xy}) - \frac{\rho g}{\mu} = 0 \quad \text{in } \mathscr{V}_0 \end{aligned} \tag{2.13a}$$

and

$$w^{\langle 3\rangle}|_{\mathscr{S}} = 0. \tag{2.13b}$$

On $y=0$

$$S^{[3]}_{yz} - 3h^{\langle 2\rangle}_{,x} S^{\langle 1\rangle}_{xz} = S^{\langle 3\rangle}_{yz} + 3h^{\langle 2\rangle} S^{\langle 1\rangle}_{yz,y} - 3h^{\langle 2\rangle}_{,x} S^{\langle 1\rangle}_{xz} = 0.$$

We may rewrite this last equation:

$$w^{\langle 3\rangle}_{,y} + \frac{12\tilde{\beta}}{\mu} w^{\langle 1\rangle}_{,y}|\nabla w^{\langle 1\rangle}|^2 + 3(h^{\langle 2\rangle} w^{\langle 1\rangle}_{,yy} - h^{\langle 2\rangle}_{,x} w^{\langle 1\rangle}_{,x}) = 0. \tag{2.13c}$$

The second term of (2.13c) vanishes because of (2.5c).

$$n=4:\quad S^{\langle 4\rangle}=S_1^{\langle 4\rangle}+S_2^{\langle 4\rangle}+S_3^{\langle 4\rangle}+S_4^{\langle 4\rangle}=\mu A_1^{\langle 4\rangle}+\alpha_1 A_2^{\langle 4\rangle}+\alpha_2(A_1^2)^{\langle 4\rangle}$$
$$+\beta_2[A_2\cdot A_1+A_1\cdot A_2]^{\langle 4\rangle}+\beta_3[A_1\operatorname{tr}A_2]^{\langle 4\rangle}+\gamma_3(A_2^2)^{\langle 4\rangle}$$
$$+\gamma_4[A_2\cdot A_1^2+A_1^2\cdot A_2]^{\langle 4\rangle}+\gamma_5[A_2\operatorname{tr}A_2]^{\langle 4\rangle}+\gamma_6[A_1^2\operatorname{tr}A_2]^{\langle 4\rangle}$$
$$+\gamma_8[A_1\operatorname{tr}(A_1\cdot A_2)]^{\langle 4\rangle} \tag{2.14}$$
$$=e_x e_x[2\mu u_{,x}^{\langle 4\rangle}+8\tilde{N}_2 w_{,x}^{\langle 1\rangle}w_{,x}^{\langle 3\rangle}+96\tilde{\gamma}\, w_{,x}^{\langle 1\rangle^2}|\nabla w^{\langle 1\rangle}|^2]$$
$$+e_y e_y[2\mu v_{,y}^{\langle 4\rangle}+8\tilde{N}_2 w_{,y}^{\langle 1\rangle}w_{,y}^{\langle 3\rangle}+96\tilde{\gamma}\, w_{,y}^{\langle 1\rangle^2}|\nabla w^{\langle 1\rangle}|^2]$$
$$+(e_x e_y+e_y e_x)[\mu(u_{,y}^{\langle 4\rangle}+v_{,x}^{\langle 4\rangle})+4\tilde{N}_2(w_{,x}^{\langle 1\rangle}w_{,y}^{\langle 3\rangle}+w_{,y}^{\langle 1\rangle}w_{,x}^{\langle 3\rangle})$$
$$+96\tilde{\gamma}\, w_{,x}^{\langle 1\rangle}w_{,y}^{\langle 1\rangle}|\nabla w^{\langle 1\rangle}|^2]+8e_z e_z[6\gamma_6|\nabla w^{\langle 1\rangle}|^4+\alpha_2\nabla w^{\langle 1\rangle}\cdot\nabla w^{\langle 3\rangle}].$$

Equation (2.2a) may be written as

$$\mu\nabla^2 u^{\langle 4\rangle}+e_x\{4\tilde{N}_2[(2w_{,x}^{\langle 1\rangle}w_{,x}^{\langle 3\rangle})_{,x}+(w_{,x}^{\langle 1\rangle}w_{,y}^{\langle 3\rangle}+w_{,y}^{\langle 1\rangle}w_{,x}^{\langle 3\rangle})_{,y}]$$
$$+96\tilde{\gamma}[(w_{,x}^{\langle 1\rangle^2}|\nabla w^{\langle 1\rangle}|^2)_{,x}+(w_{,x}^{\langle 1\rangle}w_{,y}^{\langle 1\rangle}|\nabla w^{\langle 1\rangle}|^2)_{,y}]\}$$
$$+e_y\{4\tilde{N}_2[(2w_{,y}^{\langle 1\rangle}w_{,y}^{\langle 3\rangle})_{,y}+(w_{,x}^{\langle 1\rangle}w_{,y}^{\langle 3\rangle}+w_{,y}^{\langle 1\rangle}w_{,x}^{\langle 3\rangle})_{,x}]$$
$$+96\tilde{\gamma}[(w_{,y}^{\langle 1\rangle^2}|\nabla w^{\langle 1\rangle}|^2)_{,y}+(w_{,x}^{\langle 1\rangle}w_{,y}^{\langle 1\rangle}|\nabla w^{\langle 1\rangle}|^2)_{,x}]\}-\nabla\Phi^4=0\quad\text{in }\mathscr{V}_0 \tag{2.15a}$$

where

$$u^{\langle 4\rangle}|_{\mathscr{S}}=e_x u^{\langle 4\rangle}|_{\mathscr{S}}+e_y v^{\langle 4\rangle}|_{\mathscr{S}}=0. \tag{2.15b}$$

The free surface equation (2.2c) on $y=0$ becomes

$$v^{\langle 4\rangle}=0, \tag{2.15c}$$

$$u_{,y}^{\langle 4\rangle}+v_{,x}^{\langle 4\rangle}=0, \tag{2.15d}$$

$$S_{yy}^{\langle 4\rangle}+6h^{\langle 2\rangle}S_{yy,y}^{\langle 2\rangle}-6h_{,x}^{\langle 2\rangle}S_{xy}^{\langle 2\rangle}-\Phi^{\langle 4\rangle}-6h^{\langle 2\rangle}\Phi_{,y}^{\langle 2\rangle}$$
$$=\sigma h_{,xx}^{\langle 4\rangle}-\rho g h^{\langle 4\rangle}+6\rho g h^{\langle 2\rangle} \tag{2.15e}$$

where substantial derivatives have been evaluated using (2.3). Equation (2.15d) is deduced from the equation

$$6h_{,x}^{\langle 2\rangle}(S_{yy}^{\langle 2\rangle}-S_{xx}^{\langle 2\rangle})+S_{xy}^{\langle 4\rangle}+6h^{\langle 2\rangle}S_{xy,y}^{\langle 2\rangle}=0$$

and (2.12), (2.13c) and (2.14).

Since the boundary conditions (2.15b, c, d) are satisfied when $u^{\langle 4\rangle}=0$, secondary motions are not forced from the boundary. The solution of (2.15a, b, c, d) is unique and $u^{\langle 4\rangle}\equiv 0$ if and only if the inhomogeneous terms in (2.15a) have a potential. We may write (2.15a) as

$$\mu\nabla^2 u^{\langle 4\rangle}+\nabla[4\tilde{N}_2\nabla w^{\langle 1\rangle}\cdot\nabla w^{\langle 3\rangle}+120\tilde{\gamma}|\nabla w^{\langle 1\rangle}|^4-\Phi^{\langle 4\rangle}]+e_x X+e_y Y=0 \tag{2.16}$$

where

$$X=4\tilde{N}_2[w_{,x}^{\langle 1\rangle}\nabla^2 w^{\langle 3\rangle}+w_{,x}^{\langle 3\rangle}\nabla^2 w^{\langle 1\rangle}]$$
$$+96\tilde{\gamma}[w_{,x}^{\langle 1\rangle}(w_{,x}^{\langle 1\rangle^2}+3w_{,y}^{\langle 1\rangle^2})(w_{,yy}^{\langle 1\rangle}-w_{,xx}^{\langle 1\rangle})-4w_{,y}^{\langle 1\rangle^3}w_{,xy}^{\langle 1\rangle}]$$

and

$$Y = 4\tilde{N}_2[w^{\langle 1\rangle}_{,y}\,\nabla^2 w^{\langle 3\rangle} + w^{\langle 3\rangle}_{,y}\,\nabla^2 w^{\langle 1\rangle}] + 96\tilde{\gamma}[w^{\langle 1\rangle}_{,y}(w^{\langle 1\rangle 2}_{,y} + 3w^{\langle 1\rangle 2}_{,x})(w^{\langle 1\rangle}_{,xx} - w^{\langle 1\rangle}_{,yy}) - 4w^{\langle 1\rangle 3}_{,x}\,w^{\langle 1\rangle}_{,xy}].$$

In important special cases which include a deep trough ($D/d\to\infty$) and flow down troughs of semicircular cross-section, $\boldsymbol{e}_x X + \boldsymbol{e}_y Y = \nabla F$ and there is no secondary flow at order four.* The inhomogeneous terms in (2.15a) may then be balanced by the reaction pressure $\Phi^{\langle 4\rangle}$ and, using (2.5_2), the differential equation (2.15e) for the free surface reduces to

$$\sigma h^{\langle 4\rangle}_{,xx} - \rho g h^{\langle 4\rangle} = -\Phi^{\langle 4\rangle} - 6\rho g h^{\langle 2\rangle}. \tag{2.17}$$

The functions on the right of (2.17) are evaluated on $y=0$.

