Slow Motion and Viscometric Motion; Stability and Bifurcation of the Rest State of a Simple Fluid

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Contents

1. Introduction

This paper is divided into four loosely-connected parts whose common thread is the study of slow steady motion of a simple fluid. The motions to be considered are those which can be constructed as a perturbation series pivoted about a state of rest. The Rivlin-Ericksen fluids of successively higher orders appear sequentially in the construction. The perturbation solutions are motivated by the desire to create a practical theory of viscometry for the Rivlin-Ericksen constants.

The first part of the paper concerns the relation of the Rivlin-Ericksen constants to the viscometric constants which arise as coefficients in the power series expansion of the three viscometric functions. The viscometric constants can be expressed in terms of the Rivlin-Ericksen constants but at each order larger than two there are more Rivlin-Ericksen constants than viscometric constants. The construction of slow steady motions which are not viscometric will naturally depend on the Rivlin-Ericksen constants which are not viscometric. However, at the lowest orders in the perturbation series for slow flow only viscometric constants appear. More precisely, if the N^{th} partial sum of the series for the velocity field is itself a viscometric field, then, whether or not the $N+1^{\text{st}}$ partial sum be a viscometric field, all partial sums up to and including the $N+2^{\text{nd}}$ depend upon the constitutive relation of the fluid only through its viscometric constants. This result gives a definite answer to the question: to what extent can the viscometers which are based on viscometric flow theory be used, in principle, to do the viscometry job required for slow flows?

The second part of the paper is about the problem of stability and bifurcation of the rest state of a simple fluid. The deformation histories which are required to test the stability of the state of rest are different from those which must be used to construct the bifurcating solution; different constitutive relations are required to study stability, on the one hand, and bifurcation, on the other, in one and the same simple fluid. A criterion is derived for instability of the rest state in simple fluids which are assumed to satisfy a conditional stability theorem. This criterion follows when the stress response is linearized for small amplitude motions of arbitrary frequency. Small amplitude motions of arbitrary frequency do not lead to the Rivlin-Ericksen fluids. Though an explicit criterion for the instability of the Rivlin-Ericksen fluid of order n can be derived, its relevance to real simple fluids is very limited. Bifurcation of the rest state is altogether different. When the spectrum of the linearized stability operator for the rest state is real-valued and discrete the unstable rest state will give way to steady motions which can be constructed using bifurcation theory and the Rivlin-Ericksen fluids. The bifurcation of a simple fluid heated from below is an example. The heat transported across the fluid and the shape of the free surface on top of the fluid are computed observables with a potential for use as a heat transport viscometer for the Rivlin-Ericksen constants.

The third part of the paper is about the problem of the change in diameter of a horizontal capillary jet with gravity neglected. A unique motionless jet in the form of a straight round cylinder held together by surface tension is an exact solution of the jet problem. This motionless solution is the pivot for constructing a formal perturbation solution in powers of a speed parameter. The "stick-slip" solution of RICHARDSON (1969) appears at the first order in the perturbation analysis. This solution has a lip singularity in the stresses and pressures. A momentum analysis of the jet leads to a relation giving the ratio of the final to initial diameter of the jet in terms of the momentum deficit, the thrust due to surface tension and the skin friction in the exit region of the capillary tube, and it leads to the limiting form of this relation for small Reynolds numbers. With the help of curve fitting from experiments, I show that the final diameter of a Newtonian jet decreases as const.$/R^{\frac{2}{3}}$ when the Reynolds number R is large.

The fourth part of the paper is about the problem of flow of a simple fluid confined between aligned horizontal disks which rotate at different rates around a common axis. The fluid is held in the space between the disks by a vertical surface film which is attached to the edge of the disks. The flow and the shape of the free surface is constructed as a perturbation series in the difference 2ω of the two disks. The solution is carried through second order by reducing the second-order perturbation problem to a type which is related to the semi-infinite strip problem of the classical theory of elasticity. This problem concerns an edge that goes all the way around, but it is possible to solve it by a rapidly convergent "Fourier series" of Papkovich-Fadle eigenfunctions. The solution, described in relatively elementary terms, has a complicated structure exhibiting motion in cells; these cells are required to turn the flow around at the edge and they decay rapidly with distance from the edge. The motion is driven by torques generated by vertical gradients of the centripetal acceleration associated with the basic shearing motion.

It is of interest that the basic theory for the torsion flow viscometer starts by neglecting inertia and the effects of finite edges. This approximation, which makes torsion flow a viscometric flow, was introduced in the papers of RIVLIN (1948), GREENSMITH & RIVLIN (1953) and RIVLIN (1956). Viscometric torsion flow has been studied in some detail by COLEMAN, MARKOVITZ & NOLL (1966), J. R. A. PEARSON (1966), PIPKIN & TANNER (1972), TRUESDELL & NOLL (1965) and by many other authors.

The neglect of inertia is justifiable in several different limits. In my analysis the secondary motion appears first at order ω^2 and the deviation of the shearing flow from a viscometric flow appears first at order ω^3. It follows that in the limit $\omega \to 0$ the extra effects, beyond viscometric flow, vanish. For this reason it is reasonable to neglect inertia, when ω is small, in the applications of torsion flow involving computations of the torque; for example, in the applications leading to the determination of the shear viscosity function. However, the neglect of inertia cannot be justified for the computation of the normal stress since these, like inertia, are proportional, at lowest order, to ω^2. Indeed this analysis shows that the full contribution of the radial variation of the normal stresses away from the edges in a disk viscometer with a stationary bottom plate is

$$\omega^2\left(\frac{8\alpha_1}{d^2}+\frac{6\alpha_2}{d^2}-\frac{3\rho}{5}\right)r^2 \qquad (**)$$

where r is the radius, d the distance between plates and α_1 and α_2 are Rivlin-Ericksen constants. No matter how small ω might be, (**) will not reduce to the expression

$$\omega^2\left(\frac{8\alpha_1}{d^2}+\frac{6\alpha_2}{d^2}\right),$$

which arises from viscometric analysis. Equation (**) shows that when $4\alpha_1+3\alpha_2>0$ there is a critical separation distance*

$$d=d_c=[(\tfrac{40}{3}\alpha_1+10\alpha_2)/\rho]^{\frac{1}{2}}$$

* In STP at about room temperature $d_c \simeq 4.5$ cm.

which depends on material parameters alone. When $d \ll d_c$ and ω is small, inertia is a small effect which may be neglected and the normal pressures which give the normal stress with reversed sign are largest at the center of the disk and decrease with radius. When $d > d_c$ the normal pressures are smallest at the center and increase with radius.

Analysis shows that when ω is small the free surface and the shape of the streamlines are the same for all fluids having the same surface tension and density. The speed of the secondary flow, however, is proportional to the fluidity (the reciprocal of the viscosity). I believe that this dependence of speed on fluidity ultimately explains why it is so much easier to hold very viscous fluids between rotating plates; the forces which are associated with high speed motions of relatively inviscid fluids rupture the free surface.

Part I. Slow Motions and Viscometry

2. General and Restricted Problems of Viscometry

TRUESDELL & NOLL [1965] define an incompressible simple fluid through the constitutive equation for the extra stress

$$S = \mathscr{F}\left[\overset{\infty}{\underset{s=0}{G(s)}}\right], \qquad G(s) = C_t(s) - 1, \qquad \operatorname{trace} S = 0, \tag{2.1}$$

C_t being the relative right Cauchy-Green strain tensor

$$C_t = F_t^T F_t, \qquad F_t = \operatorname{grad} \chi_t, \qquad G(0) = 0 \tag{2.2}$$

where $\zeta = \chi_t(x, \tau)$ is the point occupied at time $\tau = t - s$ by the particle which at time t is at point x. $G(s)$ is called the history and $\mathscr{F}$ is a hereditary stress response operator. The general problem of viscometry of a simple fluid is to ascertain the form of the operator $\mathscr{F}$. The solution of the general problem of viscometry is known for the important special case of the Newtonian fluid; in this special case the operator $\mathscr{F} = \mu(\nabla u + \nabla u^T) \equiv \mu A_1$, where $u(x, t)$ is the velocity at present time, is specified when the viscosity μ is given. The form of $\mathscr{F}$ is not known, even in special cases, for real non-Newtonian fluids.

To circumvent the difficulty of doing practical mechanics with a general but unknown $\mathscr{F}$, it is useful to define restricted problems of viscometry. These take form by first specifying classes of motion or histories on which $\mathscr{F}$ reduces to something more manageable and by then defining viscometry relative to the more manageable $\mathscr{F}$. For example:

(1) Three viscometric functions are necessary to characterize $\mathscr{F}$ for all viscometric motions of a simple fluid. The restricted problem of viscometry for viscometric flows is to find the three functions.

(2) One function, the shear relaxation modulus, is necessary to characterize $\mathscr{F}$ for all sufficiently small-amplitude motions of a simple fluid (COLEMAN & NOLL, 1961). The restricted problem of viscometry for motions of arbitrary frequency but sufficiently small amplitude is to find the shear relaxation modulus.

(3) An infinite set of constants are required to characterize $\mathscr{F}$ on slow motions which may be obtained by perturbation of a state of rest. The restricted problem of viscometry for these slow motions is to find the constants.

The nature of the special class of slow motions being considered in (3) places more emphasis on the small finite number of constants which appear at the lowest orders in the perturbation expansion. From a practical point of view the determination of, say, the fourteen constants mentioned under the expression (3.2) below would suffice for an approximate description of many slow flows.

3. Slow Motions and Rivlin-Ericksen Constants

The word "motion" here has a strictly kinematic sense and is not necessarily a solution of the equations of motion. Thus, a slow steady motion is a bounded solenoidal field

$$U(\boldsymbol{x}, \varepsilon) = \varepsilon \boldsymbol{u}(\boldsymbol{x}, \varepsilon), \qquad \boldsymbol{u} \in C^\infty(\bar{\mathscr{V}}), \qquad |\varepsilon| \leqq \varepsilon_0 > 0$$

where $\bar{\mathscr{V}}$ is the closure of the domain in R^3 on which $\boldsymbol{u}(\boldsymbol{x}, \varepsilon)$ is defined.

Retarded motions are not so easily defined. If $\boldsymbol{G}(s)$ is analytic, it may be expanded into a power series whose coefficients are the Rivlin-Ericksen tensors $\boldsymbol{A}_n$,

$$\boldsymbol{G}(s) = \sum_{n=1}^{\infty} \frac{s^n}{n!} \partial^n \boldsymbol{G}(s)/\partial s^n|_{s=0} = \sum_{n=1}^{\infty} (-s)^n \boldsymbol{A}_n/n!,$$

where

$$\boldsymbol{A}_1 = \nabla \boldsymbol{U} + \nabla \boldsymbol{U}^T$$

and

$$\boldsymbol{A}_{n+1} = \frac{\partial \boldsymbol{A}_n}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{A}_n + \boldsymbol{A}_n \nabla \boldsymbol{U} + \nabla \boldsymbol{U}^T \boldsymbol{A}_n, \qquad (\boldsymbol{A}_n \nabla \boldsymbol{U})_{ij} = (\boldsymbol{A}_n)_{il} \partial_j U_l.$$

If $\mathscr{F}$ has Fréchet derivatives of all orders at the point $\boldsymbol{G}(s) = \boldsymbol{0}$, then one may formally expand $\boldsymbol{S}$ into a series

$$\boldsymbol{S} = \sum_{n=1}^{\infty} \boldsymbol{S}_n[\boldsymbol{A}_n, \boldsymbol{A}_{n-1}, \dots \boldsymbol{A}_1]. \tag{3.1}$$

The tensor-valued functions $\boldsymbol{S}_n$ of $\boldsymbol{A}_n$ can be written out explicitly [for example, see TRUESDELL & NOLL, p. 494 (1965) or TRUESDELL, p. 132 (1974)]. The first four of the $\boldsymbol{S}_n$ are

$$\boldsymbol{S}_1[\boldsymbol{A}_1] = \mu \boldsymbol{A}_1, \tag{3.2a}$$

$$\boldsymbol{S}_2[\boldsymbol{A}_1, \boldsymbol{A}_2] = \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2, \tag{3.2b}$$

$$\boldsymbol{S}_3[\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3] = \beta_1 \boldsymbol{A}_3 + \beta_2(\boldsymbol{A}_2 \boldsymbol{A}_1 + \boldsymbol{A}_1 \boldsymbol{A}_2) + \beta_3(\operatorname{tr} \boldsymbol{A}_2) \boldsymbol{A}_1, \tag{3.2c}$$

$$\begin{aligned} \boldsymbol{S}_4[\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_4] = {} & \gamma_1 \boldsymbol{A}_4 + \gamma_2(\boldsymbol{A}_3 \boldsymbol{A}_1 + \boldsymbol{A}_1 \boldsymbol{A}_3) + \gamma_3 \boldsymbol{A}_2^2 \\ & + \gamma_4(\boldsymbol{A}_2 \boldsymbol{A}_1^2 + \boldsymbol{A}_1^2 \boldsymbol{A}_2) + \gamma_5(\operatorname{tr} \boldsymbol{A}_2) \boldsymbol{A}_2 + \gamma_6(\operatorname{tr} \boldsymbol{A}_2) \boldsymbol{A}_1^2 \\ & + [\gamma_7 \operatorname{tr} \boldsymbol{A}_3 + \gamma_8(\operatorname{tr} \boldsymbol{A}_1 \boldsymbol{A}_2)] \boldsymbol{A}_1. \end{aligned} \tag{3.2d}$$

The Rivlin-Ericksen coefficients $\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \dots \gamma_8$ are constants, or more generally, functions of the temperature. We define the

$$\text{set}\{c_n\} \text{ of constants for } S_n. \tag{3.3}$$

For example, $\gamma_1, \gamma_2, \dots, \gamma_8$ are the elements of the set $\{c_4\}$. The partial sums

$$S_{(N)} = \sum_{n=1}^{N} S_n[A_n, A_{n-1} \dots A_1] \tag{3.4}$$

are called stress tensors for Rivlin-Ericksen fluids of grade N.

It is thought that fluids of grade N are valid approximations to S when the motion is slow.* COLEMAN & NOLL have shown that fluids of grade N are valid approximations to S when $\mathscr{F}$ is an operator with fading memory and provided that the history $H(s)$ of some motion is the retardation of some given history $G(s)$ such that

$$H(s) = G(\alpha s) = \sum_{n=1}^{\infty} (-\alpha s)^n A_n/n!$$

where α is the retardation factor and is to be regarded as a small parameter. COLEMAN & NOLL show that asymptotically

$$\left| \mathscr{F}\left[\overset{\infty}{\underset{s=0}{H(s)}}\right] - S_{(N)} \right| = O(\alpha^{N+1}).$$

COLEMAN & NOLL's theorem gives one sense in which the retardation approximation is valid; this is in the sense of an expansion in slow time ($s' = \alpha s$) of the history $G(s')$ of the motion whose velocity at present time is $U(x, t)$. Slow time expansions and slow motion expansions are identical when the motion is steady.

JOSEPH & FOSDICK (1973) have shown that for slow steady motions one can define a retarded history

$$H_\varepsilon = H(\varepsilon s; \varepsilon) = G(s; \varepsilon).$$

Here $H_\varepsilon = G$ is computed from $F_t = \text{grad}\,\Psi$ where

$$\xi(t-s) = \Psi(x, -\varepsilon s; \varepsilon), \qquad \xi = x|_{s=0} \tag{3.5a}$$

and

$$\Psi'[x, -\varepsilon s; \varepsilon] = u[\xi(t-s); \varepsilon]$$

where the prime denotes differentiation with respect to the argument in the second place. Note that $H_\varepsilon = G$ is the retardation of the history $H(s; \varepsilon)$ which is computed from the field $\Psi(x, -s; \varepsilon)$. The slow motion $U = \varepsilon u$ may be called the retardation of the motion $u(x; \varepsilon)$. Time retardation is implied by the retardation of the motion because the equation for the particle paths of the slow motion

$$\frac{d\xi}{d\tau} = U(\xi; \varepsilon) = \varepsilon u(\xi; \varepsilon) \tag{3.5b}$$

* Formal series expansions for the extra stress were given first by GREEN & RIVLIN (1957). Necessary conditions for the convergence of these expansions are still unknown. The Coleman-Noll retardation theorem (1960) gives sufficient conditions for convergence.

may be obtained as particle paths for the retarded motion $\boldsymbol{u}(\zeta; \varepsilon)$

$$\frac{d\xi}{d\tau'} = \boldsymbol{u}(\xi; \varepsilon), \qquad \tau' = \varepsilon\tau \tag{3.5c}$$

in slow time τ'. Integration of (3.5b) from (ξ, τ) to $(\boldsymbol{x}, t)$ shows that ξ depends on t only through $\boldsymbol{x}$ and s, as in (3.5a).