It is possible to compute the secondary motions in the rectangular trough. The mathematical problem can be formulated as a biharmonic edge problem for the stream function. The solution of this edge problem in a rectangular trough should follow, step by step, the extension of JOSEPH & STURGES (1975) to finite strips of SMITH's (1952) theory for semi-infinite strips. We expect the secondary motion to be very weak when the trough is deep and, therefore, proceed in Section 4 with the much simpler higher-order theory which arises in the limit $D/d\to\infty$.

3. Second Order Curves of Height of Rise in a Rectangular Trough

Unique solutions of equations (2.5)–(2.11) are easily obtained by routine and elementary methods. These solutions are listed below.

Introduce the dimensionless variables

$$[\bar{x}, \bar{y}, \bar{D}, \zeta, b_n] = \left[\frac{x}{d}, \frac{y}{d}, \frac{D}{d}, d\left(\frac{\rho g}{\sigma}\right)^{\frac{1}{2}}, (n-1/2)\pi\right].$$

Then

$$w^{\langle 1\rangle} = V\left\{\tfrac{1}{2}(1-\bar{x}^2) + \sum_{n=1}^{\infty} f_n \cos b_n\bar{x}\cosh b_n\bar{y}\right\}, \tag{3.1}$$

$$f_n = \frac{2(-1)^n}{b_n^3\cosh b_n\bar{D}},$$

$$\Phi^{\langle 2\rangle} = \frac{\tilde{N}_2 V^2}{d^2}\,C_1 + \tilde{N}_2[w^{\langle 1\rangle 2}_{,x} + w^{\langle 1\rangle 2}_{,y} - 2\rho g w^{\langle 1\rangle}/\mu],$$

---

* In Sections 4 and 5, we show that $\boldsymbol{u}^{\langle 4\rangle}$ vanishes in deep channels and semicircular troughs. The same demonstration holds when the trough is a ring bounded by semicircular arcs. The generation of secondary motions is a consequence of the azimuthal variation of the shearing which leads to unequilibrated normal stresses driving tangential motions. This azimuthal variation is produced by the shape of the trough walls when the walls are not concentric cylinders or limiting planes. This general mechanism, discovered by ERICKSEN (1959), produces secondary motion at order four in the flow through pipes of elliptical cross-section studied by LANGLOIS & RIVLIN (1963). The same mechanism cannot operate, at order four, in the flow down semicircular troughs where secondary motions first appear at order six.

$$h^{\langle 2\rangle}=\frac{\tilde N_2 V}{\mu}\left\{C_1+C_2\cosh\zeta\bar x+2\bar x^2+\frac{4}{\zeta^2}-1\right.$$
$$+2\zeta^2\sum_{n=1}^{\infty}\frac{f_n}{g_n}\left[b_n\bar x\sin b_n\bar x+\frac{b_n^2-\zeta^2}{g_n}\cos b_n\bar x\right] \tag{3.2}$$
$$\left.+\frac{\zeta^2}{2}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}b_n b_l f_n f_l\left[\frac{\cos(b_n-b_l)\bar x}{r_{nl}}-\frac{\cos(b_n+b_l)\bar x}{s_{nl}}\right]\right\}$$

where

C_1 and C_2 are constants of integration,

$$g_n=b_n^2+\zeta^2,$$
$$r_{nl}=(b_n-b_l)^2+\zeta^2,$$

and

$$s_{nl}=(b_n+b_l)^2+\zeta^2.$$

The constants of integration are determined from conditions (2.10_1), (2.10_2) and (2.11). Using the condition for fixed slope (2.10_2), we get

$$C_1=\tfrac{1}{3}-\tfrac{1}{2}\sum_{n=1}^{\infty}(f_n b_n)^2+\frac{\mu}{\tilde N_2 V}Q^{\langle 2\rangle},$$
$$C_2=\frac{-4}{\zeta\sinh\zeta}\left[1-\zeta^4\sum_{n=1}^{\infty}(-1)^n\frac{f_n b_n}{g_n^2}\right];$$

using the condition for fixed line (2.10_1), we get

$$C_1=-1-\frac{4}{\zeta^2}+2\zeta^2\sum_{n=1}^{\infty}\frac{(-1)^n f_n b_n}{g_n}$$
$$-\frac{\zeta^2}{2}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}b_n b_l f_n f_l(-1)^{n+l}\left(\frac{1}{r_{nl}}+\frac{1}{s_{nl}}\right)$$
$$-\frac{2\zeta\cosh\zeta}{\sinh\zeta-\zeta\cosh\zeta}G,$$
$$C_2=\frac{2\zeta G}{\sinh\zeta-\zeta\cosh\zeta}$$

where

$$G=\frac{2}{3}+\frac{\mu Q^{\langle 2\rangle}}{\tilde N_2 V}-\frac{1}{4}\sum_{n=1}^{\infty}(f_n b_n)^2-\zeta^2\sum_{n=1}^{\infty}\frac{f_n b_n(-1)^n}{g_n}\left(1-\frac{2}{g_n}\right)$$
$$+\frac{\zeta^2}{4}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}f_n f_l b_n b_l(-1)^{n+l}\left(\frac{1}{r_{nl}}+\frac{1}{S_{nl}}\right).$$

The equations of this section simplify radically for very deep troughs; as $D\to\infty$, $f_n\to 0$. We find that

$$w^{\langle 1\rangle}=\frac{V}{2}(1-\bar x^2), \tag{3.3}$$

$$h^{\langle 2\rangle}=\frac{\tilde N_2 V}{\mu}\left\{2\bar x^2+\frac{4}{\zeta^2}-\frac{2}{3}+\frac{\mu Q^{\langle 2\rangle}}{\tilde N_2 V}-4\frac{\cosh\zeta\bar x}{\zeta\sinh\zeta}\right\} \tag{3.4}$$

when $h^{\langle 2\rangle}_{,x}(\pm d)=0$, and

$$h^{\langle 2\rangle}=\frac{\tilde{N}_2 V}{\mu}\left\{2\bar{x}^2-2+\frac{\zeta(\cosh\zeta\bar{x}-\cosh\zeta)}{\sinh\zeta-\zeta\cosh\zeta}\left[\frac{4}{3}+\frac{2\mu Q^{\langle 2\rangle}}{\tilde{N}_2 V}\right]\right\} \tag{3.5}$$

when $h^{\langle 2\rangle}(\pm d)=0$. These formulas show clearly how the rise at second order depends on the parameters

(i) The magnitude of the rise is proportional to the limiting value of the second normal stress $\lim[N_2(\kappa)/\kappa^2]=\tilde{N}_2=2\alpha_1+\alpha_2$ and to the ratio, V/μ, of speed to viscosity. The product $\tilde{N}_2 V/\mu$ is also proportional to the limiting value of the ratio of the second normal stress to the product of shear rate and the shear viscosity function

(ii) The shape of the rise curve depends on the prescription of the mean height at second order and the ratio ζ of the width of the trough to capillary radius. In Figs. 2 & 3 we have plotted (3.4) and (3.5) for a fixed mean height ($Q^{\langle 2\rangle}=0$) with the dimensionless capillary radius as a parameter. The free surface bulges when $\tilde{N}_2<0$ (Wineman & Pipkin, 1966). It is worth noting that the shape of the free surfaces (Fig. 2) are independent of $Q^{\langle 2\rangle}$. Changing the mass flux here merely shifts the level to which the fluid rises in the trough but does not effect its shape. In the other case (Fig. 3) the fluid grips the sharp edges so that a change in the amount of fluid in a cross-section must also change the shape of the free surface.

In Figs. 4 and 5 we have plotted some representative graphs of the curve of height of rise (3.2) at second order with D/d as a parameter. It is apparent from these graphs that the curves of height of rise for $D/d>2$ are barely distinguishable from the limiting curves for infinitely deep channels $D/d\to\infty$. The bottom does not sensibly effect the height or shape of the free surface in troughs which are deeper than they are wide. This result justifies the use of second order theory for infinitely deep channels in experiments using rectangular channels with $D/d>2$.

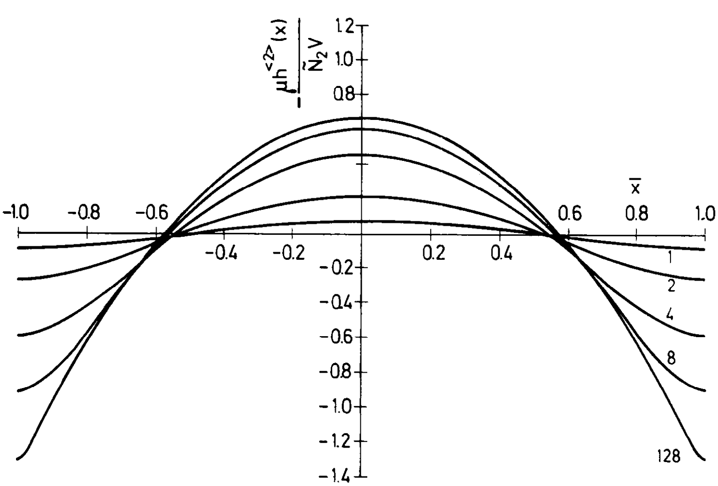

Fig. 2. Second order curves of height of rise for very deep troughs. These curves are computed from (3.3) with the dimensionless capillary radius $\zeta=d\sqrt{\rho g/\sigma}$ as a parameter.

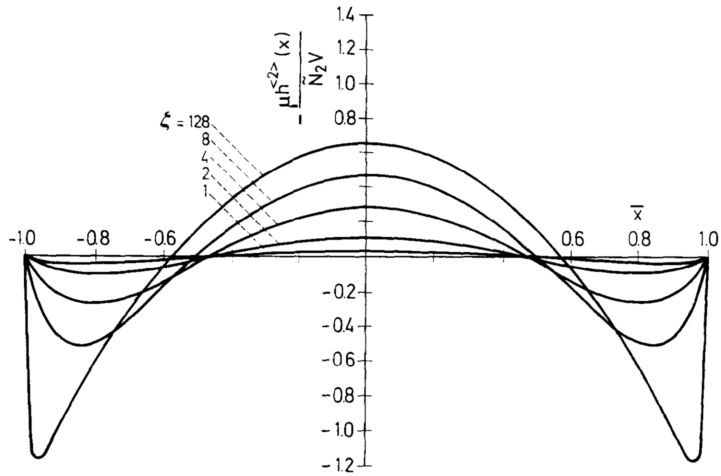

Fig. 3. Second order curves of height of rise for very deep troughs. These curves are computed from (3.4) with the dimensionless capillary radius as a parameter.

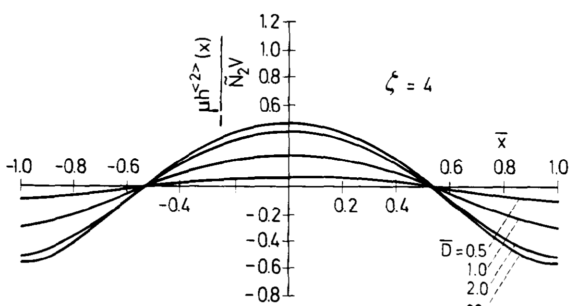

Fig. 4a. Second order curves of height of rise in rectangular channels. These curves are computed from (3.2) and the flat slope boundary condition, $h^{\langle 2\rangle}(\pm 1)=0$. The value of the ratio ζ of width/capillary radius is four for all curves shown.

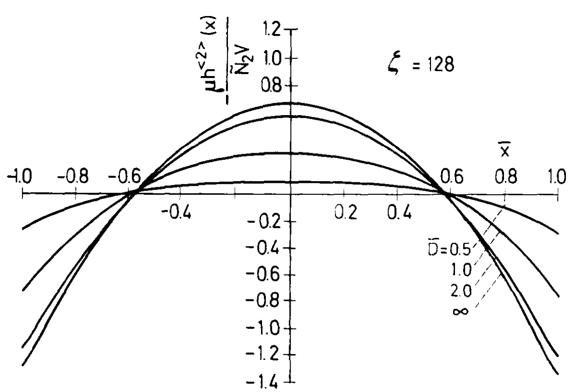

Fig. 4b. Second order curves of height of rise in rectangular channels. These curves are computed under the same conditions as those shown in Fig. 4a except that $\zeta = 128$ is large and the capillary radius relatively small. In this case the effects of the boundary conditions at $\bar{x} = \pm 1$ are felt in a boundary layer too small to be seen in this figure. The curves of rise, here and in Fig. 5b, are the same despite the fact that these satisfy a zero slope and those in Fig. 5b satisfy a zero displacement boundary condition.

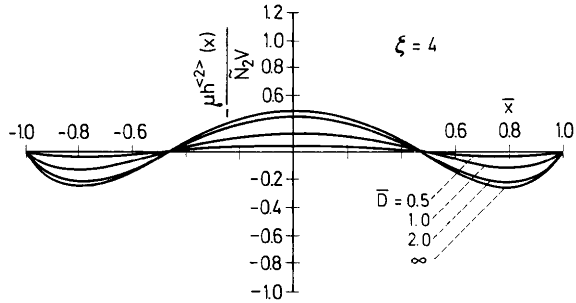

Fig. 5a. Second order curves of height of rise in rectangular channels. These curves are computed from (3.2) and the zero displacement boundary condition $h^{\langle 2\rangle}(\pm 1)=0$ with $\zeta=4$.

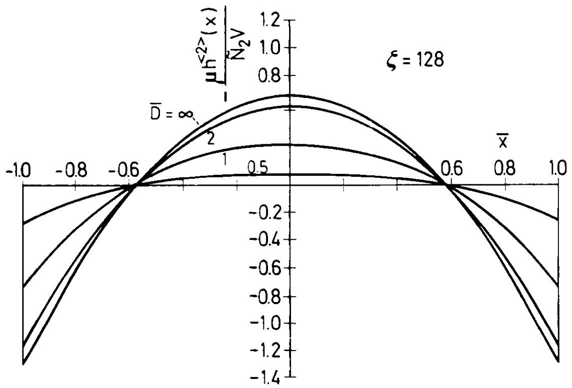

Fig. 5b. Second order curves of rise in rectangular channels with $h^{\langle 2\rangle}(\pm 1)=0$ and $\zeta=128$. The boundary condition is satisfied in a layer too small to be seen in these graphs.

Comparison of Figs. 4b and 5b show that the boundary conditions have little effect on the interior values of the height of rise when the ratio ζ of width to capillary radius is large.

4. Fourth Order Curves of Height of Rise in a Deep Trough ($\bar{D}\to\infty$)

From (2.13a) and (3.3) we find that

$$\mu V^2 w^{\langle 3\rangle} - 36\tilde{\beta}\left(\frac{\rho g}{\mu}\right)^3 x^2 - \rho g = 0. \tag{4.1}$$

The function

$$w^{\langle 3\rangle} = \frac{V}{2}(\bar{x}^2-1) + 3\frac{\tilde{\beta}V^3}{\mu d^2}(\bar{x}^4-1) + \frac{\tilde{N}_2 V^2}{\mu d}\sum_{n=1}^{\infty} c_n e^{b_n \bar{y}} \cos b_n \bar{x} \tag{4.2}$$

satisfies (4.1), (2.13b) and (2.13c) where

$$c_n = 24(-1)^n\{b_n^{-2} - 2b_n^{-4} - (\zeta^2+b_n^2)^{-2}[\zeta^2+b_n^2-2\zeta\coth\zeta]\}$$

when $h^{\langle 2\rangle}_{,x}(\pm d)=0$, and

$$c_n=8(-1)^n\left[3b_n^{-2}-6b_n^{-4}+\frac{\zeta^2\left[1+\frac{3}{4}\frac{\mu Q^{\langle 2\rangle}}{\tilde{N}_2 V}\right]}{\sinh\zeta-\zeta\cosh\zeta}\left(\frac{\sinh\zeta}{\zeta^2+b_n^2}-\frac{2\zeta\cosh\zeta}{(\zeta^2+b_n^2)^2}\right)\right]$$

when $h^{\langle 2\rangle}_{,x}(\pm d)=0$.

Using (3.3), we may rewrite (2.15a) as

$$\begin{aligned}\mu\nabla^2\boldsymbol{u}^{\langle 4\rangle}&+\boldsymbol{e}_x\{4\tilde{N}_2[2(w^{\langle 1\rangle}_{,x}w^{\langle 3\rangle}_{,x})_{,x}+w^{\langle 1\rangle}_{,x}w^{\langle 3\rangle}_{,yy}]\\&+96\tilde{\gamma}(w^{\langle 1\rangle^4}_{,x})_{,x}\}+\boldsymbol{e}_y\{4\tilde{N}_2(w^{\langle 1\rangle}_{,x}w^{\langle 3\rangle}_{,y})_{,x}\}-\nabla\Phi^{\langle 4\rangle}=0.\end{aligned}\tag{4.3}$$

The inhomogeneous terms in (4.3) can be expressed as a gradient. Using (4.1), (3.3) and $w^{\langle 1\rangle}_{,x}=-\dfrac{\rho g x}{\mu}$, we find that

$$\begin{aligned}\mu\nabla^2\boldsymbol{u}^{\langle 4\rangle}+\nabla\Big\{4\tilde{N}_2&\left[w^{\langle 1\rangle}_{,x}w^{\langle 3\rangle}_{,x}+w^{\langle 1\rangle}_{,xx}w^{\langle 3\rangle}-\tfrac{1}{2}w^{\langle 1\rangle^2}_{,x}\right.\\&\left.-9\frac{\tilde{\beta}}{\mu}w^{\langle 1\rangle^4}_{,x}\right]+96\tilde{\gamma}w^{\langle 1\rangle^4}_{,x}-\Phi^{\langle 4\rangle}\Big\}=0,\end{aligned}\tag{4.4}$$

$$\boldsymbol{u}^{\langle 4\rangle}=0,\tag{4.5}$$

and

$$\Phi^{\langle 4\rangle}=C_3\rho g+4\tilde{N}_2\left[w^{\langle 1\rangle}_{,x}w^{\langle 3\rangle}_{,x}+w^{\langle 1\rangle}_{,xx}w^{\langle 3\rangle}-\tfrac{1}{2}w^{\langle 1\rangle^2}_{,x}-9\frac{\tilde{\beta}}{\mu}w^{\langle 1\rangle^4}_{,x}\right]+96\tilde{\gamma}w^{\langle 1\rangle^4}_{,x}.\tag{4.6}$$