For slow steady motions $\partial A_n/\partial t = 0$; then A_n *and* S_n *are homogeneous tensor polynominals of degree* n *in* U. Thus, for slow motions, using (3.1), we have $A_n = \varepsilon^n \boldsymbol{a}_n$ and

$$S = \sum_{n=1}^{\infty} \varepsilon^n \bar{S}_n[\underbrace{\boldsymbol{u}, \ldots, \boldsymbol{u}}_{n \text{ times}}] \tag{3.6a}$$

where

$$\bar{S}_n[\underbrace{\boldsymbol{u}, \ldots, \boldsymbol{u}}_{n \text{ times}}] = S_n[\boldsymbol{a}_n, \boldsymbol{a}_{n-1}, \ldots, \boldsymbol{a}_1].$$

Suppose now that the slow steady motion is analytic in ε and that

$$\boldsymbol{u}(\boldsymbol{x}, \varepsilon) = \sum_{l=1}^{\infty} \varepsilon^{l-1} \boldsymbol{u}^{\langle l \rangle}.$$

Then

$$\bar{S}_n[\boldsymbol{u}, \ldots, \boldsymbol{u}] = \bar{S}_n\left[\sum_{r_1=1}^{\infty} \varepsilon^{r_1-1} \boldsymbol{u}^{\langle r_1 \rangle}, \ldots, \sum_{r_n=1}^{\infty} \varepsilon^{r_n-1} \boldsymbol{u}^{\langle r_n \rangle}\right] = \sum_{l=1}^{\infty} \varepsilon^{l-1} S_n^{\langle l \rangle} \tag{3.6b}$$

where

$$S_n^{\langle l \rangle} = \sum^{(l)} \bar{S}_n[\boldsymbol{u}^{\langle r_1 \rangle}, \boldsymbol{u}^{\langle r_2 \rangle}, \ldots, \boldsymbol{u}^{\langle r_n \rangle}] \tag{3.6c}$$

and $\sum^{(l)}$ is a summation for a fixed integer l over all sets of integers $r_i \geqq 1$ such that

$$l = \sum_{i=1}^{n} r_i + 1 - n.$$

For example,

$$S_2^{\langle l \rangle} = \sum^{(l)} \bar{S}_2[\boldsymbol{u}^{\langle r_1 \rangle}, \boldsymbol{u}^{\langle r_2 \rangle}]$$

where $\sum^{(l)}$ is a summation over all sets of integers $r_1 \geqq 1$, $r_2 \geqq 1$ such that $r_1 + r_2 = l + 1$.

It follows from (3.6) that

$$S = \sum_{n=1}^{\infty} \varepsilon^n S^{\langle n \rangle}, \tag{3.7a}$$

where the $S^{\langle n \rangle}$ are partial derivatives

$$S^{\langle n \rangle} = \sum_{q+l=1+n} S_q^{\langle l \rangle} = \frac{1}{n!} \left.\frac{\partial^n S}{\partial \varepsilon^n}\right|_{\varepsilon=0} \tag{3.7b}$$

evaluated on the slow, steady, analytic (in ε) motion (3.1).

4. Viscometric Motions, Viscometric Constants

There are several equivalent definitions of viscometric motions. Following Pipkin (1967) and Yin & Pipkin (1970), we say that $\boldsymbol{u}(\boldsymbol{x}, t)$ is a viscometric flow if, apart from rigid body motions, each material element is undergoing some simple shearing motion with a *constant* rate of shear κ. We are interested in steady flows which are viscometric throughout the region $\mathscr{V}$ of flow. When evaluated on a viscometric motion (we recall that viscometric motion is defined kinematically and need not satisfy equations of motion) the extra stress $\mathscr{F}$ is completely specified in terms of three scalar functions of the rate of shearing: $\tau(\kappa)/\kappa$, $\sigma_1(\kappa^2)$ and $\sigma_2(\kappa^2)$. Viscometric constants are defined by derivatives of the viscometric functions evaluated at $\kappa=0$. Since $\mathscr{F}$ depends exclusively on the three viscometric functions, the slow motion expansion (3.1) of $\mathscr{F}$ evaluated on a viscometric motion must necessarily be expressible exclusively in terms of those Rivlin-Ericksen constants which are also viscometric constants.

Assuming that the viscometric functions can be expanded as a power series, we may define the viscometric constants as the Taylor coefficients in the series pivoted around $\kappa=0$. *The members of the set* $\{d_n\}$ *are the viscometric constants which appear as the coefficients of* κ^n *in the Taylor series for* $\tau(\kappa)$, $\sigma_1(\kappa^2)$ *and* $\sigma_2(\kappa^2)$. To compare the set $\{d_n\}$ of viscometric constants with the set $\{c_n\}$ of Rivlin-Ericksen constants, we evaluate the tensors S_n on a steady viscometric motion and find (TRUESDELL & NOLL, 1965, p. 495):

$$\tau=\mu\kappa+2(\beta_2+\beta_3)\kappa^3+O(\kappa^5), \tag{4.1a}$$

$$\sigma_1=(2\alpha_1+\alpha_2)\kappa^2+[4(\gamma_3+\gamma_4+\gamma_5)+2\gamma_6]\kappa^4+O(\kappa^6), \tag{4.1b}$$

$$\sigma_2=\alpha_2\kappa^2+2\gamma_6\kappa^4+O(\kappa^6). \tag{4.1c}$$

Viscometric constants are fewer in number than Rivlin-Ericksen constants. For example, β_1 does not appear in (4.1) but does appear in (3.2c); β_2 and β_3 appear independently in (3.2c) but they appear only in the combination $\beta_2+\beta_3$ in (4.1a).

5. Perturbations of the Rest State

Until now we have been considering motions $\boldsymbol{u}$ which are defined kinematically without reference to the equations of motion. Now we shall consider the possibility of constructing solutions to the equations governing steady motion of a simple fluid as perturbation series pivoted about the rest state. By the "rest state" we shall understand a steady motionless solution $U(x)=0$ of the equations of motion, boundary conditions and auxilliary conditions.

When evaluated on a motionless solution, $S=0$. If we suppose that there is a family of steady solutions $U(\boldsymbol{x}, \varepsilon)$ of the governing equations of motion which is analytic in ε and such that $U(\boldsymbol{x}, 0)=0$, it is natural to seek this solution as a power series

$$U(\boldsymbol{x}, \varepsilon)=\sum_{n=1}^{\infty} \boldsymbol{u}^{\langle n\rangle}(\boldsymbol{x}, \{c_\nu\})\varepsilon^n, \quad 1\leqq\nu\leqq n \tag{5.1}$$

whose coefficients depend on the Rivlin-Ericksen constants $\{c_\nu\}$.

We are going to study how the partial sums

$$U_{(N)}(x, \varepsilon, \{c_l\})=\sum_{n=1}^{N} \boldsymbol{u}^{\langle n\rangle}(\boldsymbol{x}, \{c_\nu\})\varepsilon^n, \quad 1\leqq\nu\leqq n,\ 1\leqq l\leqq N \tag{5.2}$$

depend on the constants $\{c_l\}$. To carry out this study, we must first state how the $\{c_\nu\}$ enter into the expressions for the $\boldsymbol{u}^{\langle n\rangle}$. We shall assume that the perturbation field at order n depends on lower-order fields through the hypothesis (5.3) below:

$$\boldsymbol{u}^{\langle n\rangle}(\boldsymbol{x},\{c_\nu\})=\boldsymbol{f}_n(\boldsymbol{u}^{\langle l\rangle},\boldsymbol{S}_q^{\langle l\rangle},\mu),\qquad 1\leqq\nu\leqq n,\ 1\leqq l<n,\ q+l=n+1. \tag{5.3}$$

Hypothesis (5.3) implies that

$$\begin{aligned}&\boldsymbol{u}^{\langle 1\rangle}=\boldsymbol{f}_1(0,0,\mu),\\&\boldsymbol{u}^{\langle 2\rangle}=\boldsymbol{f}_2(\boldsymbol{u}^{\langle 1\rangle},\boldsymbol{S}_2^{\langle 1\rangle},\mu),\\&\boldsymbol{u}^{\langle 3\rangle}=\boldsymbol{f}_3(\boldsymbol{u}^{\langle 1\rangle},\boldsymbol{u}^{\langle 2\rangle},\boldsymbol{S}_2^{\langle 2\rangle},\boldsymbol{S}_3^{\langle 1\rangle},\mu),\qquad \text{etc.}\end{aligned}$$

The hypothesis (5.3) states that apart from μ, the Rivlin-Ericksen constants ultimately enter into $\boldsymbol{u}^{\langle n\rangle}$ through the tensors $\boldsymbol{S}_q^{\langle l\rangle}$.

It is merely convenient to state (5.3) as a hypothesis; (5.3) holds for all of the solutions known to me which can be constructed as a perturbation of the rest state. For example, consider the forced motion of a simple fluid in a bounded domain $\mathscr{V}$:

$$\rho\,\boldsymbol{U}\cdot\nabla\boldsymbol{U}+\nabla P-\nabla\cdot\boldsymbol{S}=\boldsymbol{f}(\boldsymbol{x},\varepsilon),\qquad \operatorname{div}\boldsymbol{U}=0\ \text{ in }\mathscr{V} \tag{5.4a, b}$$

and

$$\boldsymbol{U}=\varepsilon\boldsymbol{q}(\boldsymbol{x};\varepsilon)\qquad\text{on }\partial\mathscr{V} \tag{5.4c}$$

where $\boldsymbol{f}$ and $\boldsymbol{q}$ are prescribed vector fields which are analytic in ε and $\boldsymbol{f}(\boldsymbol{x},0)$ is expressible as a gradient. The rest solution of (5.4) is associated with the values

$$\varepsilon=0{:}\ \boldsymbol{U}=\boldsymbol{U}^{\langle 0\rangle}=0,\qquad \boldsymbol{S}^{\langle 0\rangle}=0,\qquad \nabla p^{\langle 0\rangle}=\boldsymbol{f}(\boldsymbol{x},0). \tag{5.5}$$

Assuming now series solutions in the form (5.1) with a similar series for $p(\boldsymbol{x},\{c_\nu\})-p^{\langle 0\rangle}$ and expressing $\boldsymbol{S}$ as in (3.7a, b), we find the perturbation problems:

$$\begin{aligned}\rho\sum_{m+l=n}&\boldsymbol{u}^{\langle m\rangle}\cdot\nabla\boldsymbol{u}^{\langle l\rangle}+\nabla p^{\langle n\rangle}-\mu\Delta\boldsymbol{u}^{\langle n\rangle}\\&-\nabla\cdot[\boldsymbol{S}_2^{\langle n-1\rangle}+\boldsymbol{S}_3^{\langle n-2\rangle}+\cdots+\boldsymbol{S}_{n-1}^{\langle 2\rangle}+\boldsymbol{S}_n^{\langle 1\rangle}]=\boldsymbol{f}^{\langle n\rangle}\end{aligned} \tag{5.6a}$$

and

$$\operatorname{div}\boldsymbol{u}^{\langle n\rangle}=0\qquad\text{in }\mathscr{V} \tag{5.6b}$$

whereas

$$\boldsymbol{u}^{\langle n\rangle}=q^{\langle n-1\rangle}(x)\qquad\text{on }\partial\mathscr{V} \tag{5.6c}$$

for all $n\geqq 1$.* It is apparent that (5.3) holds for solutions of (5.6) .

6. Viscometric Constants and Rivlin-Ericksen Constants

TRUESDELL [1974] has noted that there are cases in which knowledge of the viscometric constants up to and through order N completely characterizes the N^{th} partial sum (5.2). In this case the Rivlin-Ericksen constants $\{c_l\}$, $1\leqq l\leqq N$ may be replaced with the same number (when $N\leqq 2$) or with a smaller number (when $N>2$) of viscometric constants $\{d_l\}$, $1\leqq l\leqq N$. The interesting aspect of

* The summation in the term on the left of (5.6a) is to be carried out to $n-1$ terms.

this observation is that the field

$$U_{(N)}(x, \varepsilon, \{d_l\}),$$

which depends exclusively on the viscometric constants, need not itself be a viscometric motion in the kinematic sense. There are, therefore, cases in which non-viscometric approximations $\boldsymbol{U}_{(N)}$ to non-viscometric and dynamically admissible* slow flows are completely characterized by the viscometric constants. In these situations complete knowledge of the three viscometric functions for small shearing is sufficient to determine non-viscometric approximations to non-viscometric slow flows.

TRUESDELL's observation may be more precisely stated as follows: If a given motion is viscometric up to and through order N, then $\boldsymbol{U}_{(l)}$ depends exclusively on viscometric constants when $l \leqq N+2$.

Theorem. *Let* (5.3) *hold. Suppose that* $\boldsymbol{u}^{\langle 1\rangle}, \ldots, \boldsymbol{u}^{\langle N\rangle}$ *and* $\boldsymbol{U}_{(N)}$ *are viscometric motions in the kinematic sense. Then*

$$U_{(N+1)} = U_{(N+1)}(\boldsymbol{x}, \varepsilon, \{d_l\}), \qquad 1 \leqq l \leqq N+1$$

and

$$U_{(N+2)} = U_{(N+2)}(\boldsymbol{x}, \varepsilon, \{d_l\}), \qquad 1 \leqq l \leqq N+2$$

depend exclusively on the viscometric constants $\{d_l\}$.

To prove the theorem we need first to specify the way in which viscometric constants enter into the perturbation fields.

Lemma 1. *Suppose that* $\boldsymbol{u}^{\langle 1\rangle}, \ldots, \boldsymbol{u}^{\langle l\rangle}$ *depend only on those Rivlin-Ericksen constants which are also viscometric constants. Then* $\boldsymbol{S}_1^{\langle l\rangle}$ *and* $\boldsymbol{S}_2^{\langle l\rangle}$ *depend only on the viscometric constants.* Using (3.7) and (3.2a, b) we note that $\boldsymbol{S}_1^{\langle l\rangle}$ and $\boldsymbol{S}_2^{\langle l\rangle}$ depend explicitly only on the viscometric constants μ, α_1 and α_2 and implicitly on the parameters on which $\boldsymbol{u}^{\langle 1\rangle}, \ldots, \boldsymbol{u}^{\langle l\rangle}$ depend. The hypothesis of Lemma 1 excludes this implicit dependence on non-viscometric constants.

Lemma 2. *If* $\bar{\boldsymbol{S}}_n[\boldsymbol{v}, \ldots, \boldsymbol{v}]$ *is evaluated on a viscometric motion* $\boldsymbol{v}$, *then* $\bar{\boldsymbol{S}}_n$ *depends only on the viscometric constants.* This follows from the fact that $\bar{\boldsymbol{S}}^n$ is one realization of the stress response operator for a simple fluid. Such operators cannot involve non-viscometric parameters when evaluated on viscometric flows.