The solution to (2.15e) may now be completed. We find that

$$h^{\langle 4\rangle}=\frac{\tilde{N}_2 V}{\mu}H_a+\frac{(\tilde{N}_2 V)^2}{\mu^2 d}H_b+\frac{\tilde{N}_2\tilde{\beta}V^3}{\mu^2 d^2}H_c-\frac{\tilde{\gamma}V^3}{\mu d^2}H_c\tag{4.7}$$

where

$$H_a=4\bar{x}^2-4+\frac{32}{\zeta^2}+6C_1-3C_2\zeta\bar{x}\sinh\zeta\bar{x}+C_{3a}+C_{4a}\cosh\zeta\bar{x},\tag{4.8a}$$

$$H_b=4\zeta^2\sum_{n=1}^{\infty}c_n\left(\frac{b_n\bar{x}\sin b_n\bar{x}-\cos b_n\bar{x}}{\zeta^2+b_n^2}+\frac{2b_n^2\cos b_n\bar{x}}{(\zeta^2+b_n^2)^2}\right)+C_{3b}+C_{4b}\cosh\zeta\bar{x},\tag{4.8b}$$

$$H_c=12-96\left[\bar{x}^4+\frac{12}{\zeta^2}\bar{x}^2+\frac{24}{\zeta^4}\right]+C_{3c}+C_{4c}\cosh\zeta\bar{x}\tag{4.8c}$$

and

$$C_3=\frac{\tilde{N}_2 V}{\mu}C_{3a}+\frac{(\tilde{N}_2 V)^2}{\mu^2 d}C_{3b}+\frac{\tilde{N}_2\tilde{\beta}V^3}{\mu^2 d^2}C_{3c}-\frac{\tilde{\gamma}V^3}{\mu d^2}(C_{3c}+12),$$

where $C_{3a},\ldots,C_{4c}$ are constants of integration. Application of the condition for fixed contact line (2.10_1) gives

$$\begin{aligned}C_{3a}=-\frac{32}{\zeta^2}-6C_1+3C_2(\zeta\sinh\zeta+\cosh\zeta)-\frac{\zeta\cosh\zeta}{\sinh\zeta-\zeta\cosh\zeta}\\ \cdot\left[\frac{8}{3}-3C_2\zeta\sinh\zeta+\frac{\mu}{\tilde{N}_2 V}Q^{\langle 4\rangle}\right],\end{aligned}$$

$$C_{3b}=4\zeta^2\sum_{n=1}^{\infty}\frac{c_n b_n(-1)^n}{\zeta^2+b_n^2}\left\{1+\frac{\zeta\cosh\zeta}{\sinh\zeta-\zeta\cosh\zeta}\left[1-\frac{2}{\zeta^2+b_n^2}\right]\right\},$$

$$C_{3c}=96\left\{\frac{7}{8}+\frac{12}{\zeta^2}+\frac{24}{\zeta^4}+\frac{\zeta\cosh\zeta}{\sinh\zeta-\zeta\cosh\zeta}\left[\frac{4}{5}+\frac{8}{\zeta^2}\right]\right\},$$

$$C_{4a}=-3C_2+\frac{\zeta}{\sinh\zeta-\zeta\cosh\zeta}\left[\frac{8}{3}-3C_2\zeta\sinh\zeta+\frac{\mu}{\tilde{N}_2 V}Q^{\langle 4\rangle}\right],$$

$$C_{4b}=\frac{-4\zeta^3}{\sinh\zeta-\zeta\cosh\zeta}\sum_{n=1}^{\infty}\frac{c_n b_n(-1)^n}{\zeta^2+b_n^2}\left[1-\frac{2}{\zeta^2+b_n^2}\right],$$

and

$$C_{4c}=-\frac{96\zeta}{\sinh\zeta-\zeta\cosh\zeta}\left[\frac{4}{5}+\frac{8}{\zeta^2}\right],$$

while application of the prescribed condition for angle of contact (2.10_2) gives

$$C_{3a}=6-6C_1-6C_2\frac{\sinh\zeta}{\zeta}-\frac{10}{3}-\frac{24}{\zeta^2}+\frac{\mu}{\tilde{N}_2 V}Q^{\langle 4\rangle},$$

$$C_{3b}=0,\qquad C_{3c}=\frac{36}{5},$$

$$C_{4a}=\frac{-1}{\zeta\sinh\zeta}[8-3C_2\zeta(\sinh\zeta+\zeta\cosh\zeta)],$$

$$C_{4b}=\frac{8\zeta^4}{\zeta\sinh\zeta}\sum_{n=1}^{\infty}\frac{c_n b_n(-1)^n}{(\zeta^2+b_n^2)^2}$$

and

$$C_{4c}=\frac{96}{\zeta\sinh\zeta}\left[4+\frac{24}{\zeta^2}\right].$$

In Fig. 6, we have sketched the coefficients of height of rise (4.8) for the case in which a horizontal angle of contact is prescribed, $Q^{\langle 4\rangle}=0$ and $\zeta=1$. In Fig. 7, we have sketched (4.8) for the case in which the line of contact is fixed, $Q^{\langle 4\rangle}=0$ and $\zeta=1$.

We shall not carry the analysis further than fourth order. At this order, we have already good expressions for the shape of the free surface which depend on the RE constants of the first, second, third and fourth order fluids. The computation of the change in the axial velocity at fifth order is straightforward but tedious and, up to the time of writing, we did not find a clear motivation for writing down the results. Computations at sixth order become very complicated. We carried the calculations far enough to satisfy ourselves that secondary motions do appear at this order. It is necessary to add, however, that the heavy labor of computation at sixth order reduced our standard of self-satisfaction below the level required for categorical statements.

The results achieved so far are collected into partial sums for series giving the axial flow and the shape of the free surface in Section 6.

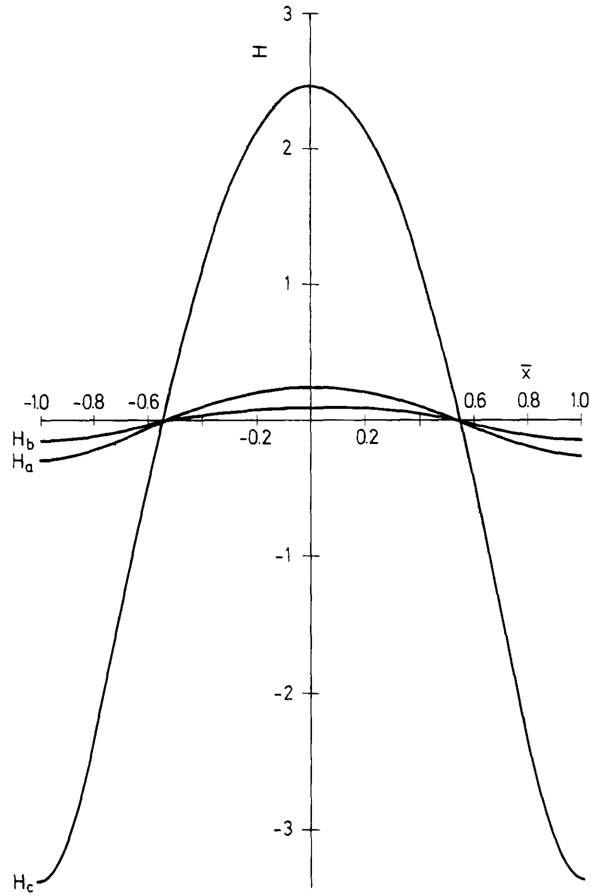

Fig. 6. Fourth order curves of height of rise for very deep troughs. These curves are calculated from (4.8) with $Q^{\langle 4\rangle}=0$, $\zeta=1$ and the condition of flat angle of contact (2.10_2).

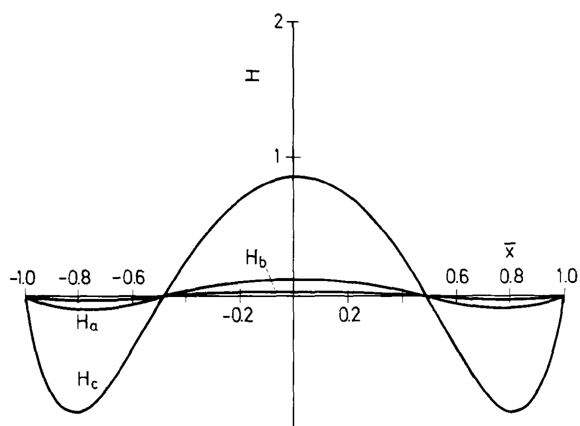

Fig. 7. Fourth order dimensionless curves of height of rise for very deep troughs. These curves are calculated from (4.8) with $Q^{\langle 4\rangle}=0$, $\zeta=1$ and the condition of fixed line of contact (2.10_1).

5. The Semicircular Trough

Most of the equations given in Section 2 hold when the cross-section of the trough is arbitrary. Of course, it is then necessary to replace (1.2) with a mapping appropriate to the given boundary. We may find one to one mappings $x=x_0$ and $y=\mathscr{Y}(y_0, x_0, h(x_0, \varepsilon))$ which are linear in y_0 and take

$$\mathscr{V}_\beta \leftrightarrow \mathscr{V}_0; \quad \mathscr{S}_\beta \leftrightarrow \mathscr{S}_0, \quad y=h(x;\varepsilon) \rightarrow y_0=0.$$

In some cases, it may also be necessary to modify slightly the mean height to account for features of the changed cross-section.