Lemma 3. *If* $\boldsymbol{u}^{\langle 1\rangle}, \ldots, \boldsymbol{u}^{\langle N\rangle}$ *and* $\boldsymbol{U}_{(N)} = \sum_{l=1}^{N} \boldsymbol{u}^{\langle l\rangle}$ *are viscometric motions in the kinematic sense, then, for all* $m \geqq 1$, $\boldsymbol{S}_m^{\langle 1\rangle}, \ldots, \boldsymbol{S}_m^{\langle N\rangle}$ *depend on the viscometric constants only.* To prove Lemma 3 we note, using (3.6) and (3.7), that

$$\bar{\boldsymbol{S}}_m[\underbrace{\boldsymbol{U}_{(N)}, \ldots, \boldsymbol{U}_{(N)}}_{m\,\text{times}}] = \sum_{l=1}^{N} \varepsilon^{l-1} \boldsymbol{S}_m^{\langle l\rangle} + O(\varepsilon^{N+1}). \tag{6.1}$$

Since, by hypothesis, $\boldsymbol{U}_{(N)}$ is a viscometric motion, Lemma 2 applies to the left side of (6.1) and implies that the left side of (6.1) depends only on viscometric constants. Differentiating (6.1) repeatedly with respect to ε at $\varepsilon=0$, we demonstrate that the $\boldsymbol{S}_m^{\langle l\rangle}$ depend only on the viscometric constants when $1 \leqq l \leqq N$.

* Dynamically admissible flows satisfy the equations of motion.

With these preliminary lemmas aside, we may now turn to the proof of the theorem. If $\boldsymbol{u}^{\langle 1\rangle}, \ldots, \boldsymbol{u}^{\langle N\rangle}$ are viscometric then, by Lemma 3 and hypothesis (5.3) (with $l=N+1$), we have that

$$\boldsymbol{u}^{\langle N+1\rangle} \textit{ depends exclusively on viscometric constants.} \tag{6.2}$$

Applying (5.3) again, we find that

$$\boldsymbol{u}^{\langle N+2\rangle}=f_{N+2}(\boldsymbol{u}^{\langle l\rangle}, S_q^{\langle l\rangle}, \mu)$$

where

$$1 \leqq l < N+2, \qquad q+l=N+3.$$

When $l=N+1$, $q=2$. Then, by Lemma 1 and (6.2), $S_2^{\langle N+1\rangle}$ depends exclusively on the viscometric constants. Using Lemma 3 once again, we find that

$$\boldsymbol{u}^{\langle N+2\rangle} \textit{ depends exclusively on the viscometric constants.} \tag{6.3}$$

It follows from the hypothesis of the theorem and from (6.2) and (6.3) that the partial sums $U_{(l)}$ through order $l=N+2$ depend only on those Rivlin-Ericksen constants which are also viscometric constants.

7. Applications and Remarks about Viscometry

Consider perturbation solutions pivoted about the rest state and such that $\boldsymbol{u}^{\langle 1\rangle}$ is not a viscometric motion. Such solutions can be associated with the bifurcation problem for a simple fluid heated from below and with the die swell problem treated in Part III of this paper. For these problems the theorem proved in Section 6 guarantees that $\boldsymbol{u}^{\langle 1\rangle}$ and $\boldsymbol{u}^{\langle 2\rangle}$ depend only on the viscometric constants. $U_{(3)}$ and $\boldsymbol{u}^{\langle 3\rangle}$ could depend on non-viscometric constants.

In the problem of a steadily rotating sphere which was treated first by GIESEKUS (1963) and, more completely, by FOSDICK & KAO, (1973), $\boldsymbol{u}^{\langle 1\rangle}$ is a viscometric motion but $\boldsymbol{u}^{\langle 2\rangle}$ is not viscometric. The same fact, $\boldsymbol{u}^{\langle 1\rangle}$ is a viscometric flow and $\boldsymbol{u}^{\langle 2\rangle}$ is not, holds for the flow in a torsion viscometer with a free surface on the circumferential edge, which is studied in Part IV of this paper. For these problems the theorem proved in Section 6 guarantees that $\boldsymbol{u}^{\langle 2\rangle}$ and $\boldsymbol{u}^{\langle 3\rangle}$ depend exclusively on the viscometric constants. The velocity field $\boldsymbol{u}^{\langle 4\rangle}$ may depend on non-viscometric constants; in the flow studied by FOSDICK & KAO $\boldsymbol{u}^{\langle 4\rangle}$ does depend on a non-viscometric constant of the third order fluid.

In the problem of rod climbing considered by JOSEPH & FOSDICK (1973) $u^{\langle 1\rangle}$ is viscometric (Couette flow), $\boldsymbol{u}^{\langle 2\rangle}=0$, $\boldsymbol{u}^{\langle 3\rangle}$ is again viscometric (a vertically stratified Couette flow) but $\boldsymbol{u}^{\langle 4\rangle}$ is not a viscometric flow. The theorem and explicit computation of JOSEPH & FOSDICK shows that $u^{\langle 4\rangle}$ depends exclusively on the viscometric constants. The theorem also asserts that $U^{\langle 5\rangle}$, which has not yet been studied, necessarily depends exclusively on the viscometric constants.

In the swirling flow in pipes studied by LANGLOIS & RIVLIN (1963), $\boldsymbol{u}^{\langle 1\rangle}$ is a rectilinear flow of Newtonian fluid through the pipe, $\boldsymbol{u}^{\langle 2\rangle}=0$, $u^{\langle 3\rangle}$ is another rectilinear flow. These flows and their sum

$$u^{\langle 1\rangle}\varepsilon+u^{\langle 2\rangle}\varepsilon^2+u^{\langle 3\rangle}\varepsilon^3=U_{(3)}$$

are viscometric. Hence, by the theorem of Section 6,

$U_{(4)}$ and $U_{(5)}$ depend exclusively on viscometric constants.

The exclusive dependence of $U_{(4)}$ on $\{d_l\}$, $l \leqq 4$, was known by direct computations but the assertion about $U_{(5)}$ is made first here.

Nearly all of the existing viscometers are based on the theory of viscometric flows. These viscometers, at best, can give the three viscometric functions and, in principle, determine the set $\{d_l\}$ of viscometric constants. Speaking practically, the viscometers based on viscometric flow theory are not good instruments for determining the viscometric constants. These viscometers are not accurate at low rates of shearing, and it is not possible to find the values of derivatives of the three functions at $\kappa=0$ by backward extrapolation.

The viscometry of slow steady motion has no intrinsic relation to the viscometry of viscometric flow. For slow steady flows, it is necessary to determine the Rivlin-Ericksen constants and not just the viscometric constants. This determination may perhaps be made using various free surface viscometers (see WINEMAN & PIPKIN (1966), TANNER (1970), JOSEPH, BEAVERS & FOSDICK (1973) and BEAVERS & JOSEPH (1974) and Parts II and IV of this paper) which use slow motion theory instead of viscometric flow theory.

Part II. Stability and Bifurcation

8. Stability of the Rest State

The study of flow of viscoelastic fluids, and the study of the stability of flow, is made difficult by the complexity of the response. Apart from the tremendously important special case of a Newtonian fluid, there is no real viscous fluid for which the form of the response operator $\mathscr{F}$ is known for all possible motions. To study stability, it is useful to restrict consideration to histories about which something more definite than (2.1) and (2.2) can be said about $\mathscr{F}$. Identification of classes of history which are appropriate for stability studies is a still open and fundamental question in the theory of the viscoelastic fluids. Some progress in the stability theory can be made for the simplest situation – the study of the stability of the rest state. This problem, which is trivial for Newtonian fluids, has many interesting and complex features and may even be of some value in separating the physically realizable simple fluids from the others.

To study the stability of the rest state, we consider the initial value problem for a disturbance of the rest state in a closed container $\mathscr{V}$; thus,

$$\rho\left[\frac{\partial U}{\partial t}+U\cdot\nabla U\right]+\nabla P=\nabla\cdot S, \qquad \nabla\cdot U=0, \qquad U|_{\partial\mathscr{V}}=0 \tag{8.1}$$

where $U(x, t)$ is the disturbance and S is evaluated on the history of U which is presumed given up to time t. The energy of this disturbance satisfies

$$\frac{\rho}{2}\frac{d}{dt}\langle|U|^2\rangle=-\langle A_1[U]:S\rangle, \qquad A_1[U]=\nabla U+\nabla U^T \tag{8.2}$$

where the integral $\langle A_1 : S\rangle$ is called the stress-power. For Newtonian fluids this integral may be written as

$$\langle A_1 : S\rangle = \mu\langle A_1 : A_1\rangle = 2\mu\langle |\nabla U|^2\rangle > c^2\langle |U|^2\rangle.$$

It follows that $\langle |U|^2\rangle \leqq \langle |U_0|^2\rangle \exp\{-2c^2 t/\rho\}$ where $U = U_0$ at $t=0$. The rest state of a Newtonian fluid is globally and monotonically stable because the stress-power is positive definite.

It has been shown by TRUESDELL (1952) and by NOLL (unpublished; see TRUESDELL & NOLL [1965, p. 511]) that it is impossible for the stress power to be positive for all kinematically admissible motions and for all possible fluids. Their results and those of COLEMAN (1962) indicate that there are kinematically admissible motions which lead to negative values for the stress-power even when the response operator is of a realistic type.

It is not possible, of course, to decide about the asymptotic stability of the rest state without considering the destiny of disturbances which satisfy equations (8.1). However, since kinematically admissible disturbances are admissible as initial conditions for (8.1), it does not seem possible to guarantee monotonic stability even in the class of dynamically admissible solutions of (8.1).*

It appears unquestionably true that response operators which lead to the instability of the rest state give an incorrect description of physical fluids. This point of view seems to have been clearly expressed first by A. CRAIK (1968). CRAIK considers the stability of the rest state of a viscoelastic fluid, confined between horizontally infinite parallel planes, to infinitesimal, two-dimensional disturbances. He remarks that, "On physical grounds, we may assert that any physically realistic models should possess the property that a layer of fluid at rest between horizontal plane rigid boundaries is in stable equilibrium." This is an intuitively correct idea; the state of rest should be the terminal form of every solution of the initial value problem for a fluid which fills a container whose walls are at rest when there are no body forces present to drive a motion. I would expect that only those response operators which lead to the stability of the rest state describe physical fluids.

It is not possible to study (8.1) without knowing something about the stress operator $\mathscr{F}$. We have already noted that there is no real non-Newtonian fluid for which $\mathscr{F}$ is known. Even if $\mathscr{F}$ could be specified for some particular fluid, the general stability problem might be unmanageably difficult. It seems better to restrict the histories to motions on which $\mathscr{F}$ reduces to something manageable. Manageable $\mathscr{F}$'s arise from slow motion expansions and under the assumption of complete linearization. The latter assumption seems to be unavoidable in a discussion of a linear theory of stability and it will form the basis for the analysis given here. But a retardation of the time does not stem from the requirements of the stability problems and the time should not be retarded. In fact, in stability studies we must allow at least for the possibility of small-amplitude oscillations of arbitrary frequency; these certainly need not imply slow times.

* There is no existence theory for the initial boundary value problem. There is no existence theory for steady motions though important special solutions are known. It is not yet clear how one should proceed with general dynamical problems for simple fluids, or even how correctly to pose conditions at the boundary. Our formulation (8.1) is merely a guess based on the formulation known to work for Newtonian fluids and for general fluids in special flows.

The complete linearization of $\mathcal{F}$ can be expressed by constitutive equations of the infinitesimal theory of linear viscoelasticity. COLEMAN & NOLL (1961) have shown that the constitutive equation of the infinitesimal theory of linear viscoelasticity is the general form which $\mathcal{F}$ must take for a simple fluid with fading memory when the strain relative to some fixed configuration is small (see MARKOVITZ & COLEMAN, 1964, p. 90). This constitutive equation is in the form

$$S=\varepsilon\int_0^\infty m(s)\big(E(t-s)-E(t)\big)\,ds \tag{8.3a}$$

where $E(t-s)$ is the infinitesimal strain tensor at time $t-s$ relative to a fixed configuration and

$$m(s)=\frac{dG}{ds} \tag{8.3b}$$

where $G(s)$ is the shear relaxation modulus. The constitutive equation

$$S=\varepsilon\int_0^\infty G(s)\,\boldsymbol{a}_1(t-s)\,ds \tag{8.3c}$$

follows from an integration of (8.3a) by parts using (8.3b), the property that $G(s)\to 0$ as $s\to\infty$ and the relation

$$\boldsymbol{x}(t)=\xi^0+\varepsilon\boldsymbol{v}(\xi^0,t)$$

between the current coordinates, the fixed coordinates, and the small deformation.

We have now completed the preliminaries to the stability analysis of the rest state of a simple fluid. The assumptions which we make in the analysis are essentially statements about the spectral problem of linearized theory. The spectral problem may be obtained by copying the procedure which is used and is correct for nonlinear problems when there is no memory: the recipe is to linearize, then to substitute exponential solutions proportional to $e^{-\sigma t}$.

Linearization of (8.1), using $(U,P)=\varepsilon(\boldsymbol{u},p)$ and (8.3c), gives

$$\rho\frac{\partial\boldsymbol{u}}{\partial t}+\nabla p=\int_0^\infty G(s)\,\nabla^2\boldsymbol{u}(t-s)\,ds,\qquad \nabla\cdot\boldsymbol{u}=0,\qquad \boldsymbol{u}|_{\partial\mathscr{V}}=0. \tag{8.4}$$

Substituting

$$\boldsymbol{u}(\boldsymbol{x},t)=e^{-\sigma t}\,\hat{\boldsymbol{u}}(x),\qquad p(\boldsymbol{x},t)=e^{-\sigma t}\,\hat{p}(x) \tag{8.5}$$

into (8.4), we have

$$-\rho\sigma\hat{\boldsymbol{u}}+\nabla\hat{p}=k(\sigma)\,\nabla^2\hat{\boldsymbol{u}},\qquad \nabla\cdot\hat{\boldsymbol{u}}=0,\qquad \hat{\boldsymbol{u}}|_{\partial\mathscr{V}}=0 \tag{8.6}$$

where

$$k(\sigma)=\int_0^\infty G(s)\,e^{\sigma s}\,ds. \tag{8.7}$$

Equations (8.6) and (8.7) define the *spectral problem* of the linearized theory of the rest state of a simple fluid.

We say that the rest state is stable if

$$\operatorname{re}(\sigma)>0 \tag{8.8a}$$

and is unstable if

$$\operatorname{re}(\sigma)<0. \tag{8.8b}$$

We are assuming a principle of linearized stability. When (8.8b) holds, the rest state is unstable. The principle of linearized stability holds for systems of nonlinear ordinary differential equations as well as for nonlinear partial differential equations of the Navier-Stokes type.

This principle has been partially established for nonlinear ordinary differential equations of the functional-differential equation type analogous to (8.4).* Pending a deeper justification we shall assume this principle.

It is convenient to regard (8.6) as an eigenvalue problem with eigenvalues

$$\hat{\Lambda}=\rho\,\sigma/k(\sigma). \tag{8.9}$$

The eigenvalues $\hat{\Lambda}$ *of* (8.6) *are real-valued: they form a discrete, denumerable set* $\hat{\Lambda}=\hat{\Lambda}_n$ $(n=1, 2, \ldots\infty)$ *which may be arranged as an increasing sequence*

$$\hat{\Lambda}_1\leqq\hat{\Lambda}_2\leqq\cdots\hat{\Lambda}_n, \qquad \lim_{n\to\infty}\hat{\Lambda}_n=\infty.$$

Proof. The eigenvalues $\hat{\Lambda}$ may be characterized as the critical points of the Rayleigh quotient

$$\hat{\Lambda}_n=\min_{H_n}\langle|\nabla \boldsymbol{u}|^2\rangle/\langle|\boldsymbol{u}|^2\rangle$$

where H_n is the complement of the Hilbert space of solenoidal vectors which vanishes on $\partial\mathscr{V}$ and which is orthogonal to the eigensubspaces of the first n eigenvectors. The properties asserted by the theorem are guaranteed by standard theorems about the variational characterization of eigenvalues of self-adjoint operators in Hilbert spaces.

Assuming the principle of linearized stability, we may now assert that the rest state of a simple fluid is stable if we have $\operatorname{re}(\sigma_n)<0$ for all eigenvalues σ_n, where σ_n are the possibly complex-valued roots of the equation

$$k(\sigma_n)\,\hat{\Lambda}_n=\rho\,\sigma_n. \tag{8.10}$$

The real and imaginary parts of (8.10) may be written as

$$\hat{\Lambda}_n\int_0^\infty G(s)\,e^{\xi_n s}\cos\eta_n s\,ds=\rho\,\xi_n, \tag{8.11a}$$

$$\hat{\Lambda}_n\int_0^\infty G(s)\,e^{\xi_n s}\sin\eta_n\,ds=-\rho\,\eta_n \tag{8.11b}$$

where

$$\sigma_n=\xi_n+i\,\eta_n.$$

If we consider solutions of (8.11) over all possible containers $\mathscr{V}$, we must allow $\hat{\Lambda}_n(\mathscr{V})$ to take on all positive values. Simple fluids whose shear relaxation modulus $G(s)$ is such that for some n and some container $\xi_n<0$ are not physical fluids. Such unphysical simple fluids have an unstable rest state.

* Private communication by R. K. Miller, extending the results of Miller (1971).

When $G(s)>0$ there is no non-oscillatory solution ($\eta_n=0$) of (8.4) with $\xi_n<0$. Moreover, CRAIK has shown that there can be no solution of (8.11a) with $\xi_n<0$ and $\eta_n\neq 0$ if $G(s)$ decays monotonically ($m(s)<0$). If we assume a principle of linearized stability, it follows that simple fluids with a positive and monotonically decaying shear modulus are conditionally stable.

I have argued that the histories which are appropriate in the study of the stability of the rest state of a simple fluid do not lead to a retardation approximation and to Rivlin-Ericksen fluids. Of course it is possible to study the rest state of Rivlin-Ericksen fluids of arbitrary order by the spectral method. To linearize the response of the Rivlin-Ericksen fluid, we first set $U(x,t)=\varepsilon u(x,t)$; then, in the limit $\varepsilon\to 0$, we find from the recursion formula for A_n (above (3.1)) that

$$A_n[\varepsilon u(x,t)]\to\varepsilon\partial^{n-1}a_1/\partial t^{n-1},$$

and from (3.4) that

$$S_{(N+1)}\to\varepsilon\left\{\mu a_1+\sum_{n=1}^{N}\hat{\alpha}_n(\partial^n a_1/\partial t^n)\right\}. \tag{8.12}$$

This last expression may also be obtained as a partial sum of the retardation expansion of

$$S=\mathscr{F}\left[\varepsilon\sum_{n=1}^{\infty}(-s)^n(\partial^{n-1}a_1/\partial t^{n-1})\right].$$

The fluid of grade $N+1$ is also governed by (8.6) with

$$k(\sigma)=\mu+\sum_{n=1}^{N+1}(-\sigma)^n\hat{\alpha}_n.$$

Hence, the eigenvalues σ_n are related to the eigenvalues $\hat{\Lambda}_n$:

$$\hat{\Lambda}_n=\rho\sigma_n\Big/\left(\mu+\sum_{l=1}^{N+1}(-\sigma_n)^l\hat{\alpha}_l\right). \tag{8.13}$$

The simplest case is a second-order fluid; for this,

$$\sigma_n=\mu\hat{\Lambda}_n/(\rho+\hat{\alpha}_1\hat{\Lambda}_n). \tag{8.14}$$

Since $\hat{\alpha}_1=\alpha_1$ is negative in polyisobutylene solution and $\hat{\Lambda}_n\to\infty$, we find that there are eigenvalues $\sigma_n<0$, and it follows that the rest state of the second-order retardation approximation is unstable. Similar instability results will hold for the n^{th}-order approximation if the coefficients $\hat{\alpha}_n$ lie in a certain set.*

* COLEMAN & MIZEL (1966), following earlier work of COLEMAN, DUFFIN & MIZEL (1965), have considered the stability of shearing flows of a second-order fluid with $\alpha_1<0$. The coefficient α_1 is generally believed to be negative in polyisobutylene solutions which are used in experiments. The aforementioned stability problem with $\alpha_1>0$ has been studied by TING (1963). COLEMAN, DUFFIN & MIZEL motivate their use of the second-order fluid by an appeal to the retardation theorem of COLEMAN & NOLL (1960) but are careful to emphasize that the use of a more general constitutive relation might yield different results. They show that for certain critical channel-widths, a flow which is initially a laminar shearing flow cannot remain so. COLEMAN & MIZEL show that for these critical values of h, if there is any flow at all, the departure from shearing flow must appear instantly in the first time derivative of the velocity. The rest state is a special case of shearing flow. CRAIK (1968), using a linearized theory of stability, has shown that the second-order fluid is unstable but the linear viscoelastic fluid is stable to infinitesimal two-dimensional disturbances.

If the Rivlin-Ericksen fluid of grade N is accepted as a real constitutive equation (a "model" equation) for some fluid in all motions, then one is obliged to consider stability analyses of the type just given. Certainly the always unstable rest state of the fluid of grade 2 with $\alpha_1 < 0$ would seem to eliminate these fluids as models for real fluids in all motions. Their use in the sense of retardation is, however, in no way damaged by the instability demonstrated. It is always *possible* that a fluid of grade $N > 1$ closely describes a real fluid, if not in all motions, in "most" motions of interest. Merit in the use of the fluid of grade N as a model is as much a matter for experience as for analysis. Certainly the fluid of grade N whose coefficients μ and $\hat{\alpha}_n$ lie in the stable set is a better candidate for a fluid model than the fluid of grade N whose coefficients are in the unstable set. The stable and unstable sets of coefficients can be given by analysis but the values of the coefficients in real fluids must be determined experimentally.

On the other hand, it should be clear that the concept of a fluid model is not natural to stability studies. It is better to make the theory of the simple fluid practical by restricting considerations to motions which are both appropriate for stability analysis and lead to reductions in the complexity of the response. In the linearized case this leads to the theory of infinitesimal viscoelasticity rather than to Rivlin-Ericksen fluids.

The interesting way in which a simple fluid may appear to obey different stress laws when undergoing different motions can be illustrated by considering the bifurcation problem for a simple fluid heated from below. SOKOLOV & TANNER (1972) studied the stability part of the problem by the advocated method. They find that the spectrum σ_n is real-valued for many models of the simple fluid. Given this "exchange of stability", bifurcation theory shows that steady convection bifurcates from steady conduction. The steady solution can be constructed as an analytic perturbation pivoted around the conduction state (of rest) using the retardation approximations for slow steady flow and the Rivlin-Ericksen fluids. It takes the infinitesimal viscoelastic reduction to study stability and the slow steady motion reduction to study bifurcation. We turn next to the bifurcation problem.

9. Bifurcation of the Rest State of a Simple Fluid Heated from Below

We consider a pool of liquid (a simple fluid) resting on a hot flat plate and confined by vertical insulating side walls. If the bottom plate is not too hot, the fluid will be motionless and the transport of heat from the bottom plate across the fluid will take place by heat conduction. At a critical temperature difference, the conduction solution will lose stability and some of motion will begin in the fluid. The existence of motion will alter the shape of the free surface at the top of the fluid and will change the amount of heat transported from the amount which would be transported by conduction alone. We shall use bifurcation theory to compute how the shape of the free surface and the heat transport curve depend on the Rivlin-Ericksen coefficients.

The liquid pool is confined to a cylinder whose arbitrary cross section is designated by the set of points $(x, y) \in \mathscr{A}$, independent of the vertical coordinate z. The bottom plate at $z=0$ is horizontal and is held at a fixed temperature $T_0 + \Delta T$.

The top of the fluid is given by $z=\mathscr{h}(x, y; \varepsilon)$ where $\varepsilon^2=Nu-1$ is the Nusselt number discrepancy. The simple fluid is assumed to be governed by the Oberbeck-Boussinesq equations as in the classical problem of Bénard; however, we are considering simple fluids whose stress response is expandable as in (3.6a). The region occupied by the fluid changes with the Nusselt number and is given by

$$\mathscr{V}_\varepsilon=[(x, y, z)\,|\,(x, y)\in\mathscr{A}, 0\leqq z\leqq\mathscr{h}(x, y; \varepsilon)].$$

In $\mathscr{V}_\varepsilon$ we require that the Oberbeck-Boussinesq equations (9.4a, b) hold. Here, however, $\boldsymbol{S}$ is not Newtonian and the free surface $\mathscr{f}(x, y, z, \varepsilon)=z-\mathscr{h}(x, y; \varepsilon)=0$ is to be determined as a part of the solution. Since the fluid is incompressible, the conservation of volume implies that the mean height

$$\bar{\mathscr{h}}=\iint_{\mathscr{A}}\mathscr{h}(x, y; \varepsilon)\,dx\,dy\Big/\iint_{\mathscr{A}}dx\,dy \tag{9.1}$$

is independent of ε.

The construction of a bifurcating motion can be carried out when the temperature which is prescribed on the boundary is compatible with a static solution of the Oberbeck-Boussinesq equations. A static solution is possible only in the case $T=T(z)$. In this case

$$T-T_0=\Delta T(1-z/\bar{\mathscr{h}}) \tag{9.2}$$

where, for convenience, we have set the reference temperature T_0 of T at a height $\bar{\mathscr{h}}$ above the hot plate. We shall require that (9.2) holds at all points of the free surface $z=\mathscr{h}(x, y; \varepsilon)$ of the liquid pool. We might suppose that the cylinder has a top rigid surface above the free surface and that the temperature of the rigid top is prescribed and compatible with (9.2). If the air layer between the rigid surface and the free surface of the liquid is small, convection will start first in the liquid. We assume that the static distribution (9.2) holds in the air right up to the boundary $z=\mathscr{h}(x, y; \varepsilon)$.

The boundary conditions are that $\boldsymbol{u}=0$ on $z=0$ and on the boundary $\partial\mathscr{A}$ of the cylinder. The temperature is $T_0+\Delta T$ on $z=0$ and is $T_0+\Delta T(1-\mathscr{h}/\bar{\mathscr{h}})$ on $z=\mathscr{h}(x, y; \varepsilon)$. The side walls are insulated; $\boldsymbol{n}$ is the outward normal to the side wall $\partial\mathscr{A}$ and $\boldsymbol{n}_{\mathscr{A}}\cdot\Delta T=0$ on $\partial\mathscr{A}$. On the free surface the shear stresses must vanish, the jump in the normal components of the stress must balance surface tension forces and the kinematic condition for material surfaces $\dot{\mathscr{f}}=d\mathscr{f}/dt=0$ must hold. As a compatibility equation with $\boldsymbol{u}|_{\partial\mathscr{A}}=0$, we require that $\mathscr{h}|_{\partial\mathscr{A}}=\bar{\mathscr{h}}$.

SOKOLOV & TANNER (1972) have studied the stability of the conduction solution (9.2) in the more restricted situation in which the free surface does not deform. They subject the rest state to initial histories of disturbances of infinitesimally small amplitude but arbitrary frequency. Their analysis is similar to the study of the rest state which was given in the previous section. They find that for many models of a simple fluid considered by them, a "principle of exchange of stability" holds and $\sigma(\Delta T_c)=0$ at criticality.

We are going to assume in our analysis that $\sigma(\Delta T_c)=0$ is a simple eigenvalue of the spectral problem. We then assume, following the lessons learned from bifurcation theory in the Newtonian case, that the bifurcating solution is steady.

To construct the bifurcating solution, we use the Rivlin-Ericksen approximations for perturbations of the rest state. It is interesting that entirely different stress laws are required to study stability, on the one hand, and bifurcation, on the other hand, in one and the same simple fluid.

To formulate the bifurcation theory, we first define a triad of orthonormal vectors at each point of the surface $z=\mathscr{h}$: two tangential vectors $\boldsymbol{m}/|\boldsymbol{m}|$ and $\boldsymbol{p}/|\boldsymbol{p}|$ lying in the surface $\mathscr{f}\equiv z-\mathscr{h}=0$ and the normal $\boldsymbol{n}=\nabla(z-\mathscr{h})/|\nabla(z-\mathscr{h})$ where

$$\begin{bmatrix} \nabla(z-\mathscr{h}) \\ \boldsymbol{m} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_z-\nabla_2\mathscr{h} \\ \boldsymbol{e}_y+\boldsymbol{e}_z\,\partial\mathscr{h}/\partial y \\ \boldsymbol{e}_x(1+|\partial\mathscr{h}/\partial y|^2)-\boldsymbol{e}_y\dfrac{\partial\mathscr{h}}{\partial x}\dfrac{\partial\mathscr{h}}{\partial y}+\boldsymbol{e}_z\dfrac{\partial\mathscr{h}}{\partial x} \end{bmatrix},$$

and $\nabla_2=\boldsymbol{e}_x\,\partial/\partial x+\boldsymbol{e}_y\,\partial/\partial y$.

We next introduce functions θ, ϕ, H and h:

$$\begin{aligned}
&\theta=T-T_0-\Delta T(1-z/\bar{\mathscr{h}}),\\
&\phi=\theta\sqrt{\alpha\rho_0\,g\bar{\mathscr{h}}/\Delta T},\\
&H=p-p_a+\rho_0\,g\left[z-\bar{\mathscr{h}}-\frac{\alpha\Delta T}{2}(z-\bar{\mathscr{h}})^2\right],\\
&h=\mathscr{h}(x,y;\varepsilon)-\bar{\mathscr{h}},
\end{aligned}$$

and the parameter

$$\Lambda=\sqrt{\rho_0\alpha g\Delta T/\bar{\mathscr{h}}}. \tag{9.3a}$$

Finally, we define ε^2 as the Nusselt number discrepancy

$$Nu-1=\varepsilon^2. \tag{9.3b}$$

The Nusselt number relative to the wall at $z=0$ is defined as

$$Nu=\frac{\chi\dfrac{d\bar{T}(0)}{dz}}{-\chi\dfrac{\Delta T}{\bar{\mathscr{h}}}}=1-\frac{1}{\Lambda}\frac{d\bar{\phi}(0)}{dz}$$

where the overbar designates horizontal averaging as in (9.1). To shorten the writing of problems, it is convenient to define a set

$$\begin{aligned}F=\{(u,\phi,h)\,|\,\operatorname{div}\boldsymbol{u}=0,\quad &|\boldsymbol{u}|=\phi=0\quad\text{on } z=0,\\ |\boldsymbol{u}|=\boldsymbol{n}_{\mathscr{A}}\cdot\nabla\phi=h=0\quad&\text{on } \partial\mathscr{A},\ \bar{h}=0\}.\end{aligned}$$

The boundary value problem for steady bifurcating solutions may be written as

$$\left.\begin{aligned}&-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\Lambda\phi\,\boldsymbol{e}_z-\nabla H+\nabla\cdot\boldsymbol{S}=0,\\ &-\boldsymbol{u}\cdot\nabla\phi+\Lambda w+\kappa\nabla^2\phi=0,\end{aligned}\right\}\quad\text{in } \mathscr{V}_\varepsilon \tag{9.4a, b}$$

$$\left.\begin{aligned}&\phi=\boldsymbol{u}\cdot\nabla\mathscr{f}=\nabla\mathscr{f}\cdot\boldsymbol{S}\cdot\boldsymbol{p}=\nabla\mathscr{f}\cdot\boldsymbol{S}\cdot\boldsymbol{m}=0,\\ &\sigma\nabla_2\cdot\left[\frac{\nabla_2 h}{(1+|\nabla_2 h|^2)^{\frac{1}{2}}}\right]-\rho_0\,g\left[h-\frac{\Delta T\alpha}{2}h^2\right]+H=\boldsymbol{n}\cdot\boldsymbol{S}\cdot\boldsymbol{n},\end{aligned}\right\}\quad\text{on } \mathscr{f}=0 \tag{9.4c, d}$$

$$(\boldsymbol{u}, \phi, h) \in F, \tag{9.4e}$$

$$\varepsilon^2 = Nu - 1 = -\frac{1}{\Lambda}\frac{d\bar{\phi}(0)}{dz}. \tag{9.4f}$$

The rest state is an exact solution of (9.4) which is defined by $Nu=1$ ($\varepsilon=0$) and

$$\begin{aligned}(\boldsymbol{u}, \boldsymbol{S}, \phi, h, T, p) &= (\boldsymbol{u}^{\langle 0\rangle}, \boldsymbol{S}^{\langle 0\rangle}, \phi^{\langle 0\rangle}, h^{\langle 0\rangle}, T^{\langle 0\rangle}, P^{\langle 0\rangle}) \\ &= \left(0, 0, 0, \bar{\hbar}, \Delta T(1-z/\bar{\hbar}), p_a - \rho_0 g\left[(z-\bar{\hbar}) - \frac{\alpha\Delta T}{2}(z-\bar{\hbar})^2\right]\right).\end{aligned} \tag{9.5}$$

The rest solution exists for all values of Λ. When $\Lambda=\Lambda^{\langle 0\rangle}$ (corresponding to $\Delta T=\Delta T_c$ where ΔT_c is the critical temperature difference), then the rest solution loses its stability to motion.