The semicircular trough is interesting because it belongs to the special class of problems in which secondary motions are first generated at sixth order in the series of powers of β. For the semicircular trough, it suffices to use a mapping function $\mathscr{Y}=[1-h/(d^2-x_0^2)^{\frac{1}{2}}]\,y_0+h$. Then, relative to the problem for which

$$h(\pm d)=0, \tag{5.1}$$

all of the equations of Section 2 hold and we find that

$$w^{\langle 1\rangle}=\frac{V}{4}\left(1-\frac{r^2}{d^2}\right), \tag{5.2}$$

where $x=r\cos\theta$ and $y=r\sin\theta$,

$$\Phi^{\langle 2\rangle}=\tilde{N}_2\left[|\nabla w^{\langle 1\rangle}|^2-\frac{2\rho g}{\mu}w^{\langle 1\rangle}\right]-\frac{\tilde{N}_2 V^2}{d^2}\left[\frac{1}{4}+\frac{3}{2\zeta^2}+C_6\cosh\zeta\right] \tag{5.3a}$$

and

$$h^{\langle 2\rangle}=\frac{\tilde{N}_2 V}{\mu}\left[C_6(\cosh\zeta\bar{x}-\cosh\zeta)+\tfrac{3}{4}(\bar{x}^2-1)\right], \tag{5.3b}$$

where $\bar{x}=x/d$ and ζ are the dimensionless symbols introduced in Section 3 and

$$C_6=\frac{\zeta}{\sinh\zeta-\zeta\cosh\zeta}\left(\frac{1}{2}+\frac{\mu Q^{\langle 2\rangle}}{\tilde{N}_2 V}\right).$$

At third order we have

$$\nabla^2 w^{\langle 3\rangle}=\frac{\rho g}{\mu}-\frac{48\tilde{\beta}}{\mu}|\nabla w^{\langle 1\rangle}|^2\, w^{\langle 1\rangle}_{,xx}=\frac{\rho g}{\mu}\left[1+\frac{24\tilde{\beta}}{\mu}|\nabla w^{\langle 1\rangle}|^2\right] \quad \text{in } \mathscr{V}_0, \tag{5.4a}$$

$$w^{\langle 3\rangle}=0 \quad \text{on} \quad r=d \tag{5.4b}$$

and

$$w^{\langle 3\rangle}_{,y}=\frac{\tilde{N}_2 V^2}{\mu d^2}\,\frac{3}{2}\left[C_6(\cosh\zeta\bar{x}-\cosh\zeta-\zeta\bar{x}\sinh\zeta\bar{x})-\tfrac{3}{4}\bar{x}^2-\tfrac{3}{4}\right] \quad \text{on} \quad y=0. \tag{5.4c}$$

The demonstration that $\boldsymbol{u}^{\langle 4\rangle}\equiv 0$ in $\mathscr{V}_0$ does not require any further analysis of the problem of third order. It is necessary to show that the vector $\boldsymbol{e}_x X+\boldsymbol{e}_y Y$ defined under (2.16) may be obtained from a potential. From (5.1) we learn that

$w^{\langle 1\rangle}_{,xx}=w^{\langle 1\rangle}_{,yy}=-\rho g/2\mu$ and $w^{\langle 1\rangle}_{,xy}=0$. Then using (5.1) and (5.4a), we may write

$$X=4\tilde{N}_2\frac{\rho g}{\mu}\left[w^{\langle 1\rangle}_{,x}-w^{\langle 3\rangle}_{,x}+\frac{24\tilde{\beta}}{\mu}w^{\langle 1\rangle}_{,x}|\nabla w^{\langle 1\rangle}|^2\right]$$

and

$$Y=4\tilde{N}_2\frac{\rho g}{\mu}\left[w^{\langle 1\rangle}_{,y}-w^{\langle 3\rangle}_{,y}+\frac{24\tilde{\beta}}{\mu}w^{\langle 1\rangle}_{,y}|\nabla w^{\langle 1\rangle}|^2\right].$$

Hence, X and Y can be obtained as the components of the gradient of the potential $F=4\tilde{N}_2G$

$$G=\frac{\rho g}{\mu}\left[w^{\langle 1\rangle}-w^{\langle 3\rangle}-\frac{3}{4}\frac{\tilde{\beta}}{\mu}\frac{V^3}{d^6}r^4\right] \tag{5.5}$$

and

$$\Phi^{\langle 4\rangle}=4\tilde{N}_2[\nabla w^{\langle 1\rangle}\cdot\nabla w^{\langle 3\rangle}+G]+120\tilde{\gamma}|\nabla w^{\langle 1\rangle}|^4+C_7\rho g. \tag{5.6}$$

The differential equation (2.17) for the free surface may now be written as

$$\sigma h^{\langle 4\rangle}_{,xx}-\rho g h^{\langle 4\rangle}=-4\tilde{N}_2[w^{\langle 1\rangle}_{,x}w^{\langle 3\rangle}_{,x}+G]-120\tilde{\gamma}w^{\langle 1\rangle^4}_{,x}-C_7\rho g-6\rho g h^{\langle 2\rangle}. \tag{5.7}$$

To complete the analysis at fourth order, we must first find expressions for the perturbation field $w^{\langle 3\rangle}$ by solving (5.4).

Problem (5.4) may be reduced to an edge problem for Laplace's equation. Thus

$$w^{\langle 3\rangle}=-w^{\langle 1\rangle}+\frac{\tilde{\beta}V^3}{\mu d^2}\frac{3}{8}\left[\left(\frac{r}{d}\right)^4-1\right]+\frac{\tilde{N}_2V^2}{\mu d}\phi(r,\theta) \tag{5.8}$$

where

$$\nabla^2\phi=0 \quad\text{in}\quad \mathscr{V}_0, \qquad \phi(d,\theta)=0 \tag{5.9a,b}$$

and

$$w^{\langle 3\rangle}_{,y}=\frac{\tilde{N}_2V^2}{\mu d^2}\frac{1}{x}\phi_{,\theta} \quad\text{on}\quad y=0 \tag{5.9c}$$

where $w^{\langle 3\rangle}_{,y}$ is given by (5.4c) and $\phi_{,\theta}=\partial_\theta\phi(r,\theta)$ is evaluated on $\theta=0$ for $x>0$ and $\theta=\pi$ for $x<0$. The harmonic function $\phi(r,\theta)$ satisfying problem (5.9) is easily obtained by the method of separation of variables used by KUO & TANNER (1972) in their study of the flow of a Newtonian fluid down a tilted through. We shall not here give this solution or the resulting expression for the correction for height $h^{\langle 4\rangle}$ which arises from the integration of (5.7).

We conclude this section with some remarks concerning the computation of the viscometric function $N_2(\kappa^2)$, $\kappa^2=w^2_{,x}+w^2_{,y}$, from experiments in a tilted trough. KUO & TANNER (1974) have studied this problem by the cut away method; they discard the term

$$S_{r\theta,\theta|_{y=0}}\sim O(N_2h_m^2/R_0^2) \qquad \text{(their estimate in their notation)}$$

where h_m is the maximum deflection and N_2 is the second normal stress difference. In our notation, $N_2=O(\beta^2)$, $h_m^2=O(\beta^4)$ so that the rejected term is said to be of $O(\beta^6)$. In contrast, we find that

$$S_{r\theta,\theta|_{y=0}} = S^{\langle 4\rangle}_{r\theta,\theta|_{y=0}}\beta^4 + O(\beta^6), \tag{5.10}$$

and conclude that KUO (1973), and KUO & TANNER (1974) have discarded a term of the same order as others which are retained*. Their basic equations [22] and [23] are therefore incorrect at $O(\beta^4)$ or, in their notation, at $O(N_2 h_m/R_0)$. If we assume that all of the pertinent error estimates in KUO's thesis (1973) and in the paper of KUO & TANNER (1974) have been incorrectly printed, then the theory given by them is valid up to an order of error h_m/R_0. We say that this means that the KUO-TANNER analysis is valid to $O(\beta^2)$ and that their analysis gives $N_2 = (2\alpha_1 + \alpha_2)\kappa^2$ and nothing more. TANNER (private communication) says that the revised error estimate, $O(h_m/R_0)$, holds irrespective of considerations of order in our sense and that the cut away argument was used by them, deliberately, to avoid Rivlin-Ericksen expansions.

To obtain the form of $N_2(\kappa)$ at the next higher order,

$$N_2 = (2\alpha_1 + \alpha_2)\kappa^2 + 4(\gamma_3 + \gamma_4 + \gamma_5 + \tfrac{1}{2}\gamma_6)\kappa^4 + O(\kappa^6),$$

it is necessary to determine the value the constant $\gamma_3 + \gamma_4 + \gamma_5 + \frac{1}{2}\gamma_6 = \tilde{\gamma}$. This constant appears in explicit form in the expression (4.7) for the height of rise at fourth order. In principle, we can compute $\tilde{\gamma}$ by measuring the free surface. Whether this point of principle is also of practical importance will be decided by experiments.

6. Application of the Theory to Experiments

Our present view is that deep rectangular troughs may be the most suitable configurations to use in the experimental determination of the RE constants. We propose to use the theory for the infinitely deep trough to guide the interpretation of data taken in deep troughs. For the infinitely deep trough, we have

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* To prove (5.10) we must show that

$$S^{\langle 4\rangle}_{r\theta,\theta|_{y=0}} \neq 0. \tag{5.11}$$

In the semicircle $\mathscr{V}_0$, $S_{r\theta} = \mathbf{e}_r \cdot \mathbf{S} \cdot \mathbf{e}_\theta$ is non-zero at even orders in β and is given by

$$S_{r\theta} = S^{[2]}_{r\theta}\frac{\beta^2}{2!} + S^{[4]}_{r\theta}\frac{\beta^4}{4!} + O(\beta^6) \tag{5.12}$$

where

$$S^{[2]}_{r\theta} = S^{\langle 2\rangle}_{r\theta} = 2\tilde{N}_2\left(\frac{1}{r}w^{\langle 1\rangle}_{,\theta}\right)(w^{\langle 1\rangle}_{,r}) = 0$$

because $w^{\langle 1\rangle}$ is independent of θ. Then, using (5.8), we find that

$$S^{[4]}_{r\theta} = S^{\langle 4\rangle}_{r\theta} = 4\tilde{N}_2\frac{1}{r}w^{\langle 1\rangle}_{,r}w^{\langle 3\rangle}_{,\theta} = \frac{4\tilde{N}_2^2 V^2}{\mu d^2}\frac{1}{r}w^{\langle 1\rangle}_{,r}\phi_{,\theta} \tag{5.13}$$

in the semicircle $\mathscr{V}_0$ and, using (5.9c), $\phi_{,\theta} \neq 0$ on $y=0$. Differentiating (5.13) once more we find that

$$S^{\langle 4\rangle}_{r\theta,\theta} = \frac{4\tilde{N}_2^2 V^2}{\mu d^2}\frac{1}{r}w^{\langle 1\rangle}_{,r}\phi_{,\theta\theta}.$$