To study the motion, we first define an invertible mapping which is analytic in a small parameter ε to be defined; for example,

$$\mathscr{V}_\varepsilon(x, y, z) \Leftrightarrow \mathscr{V}_0(x_0, y_0, z_0)$$

under the transformation

$$x = x_0, \qquad y = y_0, \qquad z = z(x_0, y_0, z_0; \varepsilon)$$

where the stretching function z is invertible

$$z_0 = z_0(x, y, z; \varepsilon) = z(x_0, y_0, z_0; 0)$$

and maps boundary points into boundary points

$$0 = z(x_0, y_0, 0; \varepsilon)$$

and

$$\hbar = z(x_0, y_0, \bar{\hbar}; \varepsilon).$$

We next seek solutions of the bifurcation problem pivoted around the rest state as a power series in the parameter $\varepsilon = \sqrt{Nu-1}$. First we give the form of the solution

$$\begin{bmatrix} \boldsymbol{u}(x, y, z; \varepsilon) \\ \phi(x, y, z; \varepsilon) \\ H(x, y, z; \varepsilon) \\ h(x, y, z; \varepsilon) \\ \Lambda(\varepsilon) - \Lambda^{\langle 0\rangle} \end{bmatrix} = \sum_{n=1} \varepsilon^n \begin{bmatrix} \boldsymbol{u}^{[n]}(x_0, y_0, z_0) \\ \phi^{[n]}(x_0, y_0, z_0) \\ H^{[n]}(x_0, y_0, z_0) \\ h^{\langle n\rangle}(x_0, y_0) \\ \Lambda^{\langle n\rangle} \end{bmatrix} = \sum_{n=1} \varepsilon^n \begin{bmatrix} \boldsymbol{u}^{\langle n\rangle}(x, y, z) \\ \phi^{\langle n\rangle}(x, y, z) \\ H^{\langle n\rangle}(x, y, z) \\ h^{\langle n\rangle}(x, y) \\ \Lambda^{\langle n\rangle} \end{bmatrix}. \tag{9.6}$$

In the notation of Section 3, the extra stress is given by

$$S = \sum_{n=1} \varepsilon^n \sum_{l+q=1+n} S_q^{\langle l\rangle}[\boldsymbol{u}^{\langle r_1\rangle}, \dots \boldsymbol{u}^{\langle r_q\rangle}] = \sum_{n=1} \varepsilon^n S^{\langle n\rangle}(x, y, z).$$

Here

$$(\circ)^{[n]} = \frac{1}{n!}\left(\frac{\partial}{\partial\varepsilon} + \frac{dh}{d\varepsilon}\frac{\partial}{\partial z}\right)^n (\circ) = \frac{1}{n!}\frac{d^n(\circ)}{d\varepsilon^n}$$

is the n^{th} substantial derivative following the mapping and

$$(\text{o})^{\langle n\rangle}=\frac{1}{n!}\frac{\partial^n(\text{o})}{\partial\varepsilon^n}$$

is the n^{th} partial derivative.

The recipe for the computation of these series is as follows: First, we compute the partial derivatives in the flat domain $\mathscr{V}_0(x_0, y_0, z_0)$; this gives us the functions $\boldsymbol{u}^{\langle n\rangle}(x_0, y_0, z_0)$, *etc.* The functions $\boldsymbol{u}^{\langle n\rangle}(x, y, z)$ are extensions by declaration into $\mathscr{V}_\varepsilon$ of the functions $\boldsymbol{u}^{\langle n\rangle}(x_0, y_0, z_0)$. If there is a bifurcating solution analytic in ε, then this extension will be possible (JOSEPH, 1973; JOSEPH & STURGES, 1974).

To generate the perturbation problems for the partial derivatives, we note that if $A(x, y, z; \varepsilon)$ is an identity in $\mathscr{V}_\varepsilon$, then $A^{[n]}(x, y, z; \varepsilon)=0$, and a simple induction argument gives $A^{\langle n\rangle}(x, y, z; \varepsilon)=0$. If $A(x, y, \hbar(x, y; \varepsilon))=0$, then

$$A^{[n]}=\frac{1}{n!}\left(\frac{\partial}{\partial\varepsilon}+h^{\langle 1\rangle}(x, y; \varepsilon)\frac{\partial}{\partial z}\right)^n A.$$

Using these properties, we find that when $v\geqq 1$,

$$\left.\begin{aligned}\sum_{l+n=v+1}\boldsymbol{u}^{\langle n\rangle}\cdot\nabla\boldsymbol{u}^{\langle l\rangle}-\sum_{l+n=v}\Lambda^{\langle n\rangle}\phi^{\langle l\rangle}+\nabla H^{\langle v\rangle}-\nabla\cdot\boldsymbol{S}^{\langle v\rangle}=0,\\ \sum_{l+n=v+1}\boldsymbol{u}^{\langle n\rangle}\cdot\nabla\phi^{\langle l\rangle}-\sum_{l+n=v}\Lambda^{\langle n\rangle}w^{\langle l\rangle}-\kappa\nabla^2\phi^{\langle v\rangle}=0,\end{aligned}\right\}\text{ in } \mathscr{V}_0 \quad \begin{aligned}(9.7\text{a})\\(9.7\text{b})\end{aligned}$$

$$(\boldsymbol{u}^{\langle v\rangle}, \phi^{\langle v\rangle}, h^{\langle v\rangle})\in F, \tag{9.7c}$$

$$\left.\begin{aligned}\phi^{[v]}=(\nabla f\cdot\boldsymbol{S}\cdot\boldsymbol{p})^{[v]}=(\nabla f\cdot\boldsymbol{S}\cdot\boldsymbol{m})^{[v]}=(w-\boldsymbol{u}\cdot\nabla_2 h)^{[v]}=0,\\ [\boldsymbol{n}\cdot\boldsymbol{S}\cdot\boldsymbol{n}]^{[v]}=H^{[v]}-\rho_0 g\left[h-\frac{\alpha\Delta T}{2}h^2\right]^{[v]}+\sigma\nabla_2\cdot\left[\frac{\nabla_2 h}{(1+|\nabla_2 h|^2)^{\frac{1}{2}}}\right]^{[v]}\end{aligned}\right\} h=0. \quad \begin{aligned}(9.7\text{d})\\(9.7\text{e})\end{aligned}$$

From the normalizing condition (9.4f) we find that

$$-\Lambda^{\langle v-2\rangle}=\frac{d\bar{\phi}^{\langle v\rangle}(0)}{dz}=\bar{\phi}^{\langle v\rangle}(\bar{\hbar})-\frac{1}{\kappa}\sum_{l+n=v+1}\langle w^{\langle n\rangle}\phi^{\langle l\rangle}\rangle \tag{9.7f}$$

where $\Lambda^{\langle -1\rangle}\equiv 0$ and

$$\langle\text{o}\rangle=\frac{1}{\bar{\hbar}}\int_0^{\bar{\hbar}}\bar{\text{o}}\,dz_0.$$

To derive (9.7f), take the horizontal average of (9.7b). We find that

$$\frac{d}{dz_0}\left(\sum_{l+n=v+1}\overline{w^{\langle n\rangle}\phi^{\langle l\rangle}}-\kappa\frac{d\bar{\phi}^{\langle v\rangle}(z_0)}{dz_0}\right)=0 \tag{9.8}$$

and (9.7f) follows after integrating (9.8). The boundary values for $\phi^{\langle v\rangle}(\bar{\hbar})$ may be obtained in terms of lower-order partial derivatives by unfolding the first of equations (9.7d).

When $v=1$ we find that

$$A^{[1]}=A^{\langle 1\rangle}+h^{\langle 1\rangle}\frac{\partial A^{\langle 0\rangle}}{\partial z_0}\quad\text{for all } A(x, y, \hbar(x, y; \varepsilon); \varepsilon).$$

We note that $S^{\langle 1\rangle}=S_1^{\langle 1\rangle}=\mu A_1^{\langle 1\rangle}=\mu A_1[u^{\langle 1\rangle}]$ and use (9.5) to find that

$$\left.\begin{aligned}\Lambda^{\langle 0\rangle}\phi^{\langle 1\rangle}e_z-\nabla H^{\langle 1\rangle}+\nabla\cdot S^{\langle 1\rangle}=0,\\ \Lambda^{\langle 0\rangle}w^{\langle 1\rangle}+\kappa\nabla^2\phi^{\langle 1\rangle}=0,\end{aligned}\right\}\quad\text{in } \mathscr{V}_0 \tag{9.9a, b}$$

$$(u^{\langle 1\rangle},\phi^{\langle 1\rangle},h^{\langle 1\rangle})\in F, \tag{9.9c}$$

$$\left.\begin{aligned}\phi^{\langle 1\rangle}=S_{zy}^{\langle 1\rangle}=S_{zx}^{\langle 1\rangle}=w^{\langle 1\rangle}=0,\\ S_{zz}^{\langle 1\rangle}-H^{\langle 1\rangle}=-\rho_0 g h^{\langle 1\rangle}+\sigma\nabla_2^2 h^{\langle 1\rangle}\end{aligned}\right\}\ h=0 \tag{9.9d, e}$$

and

$$-\Lambda^{\langle 0\rangle}=\bar{\phi}^{\langle 2\rangle}-\frac{1}{\kappa}\langle w^{\langle 1\rangle}\phi^{\langle 1\rangle}\rangle. \tag{9.9f}$$

The problem (9.9) is a self-contained eigenvalue problem and determines the eigenvalue $\Lambda^{\langle 0\rangle}$ which, by assumption, has multiplicity one. The eigenfunction belonging to $\Lambda^{\langle 0\rangle}$ is uniquely determined to within an arbitrary multiplicative constant which is determined uniquely by (9.9f). To see how (9.9f) determines the constant, we first note that $h^{\langle 1\rangle}$ is determined to within the same multiplicative constant by (9.9e). We then note that

$$A^{[2]}=A^{\langle 2\rangle}+h^{\langle 1\rangle}\frac{\partial A^{\langle 1\rangle}}{\partial z_0}+\left(h^{\langle 2\rangle}+\frac{h^{\langle 1\rangle^2}}{2}\frac{\partial}{\partial z_0}\right)A^{\langle 0\rangle}.$$

Therefore,

$$\phi^{[2]}=\phi^{\langle 2\rangle}+h^{\langle 1\rangle}\frac{\partial\phi^{\langle 1\rangle}}{\partial z_0}=0$$

and elimination of $\phi^{\langle 2\rangle}$ from (9.9f) gives

$$-\Lambda^{\langle 0\rangle}=-h^{\langle 1\rangle}\phi^{\langle 1\rangle}-\frac{1}{\kappa}\langle w^{\langle 1\rangle}\phi^{\langle 1\rangle}\rangle.$$

Since $\Lambda^{\langle 0\rangle}$ is known and $h^{\langle 1\rangle}$, $\phi^{\langle 1\rangle}$ and $w^{\langle 1\rangle}$ are known to within an arbitrary multiplicative constant, (9.9f) determines the multiplicative constant.

The first nonlinear effects enter at second order:

$$\begin{aligned}S^{\langle 2\rangle}&=S_1^{\langle 2\rangle}+S_2^{\langle 1\rangle}=\mu A_1[u^{\langle 2\rangle}]+\alpha_1 A_2[u^{\langle 1\rangle}]+\alpha_2 A_1^2[u^{\langle 1\rangle}]\\ &\equiv\mu A_1^{(2)}+\alpha_1 A_2^{(1)}+\alpha_2 A_1^{(1)}A_1^{(1)}\end{aligned} \tag{9.10}$$

where the symbols with superscripts in parentheses,

$$A_l^{(m)}\equiv A_l[u^{\langle m\rangle}],\qquad S_l^{(m)}=S_l[u^{\langle m\rangle}],$$

mean that the tensors A_l or S_l should be evaluated on the field $u^{\langle m\rangle}$; in particular,

$$S_1^{\langle m\rangle}=S_1^{(m)}=\mu A_1^{(m)}\quad\text{and}\quad S_n^{\langle 1\rangle}=S_n^{(1)}:$$

$$\Lambda^{\langle 0\rangle}\phi^{\langle 2\rangle}e_z+\mu\nabla\cdot A_1^{(2)}-\nabla H^{\langle 2\rangle}=u^{\langle 1\rangle}\cdot\nabla u^{\langle 1\rangle}-\Lambda^{\langle 1\rangle}\phi^{\langle 1\rangle}e_z-\nabla\cdot S_2^{\langle 1\rangle}, \tag{9.11a}$$

$$\Lambda^{\langle 0\rangle}w^{\langle 2\rangle}+\kappa\nabla^2\phi^{\langle 2\rangle}=u^{\langle 1\rangle}\cdot\nabla\phi^{\langle 1\rangle}-\Lambda^{\langle 1\rangle}w^{\langle 1\rangle}, \tag{9.11b}$$

$$(u^{\langle 2\rangle},\phi^{\langle 2\rangle},h^{\langle 2\rangle})\in F, \tag{9.11c}$$

$$\phi^{\langle 2\rangle}+h^{\langle 1\rangle}\phi^{\langle 1\rangle}_{,z_0}=w^{\langle 2\rangle}+h^{\langle 1\rangle}w^{\langle 1\rangle}_{,z_0}-\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla_2 h^{\langle 1\rangle}=0, \tag{9.11d}$$

$$A^{(2)}_{1zx}+h^{\langle 1\rangle}A^{(1)}_{1zx,z_0}-\nabla_2 h^{\langle 1\rangle}\cdot\boldsymbol{A}^{(1)}_1\cdot\boldsymbol{e}_x+A^{(1)}_{1zz}h^{\langle 1\rangle}_{,x}+\mu^{-1}S^{(1)}_{2zx}=0, \tag{9.11e}$$

$$A^{(2)}_{1zy}+h^{\langle 1\rangle}A^{(1)}_{1zy,z_0}-\nabla_2 h^{\langle 1\rangle}\cdot\boldsymbol{A}^{(1)}_1\cdot\boldsymbol{e}_y+A^{(1)}_{1zz}h^{\langle 1\rangle}_{,y}+\mu^{-1}S^{(1)}_{2xy}=0, \tag{9.11f}$$

$$\begin{aligned}\mu A^{(2)}_{1zz}+h^{\langle 1\rangle}(\mu A^{(1)}_{1zz}-H^{\langle 1\rangle})_{,z_0}+S^{(1)}_{2zz}-H^{\langle 2\rangle}\\ =\sigma\nabla_2^2 h^{\langle 2\rangle}-\rho_0 g[h^{\langle 2\rangle}-\alpha\Delta T h^{\langle 1\rangle 2}].\end{aligned} \tag{9.11g}$$

The boundary value problem (9.11) can be solved only if the inhomogeneous terms satisfy the orthogonality relation

$$\begin{aligned}&\mu\langle\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla\cdot\boldsymbol{A}^{(2)}_1-\boldsymbol{u}^{\langle 2\rangle}\nabla\cdot\boldsymbol{A}^{\langle 1\rangle}_1\rangle+\kappa\langle\phi^{\langle 1\rangle}\nabla^2\phi^{\langle 2\rangle}-\phi^{\langle 2\rangle}\nabla^2\phi^{\langle 1\rangle}\rangle\\ &\quad+\langle\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla\cdot\boldsymbol{S}^{\langle 1\rangle}_2\rangle-\langle\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla H^{\langle 2\rangle}-\boldsymbol{u}^{\langle 2\rangle}\cdot\nabla H^{\langle 1\rangle}\rangle+2\Lambda^{\langle 1\rangle}\langle w^{\langle 1\rangle}\phi^{\langle 1\rangle}\rangle\\ &=\langle\boldsymbol{u}^{\langle 1\rangle}\cdot(\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla)\cdot\boldsymbol{u}^{\langle 1\rangle}\rangle+\langle\phi^{\langle 1\rangle}(\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla)\phi^{\langle 1\rangle}\rangle=0.\end{aligned}$$

This relation may be reduced further by integrating by parts and using the condition $A^{(1)}_{1zx}=A^{(1)}_{1zy}=0$:

$$\begin{aligned}&[\mu\boldsymbol{e}_z\cdot\boldsymbol{A}^{(2)}_1\cdot\boldsymbol{u}^{\langle 1\rangle}-\kappa\phi^{\langle 2\rangle}\phi^{\langle 1\rangle}_{,z_0}]_{z=\bar{A}}-w^{\langle 2\rangle}[H^{\langle 1\rangle}-\mu A^{(1)}_{1zz}]_{z=\bar{A}}\\ &\quad-\langle\boldsymbol{S}^{\langle 1\rangle}_2\cdot\nabla\boldsymbol{u}^{\langle 1\rangle}\rangle+2\Lambda^{\langle 1\rangle}\langle w^{\langle 1\rangle}\phi^{\langle 1\rangle}\rangle=0.\end{aligned} \tag{9.12}$$

Second-order quantities $\boldsymbol{A}^{(2)}_1$, $\phi^{\langle 2\rangle}$ and $w^{\langle 2\rangle}$ may be eliminated from (9.12) using (9.11 d–g):

$$\begin{aligned}2\Lambda^{\langle 1\rangle}\langle w^{\langle 1\rangle}\phi^{\langle 1\rangle}\rangle&=\langle\boldsymbol{S}^{\langle 1\rangle}_2:\nabla\boldsymbol{u}^{\langle 1\rangle}\rangle+[\mu\{h^{\langle 1\rangle}\boldsymbol{e}_z\cdot\boldsymbol{A}^{(1)}_{1,z}-\nabla_2 h^{\langle 1\rangle}\cdot\boldsymbol{A}^{(1)}_1\\ &\quad+A^{(1)}_{1zz}\nabla_2 h^{\langle 1\rangle}\}\cdot\boldsymbol{u}^{\langle 1\rangle}+\boldsymbol{e}_z\cdot\boldsymbol{S}^{\langle 1\rangle}_2\cdot\boldsymbol{u}^{\langle 1\rangle}-\kappa h^{\langle 1\rangle}(\phi^{\langle 1\rangle}_{,z_0})^2\\ &\quad+(h^{\langle 1\rangle}w^{\langle 1\rangle}_{,z_0}-\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla_2 h^{\langle 1\rangle})(H^{\langle 1\rangle}-\mu A^{(1)}_{,zz})]_{z=\bar{A}}.\end{aligned} \tag{9.13}$$

Equation (9.13) gives the slope of the heat-transport curve evaluated on the bifurcating solution at the point of bifurcation; this slope depends on Rivlin-Ericksen constants through the tensor $\boldsymbol{S}^{\langle 1\rangle}_2=\boldsymbol{S}^{(1)}_2$; for example,

$$\begin{aligned}\langle\boldsymbol{S}^{\langle 1\rangle}_2:\nabla\boldsymbol{u}^{\langle 1\rangle}\rangle&=\alpha_1\langle\boldsymbol{A}^{(1)}_1:\boldsymbol{A}^{(1)}_2\rangle+\alpha_2\langle\boldsymbol{A}^{(1)}_1:\boldsymbol{A}^{(1)}_1\cdot\boldsymbol{A}^{(1)}_1\rangle\\ &=(\alpha_1+\alpha_2)\langle\boldsymbol{A}^{(1)}_1:\boldsymbol{A}^{(1)}_1\cdot\boldsymbol{A}^{(1)}_1\rangle\end{aligned}$$

where the last equality follows from symmetry and integration by parts using $w^{\langle 1\rangle}=0$. It follows that

$$\left.\frac{d\Lambda}{d\sqrt{Nu-1}}\right|_{Nu=1}=\Lambda^{\langle 1\rangle}(\mu,\alpha_1,\alpha_2)$$

where $\Lambda^{\langle 1\rangle}$ is given by (9.13). In passing, we note that if the deflection of free surface is neglected, then $S^{\langle 1\rangle}_{2zx}=S^{\langle 1\rangle}_{2zy}=0$ and $\Lambda^{\langle 1\rangle}=(\alpha_1+\alpha_2)\langle\boldsymbol{A}^{(1)}_1:\boldsymbol{A}^{(1)}_1\cdot\boldsymbol{A}^{(1)}_1\rangle$.

The values of $\Lambda^{\langle n\rangle}$ for $n\geqq 2$ are obtained from the solvability requirement at higher orders. These values depend on the Rivlin-Ericksen coefficients which appear at higher orders.

In the same way, the shape of the free surface, which is given by the series (9.6), depends on the Rivlin-Ericksen constants; $h^{\langle 1\rangle}$ is a function of μ alone, $h^{\langle 2\rangle}$

depends on the second-order coefficients α_1 and α_2, $h^{\langle 3 \rangle}$ depends on the third-order coefficients, and so on.

It follows, from the analysis just given, that the bifurcation theory for a simple fluid heated from below has a certain potential as a heat-transport viscometer for the Rivlin-Ericksen coefficients.

Part III. Die Swell—the Final Diameter of a Capillary Jet

Die swell is the enlargement of the diameter of a jet of non-Newtonian fluid which is extruded from a capillary tube. Extrusion processes involving non-Newtonian fluids generally, and molten polymers particularly, take on an ever increasing importance as the rheological materials replace more conventional materials. Die swell is important in the extrusion process. Understanding and control of die swell means better and cheaper fabrication of commodities ranging from plastic broom bristles to magnetic tapes.

The die swell problem is very complicated; even the small swelling observed in low speed Newtonian jets is not fully understood and the various explanations which have been suggested are controversial. Many authors have written about the die swell problem; each one of them seems to make some mistake or assume something unphysical leading to inconsistency or to results which contradict experiment. I have here studied the possibility of constructing the solution of this problem as a domain perturbation from a state of rest dominated by surface tension. I have also drawn new consequences from the requirements of a careful analysis of the global conservation of momentum of the jet.

10. The Horizontal Capillary Jet: Formulation of the Mathematical Problem

A liquid moves from left to right down a semi-infinite $(-\infty<x<0)$ pipe of radius h_0 under the driving action of a pressure gradient. It is assumed that the flow far upstream $(x\to-\infty)$ is fully developed and the pressure gradient $P'=dP/dx$ is uniform there. At $x=0$ the liquid is extruded horizontally in a zero gravity field and an axially-symmetric capillary jet of radius $h(x)$ is formed. It is assumed that the wind shear is negligible, and far downstream $(x\to\infty)$ the flow must become rectilinear, with a uniform velocity U_f and final jet diameter h_f:

$$U_f=\lim_{x\to\infty} e_x\cdot U(x,r;Q), \qquad h_f=\lim_{x\to\infty} h(x). \tag{10.1}$$

The volume flux divided by 2π for this flow,

$$Q=\int_0^{h(x)} e_x\cdot U(x,r;Q)\,r\,dr, \tag{10.2}$$

is constant and invariant for the whole flow $(-\infty<x<\infty)$. The velocity far downstream

$$U(-\infty,r;Q)=e_x\varepsilon f(r,\varepsilon), \qquad f(r,0)=2(1-r^2/h_0^2) \tag{10.3}$$

where $\varepsilon=2Q/h_0^2$ is the scale velocity. The flow configuration, the coordinates and the definitions of some symbols are given in Fig. 1.

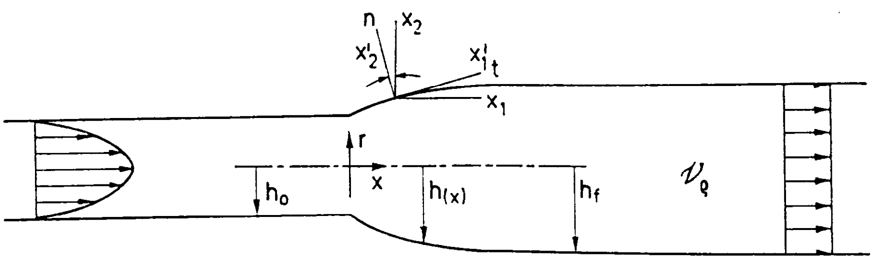

Fig. 1. The capillary jet.

The motion of the liquid in $\mathscr{V}_Q$ is governed by the following system of equations:

$$\rho\, \boldsymbol{U}\cdot \nabla \boldsymbol{U} + \nabla P - \nabla\cdot \boldsymbol{S} = 0 = \operatorname{div} \boldsymbol{U} \quad \text{in } \mathscr{V}_Q, \tag{10.4a}$$

$$2\varepsilon\, \boldsymbol{e}_x f(r,\varepsilon) - \boldsymbol{U}|_{x\to-\infty} = \varepsilon\, \boldsymbol{e}_x/\chi^2 - \boldsymbol{U}|_{x\to\infty} = \boldsymbol{U}(x, h_0; \varepsilon)|_{x\leqq 0} = 0, \tag{10.4b}$$

$$(\boldsymbol{U}, P, \boldsymbol{S}) \text{ are continuous at } x=0, \quad 0 \leqq r < h_0,$$

and, on the jet boundary $r=h(x)$, $x>0$,

$$\boldsymbol{U}\cdot \boldsymbol{n} = S_{nt} = S_{nn} - P + P_a - \sigma J = h - h_0|_{x=0} = h'|_{x=\infty} = 0 \tag{10.4c}$$

where the mean curvature of the jet boundary in the region $x>0$ is given by

$$J = \frac{-1}{h(x)\phi} + \frac{h''}{\phi^3} \quad \text{and} \quad \phi(h') = (1+h'^2)^{\frac{1}{2}}.$$

When written in coordinate form, (10.4c) may be expressed as

$$\begin{aligned} V - h'U &= h'(S_{rr} - S_{xx}) + (2-\phi^2) S_{xr} \\ &= S_{rr} - P + P_a - h' S_{xr} - \sigma J = h - h_0|_{x=0} = h'|_{x=\infty} = 0. \end{aligned} \tag{10.4d}$$

The free boundary equations (10.4d) state, respectively, that the normal component of the velocity, the shear stress, the difference between the jump in the normal stress and the surface tension force (σ = surface tension coefficient), the difference between the jet radius at $x=0$ and the pipe radius, and the slope of the free surface as $x\to\infty$ all vanish.

11. The Rest Solution and the Perturbation

When $Q=U=V=0$, the system (10.7) has a unique solution (DELAUNY, 1841). Since $S=0$ when $U=V=0$, we find from (10.7a) that $P=P_0$ where P_0 is some constant. Then equations (10.7d) reduce to

$$P_a - P_0 = \sigma J = \sigma \left\{ \frac{h''}{\phi^3} - \frac{1}{h\phi} \right\} = -\frac{\sigma}{h h'} \left(\frac{h}{\sqrt{1+h'^2}} \right)'. \tag{11.1}$$

Since $h''\to 0$ as $x\to\infty$, we have $(P_0-P_a)/\sigma = 1/h_f$ and, by integration,

$$h^2 - 2h_f h/\sqrt{1+h'^2} + h_f^2 = 0. \tag{11.2}$$

Since

$$\sqrt{1+h'^2} = 2h_f h/(h^2+h_f^2) \leqq 1,$$

we find that $h'=0$ and $h_f = h = h_0$.

A unique rest solution,

$$Q=U=V=h-h_0=h_0-\sigma/(P_0-P_a)=0, \tag{11.3}$$

could not exist without surface tension. This solution describes a motionless fluid which fills the cylinder $\mathscr{V}_0$ of radius h_0, $-\infty<x<\infty$, and is confined to the cylinder $x>0$ by the cylindrical film which balances the pressure difference $P_0-P_0>0$ by a tensile force of magnitude σ/h_0.

If the solution (U, P, h) of (10.4) were analytic in the scale velocity $\varepsilon=2Q/h_0^2$, we could construct it as a power series in ε by applying the domain perturbation method of Section 9. The same method can be used even when the solution is not analytic in ε.* Assuming only differentiability with respect to ε at $\varepsilon=0$, we may obtain expressions for $(U^{\langle 1\rangle}, P^{\langle 1\rangle}, h^{\langle 1\rangle})$ by solving the boundary value problem which arises from differentiating (10.4) with respect to ε at $\varepsilon=0$:

$$\nabla P^{\langle 1\rangle}-\nabla\cdot \boldsymbol{S}^{\langle 1\rangle}=0=\operatorname{div}\boldsymbol{U}^{\langle 1\rangle}, \quad \text{in } \mathscr{V}_0, \tag{11.4a}$$

$$2\boldsymbol{e}_x(1-r^2/h_0^2)-\boldsymbol{U}^{\langle 1\rangle}|_{x\to-\infty}=\boldsymbol{e}_x-\boldsymbol{U}^{\langle 1\rangle}|_{x\to\infty}=\boldsymbol{U}^{\langle 1\rangle}(x, h_0)|_{x\leqq 0}=0, \tag{11.4b}$$

$$(\boldsymbol{U}^{\langle 1\rangle}, P^{\langle 1\rangle}, \boldsymbol{S}^{\langle 1\rangle}) \text{ are continuous at } x=0, \quad 0\leqq r<h_0 \tag{11.4c}$$

and on the free boundary $r=h_0$, $x>0$,

$$V^{\langle 1\rangle}=S_{xr}^{\langle 1\rangle}=S_{rr}^{\langle 1\rangle}-P^{\langle 1\rangle}-\sigma\left[\frac{h^{\langle 1\rangle}}{h_0^2}+h^{\langle 1\rangle''}\right]=h^{\langle 1\rangle'}|_{x\to\infty}=0. \tag{11.4e}$$

Here $\boldsymbol{S}=\mu \boldsymbol{A}_1$ is the first term in the expansion of $\boldsymbol{S}$ around the rest state and $\nabla\cdot \boldsymbol{S}^{\langle 1\rangle}=\mu\nabla^2 U^{\langle 1\rangle}$. At this order of approximation there is no difference between Newtonian and non-Newtonian jets. Non-Newtonian effects enter at the perturbation computation at orders higher than one.

It is convenient to reformulate problem (11.4) in terms of the stream function $\psi^{\langle 1\rangle}$, $\boldsymbol{u}^{\langle 1\rangle}=-\operatorname{curl}(\boldsymbol{e}_\theta\psi^{\langle 1\rangle}/r)$. Then, (11.4a) may be reduced to

$$\mathscr{L}^2\psi=0, \quad \mathscr{L}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r} \quad \text{in } \mathscr{V}_0$$

where ψ satisfies no-slip conditions on the tube wall $r=h_0$ when $x\leqq 0$ and zero shear-stress conditions on the free boundary $x>0$, $r=h_0$.

RICHARDSON (1969), using the Wiener-Hopf method, solved the plane problem which is equivalent to (11.4). In the plane case, with walls at $y=\pm 1$, (11.4) reduces to a biharmonic problem

$$\begin{aligned} &\nabla^4\psi=0 \quad \text{in } \mathscr{V}_0, \quad \psi=\frac{\partial\psi}{\partial y}=0 \quad \text{at } y=\pm 1, \quad x\leqq 0,\\ &\psi=\frac{\partial^2\psi}{\partial y^2}=0 \quad \text{at } y=\pm 1, \quad x>0. \end{aligned} \tag{11.5}$$

RICHARDSON'S (stick-slip) problem (11.5) appears at first order in our perturbation expansion when the round capillary jet is replaced with a plane capillary jet.