Suppose $\phi_{,\theta\theta} = 0$ on $y=0$; then $\phi \equiv 0$ in $\mathscr{V}_0$, which is impossible. This completes the proof of (5.11).

found that

$$u(x_0, y_0; \beta) = O(\beta^6), \tag{6.1a}$$

$$v(x_0, y_0; \beta) = O(\beta^6), \tag{6.1b}$$

$$\begin{aligned} w(x_0, y_0; \beta) &= w^{[1]}(x_0, y_0)\,\beta + w^{[3]}(x_0, y_0)\frac{\beta^3}{3!} + O(\beta^5) \\ &= \frac{V}{2}\left(1 - \frac{x_0^2}{d^2}\right)\beta + \left[\frac{V}{2}\left(\frac{x_0^2}{d^2} - 1\right) + \frac{3\tilde{\beta}V^3}{\mu d^2}\left(\frac{x_0^4}{d^4} - 1\right)\right. \\ &\quad \left. + \frac{\tilde{N}_2 V^2}{\mu d}\sum_{n=1}^{\infty} c_n e^{\frac{b_n y_0}{d}} \cos b_n \frac{x_0}{d}\right]\frac{\beta^3}{3!} + O(\beta^5), \end{aligned} \tag{6.1c}$$

where the c_n are defined under (4.2),

$$\begin{aligned} h(x_0; \beta) &= h^{\langle 2\rangle}\frac{\beta^2}{2} + h^{\langle 4\rangle}\frac{\beta^4}{4!} + O(\beta^6) = \frac{\tilde{N}_2 V}{\mu}\left[C_1 + C_2\cosh\left(\frac{\zeta x_0}{d}\right) + 2\left(\frac{x_0}{d}\right)^2 + \frac{4}{\zeta^2} - 1\right]\frac{\beta^2}{2} \\ &\quad + \left[\frac{\tilde{N}_2 V}{\mu}H_a + \frac{\tilde{N}_2^2 V^2}{\mu^2 d}H_b + \frac{\tilde{N}_2\tilde{\beta}V^3}{\mu^2 d^2}H_c - \frac{\tilde{\gamma}V^3}{\mu d^2}H_c\right]\frac{\beta^4}{4!} + O(\beta^6), \end{aligned} \tag{6.1d}$$

where C_1 and C_2 are limiting values ($D/d \to \infty$) of the constants C_1 and C_2 defined under (3.2) and H_a, H_b and H_c are given by (4.8). We have replaced (x, y) with (x_0, y_0) in (6.1) to emphasize that the functions are defined in the reference domain under the flat free surface. To obtain the solution in the deformed domain, we invert the mapping (1.2) in the limit $D/d \to \infty$:

$$x_0 = x,$$
$$y_0 = y - h(x; \beta).$$

The formula (6.1d) is the working equation for the experiment. This formula depends on the RE constants μ, $N_2 = 2\alpha_1 + \alpha_2$, $\tilde{\beta} = \beta_2 + \beta_3$ and $\tilde{\gamma} = \gamma_3 + \gamma_4 + \gamma_5 + \frac{1}{2}\gamma_6$ in a simple way. At angles of small tilt the height of rise is linear in the bulge constant $\tilde{N}_2$.* To determine $\tilde{N}_2$ it is necessary to measure the shape of the free surface; given the shape of the free surface and its absolute height, we determine the constant $\tilde{N}_2$ and the mean height $Q^{\langle 2\rangle}$ at second order (which appears in the definition of C_1 and C_2). Of course, specification of the mean height at second order and the center line height $h^{\langle 2\rangle}(0)$ at second order are entirely equivalent.

The experimental determination of the bulge constant $\tilde{N}_2$ may be easiest when the side walls of the channel are coated to maintain a flat contact angle $h'(\pm d) = 0$. In this case the shape of the free surface is independent of the mean height and

$$\tilde{N}_2 = 2\alpha_1 + \alpha_2 = \frac{\mu}{V}\,\frac{h^{\langle 2\rangle}(d) - h^{\langle 2\rangle}(0)}{2 - \dfrac{4(\cosh\zeta - 1)}{\zeta\sinh\zeta}}. \tag{6.2}$$

Equation (6.2) shows that for a given fluid and trough, $\tilde{N}_2$ may be determined from the experimental determination of the height discrepancy $h^{\langle 2\rangle}(d) - h^{\langle 2\rangle}(0)$ at second order.

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* The bulge constant $\tilde{N}_2 = 2\alpha_1 + \alpha_2$ and the rod climbing constant $\hat{\beta} = 3\alpha_1 + 2\alpha_2$ (see JOSEPH, BEAVERS & FOSDICK, 1973; BEAVERS & JOSEPH, 1975), taken together, completely define a simple fluid in the approximation of second order for slow steady flow.

In Section 3, we showed that a deep trough, in the second order theory, has $D/d>2$. At fourth order, secondary motions are driven from the bottom of the trough and these bottom eddies may be expected to penetrate more deeply into the fluid's interior. In the different, but mathematically similar, problem which arises in the computation of the shape of a free surface on a liquid in a trench heated from its side, we showed (JOSEPH & STURGES, 1975) that the bottom eddies have sensibly vanished in a distance $D/d>3$. The effect of secondary motions driven at the bottom may possibly be reduced by making a semicircular bottom for the deep trough.

Appendix 1: A Theorem about Rivlin-Ericksen Tensors

In the study of flows of a simple fluid in which can be constructed a perturbation of the state of rest it is necessary to compute the value of derivatives of the RE tensors evaluated on the rest state. If

$$\boldsymbol{u}(x,\beta)=\sum_{j=1}^{\infty}\boldsymbol{u}^{\langle j\rangle}\frac{\beta^j}{j!} \tag{A.1}$$

is a steady flow, analytic in β, then

$$\boldsymbol{A}_n=\boldsymbol{A}_n[\boldsymbol{u},\boldsymbol{u},\dots,\boldsymbol{u}], \qquad n \text{ times},$$

is a homogeneous polynomial of degree n in $\boldsymbol{u}$ and

$$\boldsymbol{A}_n=\sum_{m=n}^{\infty}\boldsymbol{A}_n^{\langle m\rangle}\frac{\beta^m}{m!}=\boldsymbol{A}_n\left[\sum_{j=1}^{\infty}\boldsymbol{u}^{\langle j\rangle}\frac{\beta^j}{j!},\dots,\sum_{j=1}^{\infty}\boldsymbol{u}^{\langle j\rangle}\frac{\beta^j}{j!}\right]$$

where $\boldsymbol{A}_n^{\langle m\rangle}=\partial^m\boldsymbol{A}_n/\partial\beta^m$, evaluated at $\beta=0$, is given by

$$\frac{1}{m!}\boldsymbol{A}_n^{\langle m\rangle}=\sum\nolimits^{(m)}\boldsymbol{A}_n\left[\frac{\boldsymbol{u}^{\langle r_1\rangle}}{r_1!},\frac{\boldsymbol{u}^{\langle r_2\rangle}}{r_2!},\dots,\frac{\boldsymbol{u}^{\langle r_n\rangle}}{r_n!}\right]. \tag{A.2}$$

Here $\sum^{(m)}$ is a summation for a fixed integer m over all sets of positive integers such that

$$m=\sum_{i=1}^{n}r_i.$$

Since $0<r_i$, the smallest of the allowed values of r_i is 1. Suppose that $r_i=1$ for $i=1,2,\dots,n-1$. Then

$$m=\sum_{i=1}^{n}r_i=r_1+r_2+\dots+r_n=n-1+r_n.$$

Hence the largest of the allowed values of r_i is $m-(n-1)$.

We now consider the derivatives of $\boldsymbol{A}_n$ when the $\boldsymbol{A}_n$ are evaluated on the partial sums

$$\boldsymbol{u}_{(N)}=\sum_{j=1}^{N}\boldsymbol{u}^{\langle j\rangle}\frac{\beta^j}{j!}, \qquad l\geqq 1 \tag{A.3}$$

where the $\boldsymbol{u}^{\langle j\rangle}$ are the Taylor coefficients in (A.1). We find that

$$A_n[\boldsymbol{u}_{(N)},\boldsymbol{u}_{(N)},\dots,\boldsymbol{u}_{(N)}]=\sum_{m=n}^{N+n-1}A_n^{\langle m\rangle}\frac{\beta^m}{m!}+\sum_{m=n+N}^{nN}F_{mn}\beta^m \tag{A.4}$$

where the $\boldsymbol{A}_n^{\langle m\rangle}$ are the tensors defined by (A.2)

Theorem. *Suppose that* $(\boldsymbol{u}(x,\beta))$ *is given by (A.1). Then*

$$A_n^{\langle m\rangle}=0, \qquad m<n. \tag{A.5}$$

Suppose further that $\boldsymbol{u}_{(N)}(x,\beta)$ *is a viscometric flow, then*

$$A_n^{\langle m\rangle}=0, \qquad m\leqq n+N-1, \qquad n>2. \tag{A.6}$$

Proof. Equation (A.5) is obvious. To establish (A.6), we recall

$$A_n[u,u,\dots,u]=0, \qquad n>2, \qquad n \text{ times},$$

when $\boldsymbol{u}$ is a viscometric flow. Under the hypothesis of the theorem the left side of (A.4) is zero. Differentiating (A.4) repeatedly with respect to β at $\beta=0$ we prove (A.6).

The perturbation relation (A.6) is very useful in calculations; it eliminates wasted computations which, in the end, lead to zero tensors.