* Here we define a mapping into a cylinder of radius h_0 for which $x=x_0$ and

$$\frac{d}{d\varepsilon}=\frac{\partial}{\partial\varepsilon}+\frac{dh}{dx}\frac{\partial}{\partial r}.$$

The plane jet may be regarded as a limiting case of a jet which is extruded from the annular region between concentric cylinders when the gap is small.

RICHARDSON'S solution is singular at the exit lip; locally the solution is of the form

$$\psi \sim 1 - 1\cdot 162\,\hat{r}^{\frac{1}{2}} \sin\frac{\theta}{2}\sin\theta + O(\hat{r}^{\frac{3}{2}}) \quad \text{as } \hat{r}\to 0$$

where (r, θ) are polar coordinates centered at the exit lip. The stresses and pressure (obtained by differentiating ψ twice) are singular with an $r^{-\frac{1}{2}}$ singularity at the exit lip.* However, since the normal stress $S_{yy}^{\langle 1\rangle} - P^{\langle 1\rangle}$ is not singular, the curvature is bounded and $h^{\langle 1\rangle}$ is a regular function of x. Moreover, the three components of the acceleration $\boldsymbol{u}^{\langle 1\rangle}\cdot \nabla \boldsymbol{u}^{\langle 1\rangle}$ evaluated near the singularity are bounded analytic functions. RICHARDSON considers the possibility that his solutions might be regarded as the first order in a series solution of the plane jet equivalent of (11.4) in powers of $1/\sigma$; when $1/\sigma=0$, the flat jet is confined by a tight membrane. He notes that this solution would require that when $1/\sigma \neq 0$, the tangent at the free surface at the exit lip deviates from the horizontal. He notes that the local analysis of the singularity of the biharmonic shows that, when a plane solid surface and a plane zero shear stress surface meet at an angle other than π, the normal force along the zero shear stress surface necessarily becomes infinite as the edge is approached.

In a later (1970) paper, using a momentum balance in which surface tension is neglected, RICHARDSON investigates the possibility of expansions in terms of the Reynolds number R (equivalent to ε). He finds an inconsistency in the limit $R\to 0$. This inconsistency does not exist when surface tension is not neglected (see (13.4)).

NICKELL, TANNER & CASWELL (1974), using finite-element numerical methods, have studied (10.4) for a Newtonian fluid with no surface tension ($\sigma=0$) and inertia neglected. Their method cannot accurately determine the solution in the immediate neighborhood of the lip singularity. However, to convince skeptics, they compute a numerical solution of the "stick-slip" problem which is in good agreement with RICHARDSON's analytic solution away from the singularity. In this solution for the round jet, they find that the final diameter of a low Reynolds number jet is about 1.13 times the pipe diameter. They cite agreement with many experiments; in these experiments it appears that $\chi\to 1.13$ as $R\to 0$ when $\sigma\rho h_0/\mu R = \sigma\rho^2/\mu\, 2\varepsilon \ll 1$. This limit is equivalent to the limit $\mu\to\infty$ and cannot hold in the limit $\varepsilon\to 0$ when the other parameters are fixed. The jet calculated by NICKELL, TANNER & CASWELL appears to join the pipe at angle other than π.

* This singularity can be traced to the harmonic singularity $\phi_1 = \hat{r}^{\frac{1}{2}}\sin\theta/2$ which arises at a point of discontinuous prescription of boundary data for Laplace's equation. If $(-x, -y) = (\hat{x}, \hat{y}) = (\hat{r}\cos\theta, \hat{r}\sin\theta)$ are coordinates centered at the top exit lip, then, locally, $\psi = \hat{y}\,\phi_1(\hat{x}, \hat{y}) + \phi_2(\hat{x}, \hat{y})$ where ϕ_1 and ϕ_2 are harmonic, $\phi_2(\hat{x}, 0)=0$ for $-\infty \leqq x \leqq \infty$, $\phi_1(\hat{x}, 0)=0$ for $\hat{x}\geqq 0$, whereas $\partial\phi_1(\hat{x}, 0)/\partial y = 0$ for $\hat{x}<0$. Locally, the separable solutions $\phi_1 = \hat{r}^{(2n+1)/2}\sin\dfrac{2n+1}{2}\theta$ satisfy all the required local conditions when n is an integer. If the discontinuous prescription is more complicated, say with $\phi_1(\hat{x}, 0)=0$ for $x\geqq 0$ and $\partial\phi_1(\hat{x}, 0)/\partial y + \alpha\,\phi(\hat{x}, 0) = 0$ for $x<0$ and $\alpha \neq 0$, then separable solutions are not possible and the nature of the singularities at the point of discontinuous prescription will change. This is what happens in the jet problem when a Newtonian theory of surface stresses is used.

They infer from their numerical data that there is a square root singularity of the "stick-slip" type. However, local analysis of Stokes equation in the neighborhood of their singularity shows that plane analysis applies there; if the angle at the exit lip differs from π, a square root singularity is impossible, a different power singularity is required and the normal force along the zero shear stress surface necessarily becomes infinite as the exit lip is approached. Given that $T_{nn} \to \infty$ as $x \to 0$, it is not possible to satisfy the third equation of (10.4d′) which with $\sigma = 0$ may be written as $-T_{nn} + P_a = 0$. It therefore does not seem possible to assess the validity of the finite element solution without further knowledge about its behavior in the immediate vicinity of the exit ring.

The presence of an edge singularity in the perturbation analysis raises doubts about the legitimacy of the series solution for the jet problem; the rest solution has no singularity and though the velocity and acceleration tend to zero with ε, the stresses and pressure at the lip are singular for any $\varepsilon > 0$ no matter how small ε. However, it seems to me that this singularity does give expression to a real physical effect in the same sense that corner singularities in elasticity actually foretell real stress concentrations in elastic materials. Mathematically, however, the singularity is damaging and it is really impossible to proceed to higher orders with any certainty; this is certainly true in the viscoelastic case which requires at least, bounded derivatives for any expansion into Rivlin-Ericksen tensors and it may be true in the Newtonian case. On the other hand, the results achieved at first order are consistent with the results of the momentum analysis of the low speed jet and the basic dominance of surface tension on a sufficiently low speed jet with a fixed value of surface tension can scarcely be doubted. Doubting readers may be reassured by watching what happens to a water jet which is ever more slowly extruded from a water tap. I believe that the first-order perturbation solution is correct but the effect of this singularity on the perturbation analysis at higher orders needs further study.

12. The Balance of Momentum

The simplicity of the flow far upstream and far downstream of the jet make it a natural candidate for simplified analysis using the momentum theorem.* In the analysis given below, the difference between the momentum deep inside

* Momentum analyses of the jet have been given by HARMON (1955), METZNER, HOUGHTON, SAILOR & WHITE (1961), TILLET (1968), RICHARDSON (1970), MIDDLEMAN & GAVIS (1961), FREDRICKSON (1964), SLATTERY & SCHOWALTER (1964) and GAVIS & MODAN (1967). The last four of these references include the effects of surface tension but only SLATTERY & SCHOWALTER treat them correctly; the others include only one of two principal curvatures. SLATTERY & SCHOWALTER have not noted that the integral over the mean curvature (their equations (24) and (39)) could be explicitly integrated as in (12.12). Except for TILLET's, all of the analyses listed above balance momentum between stations at the end of the pipe and far downstream where the velocity is uniform. TILLET's momentum balance is restricted to the plane jet, neglects surface tension, and is used to check the consistency of his extension of GOREN's (1966) boundary layer analysis. The boundary layer analysis leads to an expression for the diameter far downstream on the plane Newtonian jet which appears to disagree with experiments on axisymmetric jets. SERRIN (1959), TRUESDELL & NOLL (1965) and COLEMAN, MARKOVITZ & NOLL (1966) have studied the swelling phenomenon under the assumption that the flow is fully developed at the exit, that surface tension is unimportant and that the total axial thrust is balanced by the pressure of the atmosphere. The momentum analysis shows that these assumptions are incompatible with swelling (see RICHARDSON, 1970).

the tube and far out on the jet is balanced by a surface tension term at the tube exit and an excess skin friction which arises on the tube wall in the neighborhood of the exit.

We consider a control volume bounded by the planes $x=-L$ and $x=L$ and an imaginary surface drawn in the liquid adjacent to the cylindrical boundary of the cylinder and the jet. The momentum theorem, relative to the axial direction, may be expressed as

$$\oint\{T_{1j}-\rho U_1 U_j\}\, n_j=0 \tag{12.1}$$

where the integration is taken over the control volume. Expanding (12.1) we find, after cancelling common factors of 2π, that

$$\begin{aligned}&\int_0^{h(L)}[T_{11}(L,r)-\rho U^2(L,r)]\,r\,dr\\ &\quad-\int_0^{h_0}[T_{11}(-L,r)-\rho U^2(-L,r)]\,r\,dr\\ &\quad+\int_{-L}^{L}T_{1j}n_j h(x)\frac{dl}{dx}\,dx=0\end{aligned} \tag{12.2}$$

where $dl=\sqrt{dh^2+dx^2}$ is the arc length along the free boundary in the plane containing the jet axis.

On the free boundary it is convenient to express stresses in terms of their normal and tangential components. Referring to Fig. 1, we find that

$$T_{11}=\cos^2\alpha T_{tt}+\sin^2\alpha T_{nn}\equiv T_{xx}, \tag{12.3}$$

$$T_{12}=\sin\alpha\cos\alpha(T_{tt}-T_{nn})=\sin\alpha\cos\alpha(S_{tt}-S_{nn}). \tag{12.4}$$

Using the relations

$$n_1=-\sin\alpha,\qquad n_2=\cos\alpha,$$

we find that

$$T_{1j}n_j=T_{11}n_1+T_{12}n_2=-\sin\alpha T_{nn}. \tag{12.5}$$

Now we shall evaluate some of the integrals which appear in (12.2). First we note that

$$\int_{-L}^{0}T_{1j}n_j h_0\,dx=h_0\int_{-L}^{0}T_{xr}\,dx \tag{12.6}$$

and

$$\int_0^L T_{1j}n_j h(x)\frac{dl}{dx}\,dx=P_a\frac{h^2(x)}{2}\bigg|_0^L+\frac{\sigma h(x)}{\sqrt{1+h'^2}}\bigg|_0^L. \tag{12.7}$$

To prove (12.7), we note that $dl/dx=1/\cos\alpha$; then using (12.5), we find that

$$\begin{aligned}\int_0^L T_{1j}n_j h\frac{dl}{dx}\,dx&=-\int_0^L \operatorname{Tan}\alpha T_{nn}h\,dx\\ &=-\int_0^L h'hT_{nn}\,dx=\tfrac{1}{2}\int_0^L(h^2)'(P_a-\sigma J)\,dx\\ &=P_a\frac{h^2}{2}\bigg|_0^L+\sigma\int_0^L\left[\frac{h'}{\phi}-\frac{hh'h''}{\phi^3}\right]dx\\ &=P_a\frac{h^2}{2}\bigg|_0^L+\sigma\int_0^L\left[\frac{h}{(1+h'^2)^{\frac{1}{2}}}\right]'dx,\end{aligned}$$

proving (12.7). We next note that

$$\begin{aligned}&\int_0^{h_0} P(-L,r)\,r\,dr+P_a\frac{h^2(x)}{2}\Big|_0^L+\frac{\sigma h(x)}{\sqrt{1+h'^2}}\Big|_0^L\\&=\int_0^{h_0}[P(-L,r)-P_0]\,r\,dr+\frac{P_a h^2(L)}{2}+\frac{\sigma h_0}{2}+\frac{\sigma h(x)}{\sqrt{1+h'^2}}\Big|_0^L\end{aligned}\tag{12.8}$$

where $P_0=P_a+\sigma/h_0$.

Combining (12.2), (12.6) and (12.8), we find that

$$\begin{aligned}&\int_0^{h(L)}[T_{11}(L,r)-\rho U^2(L,r)]\,r\,dr\\&+\int_0^{h_0}[P(-L,r)-P_0]\,r\,dr+\frac{P_a h^2(L)}{2}+\frac{\sigma h_0}{2}+\frac{\sigma h(x)}{\sqrt{1+h'^2}}\Big|_0^L\\&+\int_0^{h_0}[S_{11}(-L,r)-\rho U^2(-L,r)]\,r\,dr+h_0\int_{-L}^{0}T_{xr}\,dx=0.\end{aligned}\tag{12.9}$$

Consider the real flow in the jet. Far upstream in the pipe this flow reaches a fully developed state in which $P(-L, r)$ is independent of r. The fully developed flow can be characterized by a pressure gradient; alternately we can parameterize this flow with the volume flux which is invariant in the flow. The pressure at the exit $x=0$ is generally different than the value $P_0=P_a-\sigma/h_0$ which prevails at the exit in the rest state. We next draw a straight line through P_0 with the same slope as the constant pressure gradient which is the asymptote of the pressure profile of the true flow. The fictitious fully developed flow and the true flow have the same volume flux Q. The exit distance d_c is the horizontal distance between the pressure gradient of the true flow and the fictitious fully developed flow at a given P as $x\to-\infty$.

Returning now to (12.9), we have

$$-\int_0^{h_0}[P(-L,r)-P_0]\,r\,dr=h_0\int_{-L-d_c}^{0}\overline{T}_{rx}(h_0)\,dx \tag{12.10}$$

where $\overline{T}_{rx}$ is the shear stress in fully developed flow. In the limit $L\to\infty$, we find that

$$\begin{aligned}\int_0^{h_f}[T_{11}(L,r)-\rho U^2(L,r)]\,r\,dr&=-P_f\frac{h_f^2}{2}-\frac{\rho U_f^2 h_f^2}{2}\\&=\frac{-P_a h_f^2}{2}-\frac{\sigma h_f}{2}-\frac{\rho U_f^2 h_f^2}{2}.\end{aligned}\tag{12.11}$$

Combining (12.9), (12.10) and (12.11), we find that

$$\begin{aligned}&\frac{\sigma}{2}[h_f+h_0-2h_0(1+h_0'^2)^{-\frac{1}{2}}]-\frac{\rho U_f^2 h_f^2}{2}\\&+h_0\int_{-d}^{0}T_{rx}\,dx-\int_0^{h_0}[S_{11}(-\infty,r)-\rho U^2(-\infty,r)]\,r\,dr=0\end{aligned}\tag{12.12}$$

where we have set

$$\int_{-d}^{0} T_{rx}\,dx = \int_{-L}^{0} T_{rx}\,dx - \int_{-(L+d_c)} \overline{T}_{rx}\,dx. \quad (12.13)$$

Equation (12.12) provides an exact description without approximations. All of the complications which are associated with deviations from fully developed flow in the exit of the jet have been collapsed into the single integral giving the skin friction excess. We are assuming that $d>0$; that is, the integral (12.13) gives a skin friction excess and decreases the momentum of the jet; the other possibility, $d<0$, contradicts experiments; it leads to a decrease in the final diameter when the Reynolds number is small and an increase when the Reynolds number is large.

13. Die Swell for the Low-speed Jet; the Final Diameter of the High-speed Newtonian Jet

When the fluid is Newtonian,

$$\left[S_{11}(-\infty, r),\, T_{rx}|_{h_0},\, \int_0^{h_0} U^2(-\infty, r)\, r\, dr\right] = \left[0,\, \mu \frac{\partial U}{\partial r}\bigg|_{h_0},\, \frac{2}{3} h_0^2 \varepsilon^2\right] \quad (13.1)$$

and (12.12) may be written as

$$\sigma[\chi+1-2(1+h_0'^2)^{-\frac{1}{2}}]+\rho\varepsilon^2 h_0 \left[\frac{4}{3}-\frac{1}{\chi^2}\right]+2\mu \int_{-d}^{0} \frac{\partial U}{\partial r}\bigg|_{h_0} dx=0 \quad (13.2)$$

where $\chi = h_f/h_0$ is the ratio of the final diameter to the tube diameter.