The relation (A.6) could also be used to prove the theorem given in Part I (JOSEPH, 1974). In that theorem it was shown that if the N^{th} partial sum of the series for the velocity field is itself a viscometric field, then, whether or not the $N+1^{\text{st}}$ partial sum be a viscometric field, all partial sums up to and including the $N+2^{\text{nd}}$ depend upon the constitutive relation of the fluid only through its viscometric constants. An interesting, but esoteric, consequence of the theorem proved in Part I is that steady nonviscometric flow down infinitely deep channels and semicircular troughs depends exclusively on viscometric constants through seventh order in powers of the angle of tilt β. Though secondary flows develop at sixth order, the perturbation fields at sixth and seventh order depend exclusively on the viscometric constants.

Appendix 2. Computation of the Tensors $A_n^{\langle m\rangle}$

We now calculate the Rivlin-Erickson tensors $A_n^{\langle m\rangle}$ as required in the expressions (2.1a) and (2.2a). In the calculation we use (A.5), (A.6) and recall that $\boldsymbol{u}^{\langle 0\rangle}=0$ and $\boldsymbol{u}_{,z}^{\langle n\rangle}=0$.

If n is odd,

$$\begin{aligned}A_1^{\langle n\rangle}&=(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T)^{\langle n\rangle}=\nabla\boldsymbol{u}^{\langle n\rangle}+(\nabla\boldsymbol{u}^{\langle n\rangle})^T\\&=(\boldsymbol{e}_x\boldsymbol{e}_z+\boldsymbol{e}_z\boldsymbol{e}_x)\,w_{,x}^{\langle n\rangle}+(\boldsymbol{e}_y\boldsymbol{e}_z+\boldsymbol{e}_z\boldsymbol{e}_y)\,w_{,y}^{\langle n\rangle}.\end{aligned}$$

If n is even,

$$\begin{aligned}A_1^{\langle n\rangle}&=\boldsymbol{e}_x\boldsymbol{e}_x 2u_{,x}^{\langle n\rangle}+\boldsymbol{e}_y\boldsymbol{e}_y 2v_{,y}^{\langle n\rangle}\\&\quad+(\boldsymbol{e}_x\boldsymbol{e}_y+\boldsymbol{e}_y\boldsymbol{e}_x)\,(u_{,y}^{\langle n\rangle}+v_{,x}^{\langle n\rangle})\end{aligned}$$

At first order,

$$A_n^{\langle 1\rangle}=0, \qquad n>1.$$

At second order,

$$\begin{aligned}(A_1^2)^{\langle 2\rangle}&=2A_1^{\langle 1\rangle^2}=2\{\boldsymbol{e}_x\boldsymbol{e}_x w_{,x}^{\langle 1\rangle^2}+\boldsymbol{e}_y\boldsymbol{e}_y w_{,y}^{\langle 1\rangle^2}\\&\quad+\boldsymbol{e}_z\boldsymbol{e}_z(w_{,x}^{\langle 1\rangle^2}+w_{,y}^{\langle 1\rangle^2})+(\boldsymbol{e}_x\boldsymbol{e}_y+\boldsymbol{e}_y\boldsymbol{e}_x)\,w_{,x}^{\langle 1\rangle}w_{,y}^{\langle 1\rangle}\},\\A_2^{\langle 2\rangle}&=2\{A_1^{\langle 1\rangle}\cdot\nabla\boldsymbol{u}^{\langle 1\rangle}+(\nabla\boldsymbol{u}^{\langle 1\rangle})^T\cdot A_1^{\langle 1\rangle}\}\\&=4\{\boldsymbol{e}_x\boldsymbol{e}_x w_{,x}^{\langle 1\rangle^2}+\boldsymbol{e}_y\boldsymbol{e}_y w_{,y}^{\langle 1\rangle^2}+(\boldsymbol{e}_x\boldsymbol{e}_y+\boldsymbol{e}_y\boldsymbol{e}_x)\,w_{,x}^{\langle 1\rangle}w_{,y}^{\langle 1\rangle}\},\\A_n^{\langle 2\rangle}&=0, \qquad n>2.\end{aligned}$$

At third order (now recognizing that $u^{\langle 2\rangle}=0$),

$$(A_1^2)^{\langle 3\rangle} = A_2^{\langle 3\rangle} = 0,$$
$$A_3^{\langle 3\rangle} = 3\{A_2^{\langle 2\rangle}\cdot \nabla u^{\langle 1\rangle} + (\nabla u^{\langle 1\rangle})^T \cdot A_2^{\langle 2\rangle}\} = 0,$$
$$(A_2\cdot A_1)^{\langle 3\rangle} = 3A_2^{\langle 2\rangle}\cdot A_1^{\langle 1\rangle} = 12\{e_x e_z (w_{,x}^{\langle 1\rangle 3} + w_{,x}^{\langle 1\rangle} w_{,y}^{\langle 1\rangle 2}) + e_y e_z (w_{,x}^{\langle 1\rangle 2} w_{,y}^{\langle 1\rangle} + w_{,y}^{\langle 1\rangle 3})\},$$
$$[(\mathrm{tr}\, A_2) A_1]^{\langle 3\rangle} = 3(\mathrm{tr}\, A_2^{\langle 2\rangle}) A_1^{\langle 1\rangle} = (A_2\cdot A_1 + A_1\cdot A_2)^{\langle 3\rangle},$$
$$A_n^{\langle 3\rangle}=0, \qquad n>3.$$

At fourth order,

$$\begin{aligned}(A_1^2)^{\langle 4\rangle} &= 4(A_1^{\langle 1\rangle}\cdot A_1^{\langle 3\rangle} + A_1^{\langle 3\rangle}\cdot A_1^{\langle 1\rangle})\\ &= 4\{e_x e_x 2 w_{,x}^{\langle 1\rangle} w_{,x}^{\langle 3\rangle} + e_y e_y 2 w_{,y}^{\langle 1\rangle} w_{,y}^{\langle 3\rangle}\\ &\quad + e_z e_z 2(w_{,x}^{\langle 1\rangle} w_{,x}^{\langle 3\rangle} + w_{,y}^{\langle 1\rangle} w_{,y}^{\langle 3\rangle})\\ &\quad + (e_x e_y + e_y e_x)(w_{,x}^{\langle 1\rangle} w_{,y}^{\langle 3\rangle} + w_{,y}^{\langle 1\rangle} w_{,x}^{\langle 3\rangle})\},\end{aligned}$$
$$\begin{aligned}A_2^{\langle 4\rangle} &= 4\{A_1^{\langle 3\rangle}\cdot\nabla u^{\langle 1\rangle} + (\nabla u^{\langle 1\rangle})^T\cdot A_1^{\langle 3\rangle} + A_1^{\langle 1\rangle}\cdot\nabla u^{\langle 3\rangle} + (\nabla u^{\langle 3\rangle})^T\cdot A_1^{\langle 1\rangle}\}\\ &= 16\{e_x e_x w_{,x}^{\langle 1\rangle} w_{,x}^{\langle 3\rangle} + e_y e_y w_{,y}^{\langle 1\rangle} w_{,y}^{\langle 3\rangle}\\ &\quad + (e_x e_y + e_y e_x)(w_{,x}^{\langle 1\rangle} w_{,y}^{\langle 3\rangle} + w_{,y}^{\langle 1\rangle} w_{,x}^{\langle 3\rangle})\},\end{aligned}$$
$$A_3^{\langle 4\rangle} = (A_1\cdot A_2)^{\langle 4\rangle} = [(\mathrm{tr}\,A_2)A_1]^{\langle 4\rangle}=0,$$
$$A_4^{\langle 4\rangle} = (A_1\cdot A_3)^{\langle 4\rangle} = [(\mathrm{tr}\,A_3)A_1]^{\langle 4\rangle} = [\mathrm{tr}\,(A_1\cdot A_2)A_1]^{\langle 4\rangle}=0,$$
$$\begin{aligned}(A_2^2)^{\langle 4\rangle} &= 6A_2^{\langle 2\rangle 2} = 96\{e_x e_x (w_{,x}^{\langle 1\rangle 4} + w_{,x}^{\langle 1\rangle 2} w_{,y}^{\langle 1\rangle 2})\\ &\quad + e_y e_y (w_{,x}^{\langle 1\rangle 2} w_{,y}^{\langle 1\rangle 2} + w_{,y}^{\langle 1\rangle 4})\\ &\quad + (e_x e_y + e_y e_x)(w_{,x}^{\langle 1\rangle 3} w_{,y}^{\langle 1\rangle} + w_{,x}^{\langle 1\rangle} w_{,y}^{\langle 1\rangle 3})\},\end{aligned}$$
$$[(\mathrm{tr}\,A_2)A_2]^{\langle 4\rangle} = 6(\mathrm{tr}\,A_2^{\langle 2\rangle})A_2^{\langle 2\rangle} = (A_2^2)^{\langle 4\rangle},$$
$$(A_2\cdot A_1^2)^{\langle 4\rangle} = 12 A_2^{\langle 2\rangle}\cdot A_1^{\langle 1\rangle 2} = \tfrac12 (A_2^2)^{\langle 4\rangle},$$
$$\begin{aligned}[(\mathrm{tr}\,A_2)A_1^2]^{\langle 4\rangle} &= 12(\mathrm{tr}\,A_2^{\langle 2\rangle})A_1^{\langle 1\rangle 2}\\ &= \tfrac14 (A_2^2)^{\langle 4\rangle} + 48 e_z e_z [w_{,x}^{\langle 1\rangle 2} + w_{,y}^{\langle 1\rangle 2}]^2,\end{aligned}$$
$$A_n^{\langle 4\rangle}=0, \qquad n>4.$$

Appendix 3. Computation of the Motion at Third Order and the Correction of Height of Rise at Fourth Order for the Semi-Circular Trough

To complete the solution giving the third order correction for the axial motion, we need to solve (5.9):*

$$\nabla^2\phi = 0 \quad \text{in} \quad \mathscr{S}_0, \qquad \phi(d,\theta)=0, \tag{A.3.1}$$
$$\phi_{,\theta} = \tfrac32[C_6(\bar x\cosh\zeta\bar x - x\cosh\zeta - \zeta\bar x^2\sinh\zeta\bar x) - \tfrac34(\bar x^3+\bar x)] \quad \text{on} \quad y=0.$$

* The methods used in this section are taken from KUO & TANNER (1972).

The solution of this problem can be found by the method of separation of variables. We first change the variables

$$t=\frac{r}{d}, \qquad \tilde{\theta}=\theta+\tfrac{1}{2}\pi.$$

Then, substituting

$$\phi=R(t)\,\Theta(\tilde{\theta})$$

into $(A3.1_1)$, we choose the forms

$$\Theta(\theta)=\cosh\lambda\tilde{\theta},$$
$$R(t)=a_\lambda t^{i\lambda}+b_\lambda t^{-i\lambda}$$

where λ is a real number. The boundary condition $(A.3.1_2)$ requires that

$$a_\lambda+b_\lambda=0.$$

It follows that the general solution of $(A3.1_{1,2})$ is

$$\phi=\int_{-\infty}^{\infty} 2ia_\lambda\lambda\sin(\lambda\ln t)\cosh\lambda\tilde{\theta}\,d\lambda. \tag{A.3.2}$$

Using (A.3.2), we may express the free surface condition $(A3.1_3)$ as

$$\int_{-\infty}^{\infty} v_\lambda\sin\lambda u\,d\lambda=K(e^u), \tag{A3.3}$$

where $v_\lambda=2ia_\lambda\sinh(\tfrac{1}{2}\lambda\pi)$,

$u=\ln t$,

$K(e^u)=\tfrac{3}{2}[C_6(e^u\cosh\zeta e^u-e^u\cosh\zeta-e^{2u}\sinh\zeta e^u)-\tfrac{3}{4}(e^{3u}+e^u)]$.

Multiplying both sides of (A3.3) by $\frac{1}{\pi}\sin\lambda u$ and integrating with respect to u from $-\infty$ to 0 we find, using the Fourier integral formula*, that

$$\begin{aligned} v_\lambda&=\frac{1}{\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} v_s\cos u(s-\lambda)\,ds\,du\\ &=\frac{1}{\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} v_s\sin su\sin\lambda u\,ds\,du\\ &=\frac{1}{\pi}\int_{-\infty}^{0} K(e^u)\sin\lambda u\,du.\end{aligned}$$

The solution to (A3.1) is then given by

$$\phi=\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{0} K(e^u)\sin\lambda u\,du\right]\frac{\sin(\lambda\ln t)\cosh\lambda\tilde{\theta}}{\lambda\sinh(\tfrac{1}{2}\lambda\pi)}\,d\lambda. \tag{A3.4}$$

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* See I. S. SOKOLNIKOFF & R. M. REDHEFFER, *Mathematics of Physics and Modern Engineering*. New York: McGraw-Hill Book Company, Inc., 1958, p. 190.

We now expand $K(e^u)$ in a power series,

$$K(e^u) = \sum_{k=1}^{\infty} a_k e^{ku}, \tag{A3.5}$$

where

$$a_k = \begin{cases} 0 & \text{if } k \text{ is even}, \\ \frac{3}{2}[C_6(1-\cosh\zeta) - \frac{3}{4}] & \text{for } k=1, \\ \frac{3}{2}[-\frac{1}{2}C_6\zeta^2 - \frac{3}{4}] & \text{for } k=3, \\ \frac{3}{2}C_6\zeta^{k-1}\left[\frac{1}{(k-1)!} - \frac{1}{(k-2)!}\right] & \text{for } k=5,7,\ldots. \end{cases}$$

Then the first integral in (A3.4) is easily evaluated and gives

$$\sum_{k=1}^{\infty} a_k \int_{-\infty}^{0} e^{ku} \sin\lambda u \, du = -\sum_{k=1}^{\infty} \frac{a_k \lambda}{k^2+\lambda^2}.$$

We substitute this back into (A3.4),

$$\phi = \frac{1}{\pi} \sum_{k=1}^{\infty} a_k \int_{-\infty}^{\infty} \frac{-\sin(\lambda \ln t)\cosh\lambda\tilde{\theta}}{(k^2+\lambda^2)\sinh(\frac{1}{2}\lambda\pi)} d\lambda, \tag{A3.6}$$

and evaluate the remaining integral by the method of contour integration to get

$$\begin{aligned} \phi(r,\theta) = \frac{1}{\pi} \sum_{k=1}^{\infty} a_k \Bigg\{ & \frac{2}{k^2} - \frac{\pi}{k}\left(\frac{r}{d}\right)^k \frac{\cos[k(\theta+\frac{1}{2}\pi)]}{\sin(\frac{1}{2}k\pi)} \\ & + 4\sum_{n=1}^{\infty} (-1)^n \left(\frac{r}{d}\right)^{2n} \frac{\cos[2n(\theta+\frac{1}{2}\pi)]}{k^2-4n^2} \Bigg\}. \end{aligned} \tag{A3.7}$$

The third order correction to the axial velocity is now given by (5.8) and (A3.7).

We next give the fourth order correction to the height of rise. After much simplification (5.7) can be written

$$\begin{aligned} \sigma h^{\langle 4 \rangle}_{,xx} - \rho g h^{\langle 4 \rangle} = & \frac{\tilde{N}_2 V^2}{d^2} \left\{ -\frac{3}{2}\bar{x}^2 + \frac{5}{2} - 6C_6(\cosh\zeta\bar{x} - \cosh\zeta) \right\} \\ & + \frac{\tilde{N}_2^2 V^3}{\mu d^3} \left\{ \frac{8}{\pi} \sum_{k=1}^{\infty} a_k \left[\frac{1}{k^2} + 2\sum_{n=1}^{\infty} \frac{(n+1)\bar{x}^{2n}}{k^2-4n^2} \right] \right\} \\ & + \frac{\tilde{N}_2 \tilde{\beta} V^4}{\mu d^4} \left\{ \frac{15}{2}\bar{x}^4 - \frac{3}{2} \right\} - \frac{\tilde{\gamma} V^4}{d^4} \left\{ \frac{15}{2}\bar{x}^4 \right\} - C_7 \rho g. \end{aligned} \tag{A3.8}$$

The solution to (A3.8) has the form (4.7) where now

$$H_a = \frac{3}{2}\bar{x}^2 - \frac{5}{2} + \frac{3}{\zeta^2} - 3C_6\zeta\bar{x}\sinh\zeta\bar{x} - 6C_6\cosh\zeta + C_{7a} + C_{8a}\cosh\zeta\bar{x},$$

$$H_b = \frac{8}{\pi} \sum_{k=1}^{\infty} a_k \left\{ -\frac{1}{k^2} - 2\sum_{n=1}^{\infty} \frac{n+1}{k^2-4n^2} \left[\sum_{p=0}^{n} \frac{(2n)!}{(2n-2p)!} \frac{\bar{x}^{2(n-p)}}{\zeta^{2p}} \right] \right\} + C_{7b} + C_{8b}\cosh\zeta\bar{x},$$

(A.3.9a-c)

$$H_c = -\frac{15}{2}\left(\bar{x}^4 + \frac{12}{\zeta^2}\bar{x}^2 + \frac{24}{\zeta^4} \right) + C_{7c} + C_{8c}\cosh\zeta\bar{x},$$

and

$$C_7=\frac{\tilde{N}_2 V}{\mu}\,C_{7a}+\frac{(\tilde{N}_2 V)^2}{\mu^2 d}\,C_{7b}+\frac{\tilde{N}_2\tilde{\beta}V^3}{\mu^2 d^2}\,C_{7c}-\frac{\tilde{\gamma}V^3}{\mu d^2}(C_{7c}+\tfrac{3}{2}).$$

Finally, the constants $C_{7a}, C_{7b}, \ldots, C_{8c}$ are evaluated using the fixed contact line condition

$$h^{\langle 4\rangle}(\pm d)=0,$$

which gives

$$C_{7a}=1-\frac{3}{\zeta^2}+3C_6(\zeta\sinh\zeta+2\cosh\zeta)-C_{8a}\cosh\zeta,$$

$$C_{7b}=\frac{8}{\pi}\sum_{k=1}^{\infty}a_k\left\{\frac{1}{k^2}+2\sum_{n=1}^{\infty}\frac{n+1}{k^2-4n^2}\left[\sum_{p=0}^{n}\frac{(2n)!}{(2n-2p)!}\,\frac{1}{\zeta^{2p}}\right]\right\}-C_{8b}\cosh\zeta,$$

$$C_{7c}=\frac{15}{2}\left[1+\frac{12}{\zeta^2}+\frac{24}{\zeta^4}\right]-C_{8c}\cosh\zeta,$$

$$C_{8a}=-3C_6+\frac{\zeta}{\sinh\zeta-\zeta\cosh\zeta}\left[1-3C_6\zeta\sinh\zeta+\frac{\mu}{\tilde{N}_2 V}\,Q^{\langle 4\rangle}\right],$$

$$C_{8b}=\frac{\zeta}{\sinh\zeta-\zeta\cosh\zeta}\left\{\frac{16}{\pi}\sum_{k=1}^{\infty}a_k\sum_{n=1}^{\infty}\frac{n+1}{k^2-4n^2}\left[\sum_{p=0}^{n}\frac{(2n)!\,(2p-2n)}{(2n-2p+1)!\,\zeta^{2p}}\right]\right\},$$

$$C_{8c}=\frac{\zeta}{\sinh\zeta-\zeta\cosh\zeta}\left(-\frac{15}{2}\right)\left[\frac{4}{5}+\frac{8}{\zeta^2}\right].$$

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