At the lowest speeds $\varepsilon \to 0$ (the values of the other parameters being fixed) we assume that

$$h_0' = h_0^{\langle 1 \rangle\prime}\varepsilon + o(\varepsilon). \quad (13.3)$$

Then (13.2) reduces to

$$\chi = 1 + \varepsilon\mu K_2/\sigma + o(\varepsilon) \quad (13.4)$$

where $U=\varepsilon u+o(\varepsilon)$ and

$$K_2 = -2\int_{-d}^{0} \frac{\partial u}{\partial r}\bigg|_{h_0} dx > 0.$$

It follows that

$$\frac{d\chi}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{\mu K_2}{\sigma} > 0 \quad (13.5)$$

where K_2 is independent of ε. For the plane capillary jet, we may obtain the value $K_2=0.356$ from the stick-slip solution given by RICHARDSON. Though it seems likely that the low-speed Newtonian jet is the limiting form for viscoelastic jets in the jet interior, the nature of the low-speed flow near the lip singularity requires further study.

Since the jet is almost certainly unstable at the lowest speeds, the possibility of direct experimental confirmation of (13.5) seems remote. We note that the increase of χ given by (13.4) could not continue indefinitely since the jet diameter will decrease at higher speeds. We study the high-speed case next.

Experiments show that high speed Newtonian jets contract*; this fact together with (13.2) shows that for large values of ε,

$$\frac{4}{3}-\frac{1}{\chi^2}+\frac{2\nu}{\varepsilon^2}\int_{-d}^{0}\left.\frac{\partial U}{\partial r}\right|_{h_0} dx=O(\varepsilon^{-2}). \tag{13.6}$$

We estimate

$$-\frac{\nu}{\varepsilon^2 h_0}\int_{-d}^{0}\frac{\partial U}{\partial r}\,dx=\frac{\nu}{\varepsilon h_0}\left(\frac{u}{\delta}d\right)=\frac{2}{R}\left(\frac{u}{\delta}d\right)=K_1(R)\,R^n$$

where $R=2\varepsilon\, h_0/\nu$ is the Reynolds number and K_1 is supposed to be a weak function of R which, together with the value of the exponent n, is to be determined by comparison with all of the available experiments. This comparison (see Figs. 2 and 3) shows that n is $\frac{1}{3}$ (or very near to $\frac{1}{3}$) and $K_1=(14.4)^{\frac{2}{3}}/3=1.97$ is chosen so that it will fit the experiments of GOREN & WRONSKI (1966) when $\chi=1$ and $R=14.4$. Although K_1 is selected to fit only one point in one experiment, the relation

$$\frac{4}{3}-\frac{1}{\chi^2}-\frac{K_1}{R^{\frac{2}{3}}}=0 \tag{13.7}$$

is in agreement with all experimental data** for Reynolds numbers larger than about 10. When $R\to\infty$, $\chi^2=\frac{3}{4}$ in accord with HARMON'S (1955) result.

It is appropriate at this point to call attention to the boundary layer analyses for large R of the Newtonian jet given by GOREN (1966) and by TILLET (1968). Goren treats the formation of a boundary layer at a free surface from a uniform shear flow. TILLET considers the development of a boundary layer at the boundary of a two-dimensional jet emerging from a channel in which there is plane Poiseuille flow far upstream. TILLET'S results could be regarded as an extension of GOREN'S, and the two analyses coincide at lowest order. Interestingly enough, these boundary layers involve $\frac{1}{3}$ powers of R, but they do not seem to be consistent with experiments. In fact, for the plane case, TILLET computes

$$\chi=\frac{5}{6}+o\left(\frac{1}{R}\right)$$

and says that HARMON'S (1955) results are correct to $O(1/R)$. However, the experiments on axisymmetric jets support (13.7) which shows that HARMON'S results are correct only to $o(R^{-\frac{2}{3}})$.

The data of GOREN & WRONSKI (1966) also appear to be inconsistent with predictions of GOREN'S (1966) boundary layer analysis. For example, in their Fig. 3, we see that the jet deflection decreases with increasing Reynolds number up to about 50, but the analysis gives the opposite variation. The sign of the variation of the free surface deflection with axial distance is consistent with GOREN'S analysis for the higher Reynolds number data ($86\leqq Re\leqq 195$) shown in their Fig. 6, but their data does not scale with $R^{-\frac{1}{3}}$ when x is fixed predicted by the $(x/R)^{\frac{1}{3}}$ variation given by analysis.

* Examination of the observed profiles of jet shapes shown in Figure 2 of GOREN & WRONSKI show that $|h_0'|<\frac{1}{4}$ when $4.2<2h_0\varepsilon/\nu<47.4$. When $2h_0\varepsilon/\nu=4.2$, $h_0'\cong\frac{1}{4}$. $\chi=1.094$.

** MIDDLEMAN & GAVIS (1961), GOREN & WRONSKI (1966) and GAVIS & MODAN (1967).

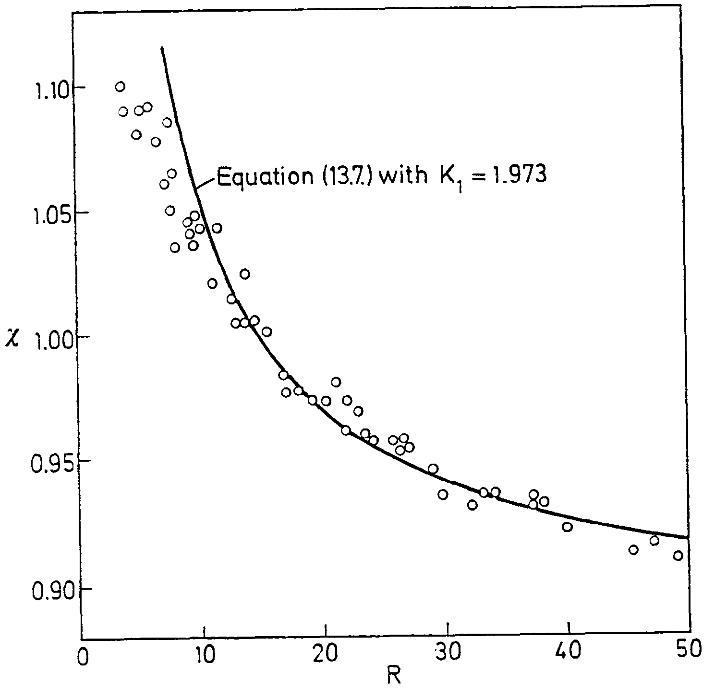

Fig. 2. The contraction ratio experiment of GOREN & WRONSKI.

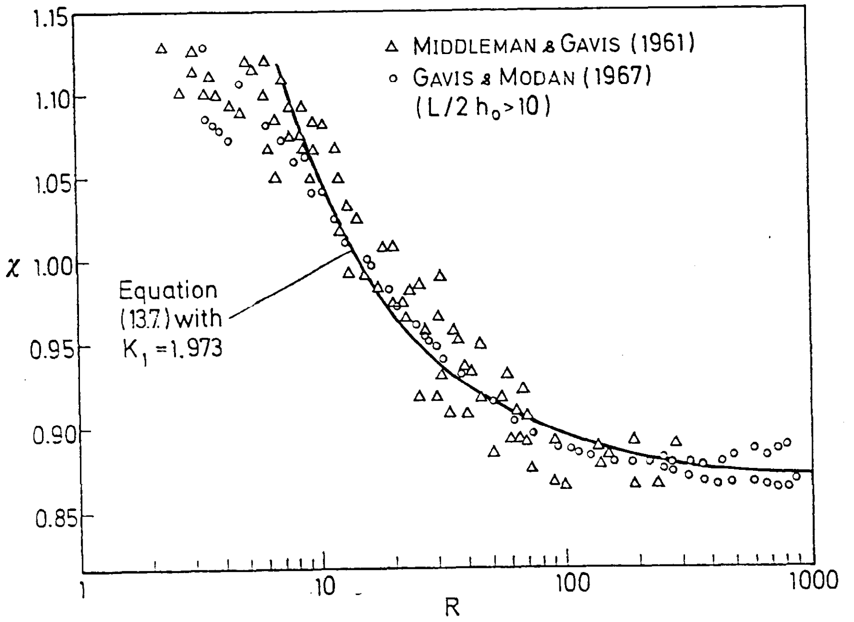

Fig. 3. Contraction ratio experiments.

The apparent inconsistency of GOREN & WRONSKI's jet deflection data with GOREN's analysis may be because the jet deflection is strongly influenced by the change in the final diameter of the jet for R less than 200. Though $R > 10$ could perhaps be considered asymptotic for the boundary layer in the exit region of the pipe, values of R larger than 200 would be required to wash out the effect of changes in the final diameter of the jet on the jet shape near the tube exit.

Part IV. The Free Surface at the Edge of a Torsion Flow Viscometer

A simple fluid, confined between two parallel disks of radius h_0 which rotate about a common x axis with angular velocities $2\omega/(1-\kappa)$ and $2\kappa\,\omega/(1-\kappa)$ at $x=d/2$ and $x=-d/2$ is held between the disks by surface tension at the free edge around the disks (see Fig. 4). We are going to derive formulas to describe the motion and the shape of the free surface as a power series in ω through terms of order ω^2. I had hoped that these formulas could be used as theory for a torsion-flow and free-surface viscometer, but analysis shows that the forces which shape the free surface and determine the streamlines at low speeds are inertial and are independent of the stress-deformation relation characterizing particular simple fluids. This motion is driven by torques generated by vertical gradients of the centripetal acceleration associated with the basic shearing motion. The torques operate in azimuthal planes and induce one big eddy from the center of the disk to the edge and a characteristic sequence of edge eddies which decay rapidly with distance from the edge. This result is all the more surprising since it is usual to study torsion flow by neglecting inertia altogether.

A brief history of theory and experiments for torsion flow can be found in the monograph of COLEMAN, MARKOVITZ & NOLL (1966). The basic viscometric approximations for this flow seem to have been given first by RIVLIN (1948) and GREENSMITH & RIVLIN (1953). They did express concern about inertia and they realized that it would induce a secondary flow; but their method of approximating this inertial effect with a constant is inadequate because it nullifies the torque

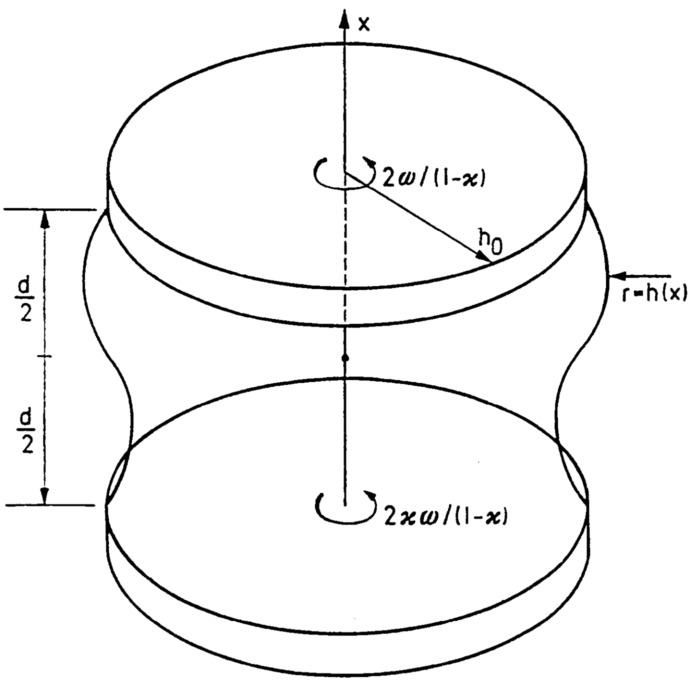

Fig. 4. The free surface on a liquid filling the space between rotating parallel disks. The domain occupied by the fluid is

$$\mathscr{V}_\omega=\{r,\theta,x\,|\,0\leqq r\leqq h(x),\ 0\leqq\theta\leqq 2\pi,\ d/2\leqq x\leqq d/2\}.$$

The motion is assumed to be axisymmetric with

$$U=e_r V(r,x)+e_\theta W(r,x)+e_x U(r,x).$$

which is produced by the stratification of the centripetal accelerations. Other authors have expressed reservations about the neglect of inertia. TRUESDELL & NOLL (p. 458) say that "Solutions can be obtained only after certain terms have been neglected. We expect that these solutions approximate the exact solutions, but a good theoretical justification is still lacking." Most authors express the belief that the neglect of inertia can be justified, possibly, for small values of ω. COLEMAN, MARKOVITZ & NOLL (1966) say that "Since the theory of torsional flow neglects inertia, it can be expected to approximate actual behavior only at low speeds of rotation." PIPKIN & TANNER (1973) say that: "In the flow between a fixed disk and a disk rotating with angular velocity ω_0, the velocity field has the form $\boldsymbol{v}=(\omega_0 rz/h)\, \boldsymbol{e}_\theta$, where h is the separation between the plates ... The shear rate is $\gamma = r\omega_0/h$. Consequently, at any given shear rate centrifugal force can be made arbitrarily small by decreasing the angular velocity ω_0 and the gap width h in the same proportion. Thus inertial effects, proportional to $\rho r \omega_0^2$, can be made negligible."

The analysis given below shows that inertial effects on the torque can be neglected when $\omega \to 0$ but that inertial effects on the normal pressures can be neglected only when the ratio $d^2\rho/10(\frac{4}{3}\alpha_1+\alpha_2)$ is much less than one; when this ratio is greater than one, the normal pressure will actually increase with radius. A brief, tight and self-contained argument about normal stresses which by-passes the heavy free-surface calculation is given in Section 20.

To simplify the analysis, it will be best to treat first the problem in which the plate speeds are equal in magnitude and opposite in sign ($\kappa=1$). The general case (any κ) is considered in Section 19.

14. Mathematical Formulation and the Rest Solution

We have chosen coordinates (r, θ, x) and the notation to make maximum use of the die-swell equations set out in Section 10. When gravity is neglected, the governing field equations (10.4a) hold for this problem. On the free surface, $r=h(x)$ the first three of equations (10.4c) and, in addition, $S_{n\theta}=0$, hold. The fluid adheres to the top and bottom plate. In the formulation which seems closest to physics, we require the fluid to be attached to the sharp edge of the disks:* $h(\pm d/2)=h_0$. Since the fluid is incompressible, the total volume of the fluid between the disks is invariant. It is essential in the analysis that the prescription of the total volume be that of a cylinder of fluid of radius h_0; that is,

$$\pi h_0^2 d = \pi \int_{-d/2}^{d/2} h^2(x)\, dx. \tag{14.1a}$$

The equations which govern the flow of a simple fluid between rotating disks, including effects of gravity, are

$$\rho \boldsymbol{U}\cdot\nabla\boldsymbol{U} + \nabla(P+\rho g x) - \nabla\cdot\boldsymbol{S} = 0 = \operatorname{div}\boldsymbol{U} \quad \text{in } \mathscr{V}_\omega, \tag{14.1b}$$

* In an alternate formulation we could require that the contact angle be fixed. The analysis could be carried out for the case in which the free and solid surfaces are perpendicular (neutral wetting). In experiments, the fluid shows strong affinity for sharp edges and does not satisfy a contact angle condition.

$$U(r, \pm d/2) = \pm e_\theta r\omega, \tag{14.1c}$$

$$\begin{aligned} V - h'U = S_{r\theta} - h' S_{x\theta} = h'(S_{rr} - S_{xx}) + (2 - \phi^2) S_{xr} \\ = S_{rr} - P + P_a - h' S_{xr} - \sigma J \\ = h(\pm d/2) - h_0 = 0 \quad \text{on } r = h(x). \end{aligned} \tag{14.1d}$$

In the rest state $\omega = 0$, the velocity and extra stresses all vanish,

$$P = -\rho g x + c_1 \tag{14.2a}$$

where g is the acceleration of gravity, c_1 is a constant and

$$\sigma J - \rho g x - P_a + c_1 = 0 \tag{14.2b}$$

where

$$J = \frac{h''}{(1+h'^2)^{\frac{3}{2}}} - \frac{1}{h(1+h'^2)^{\frac{1}{2}}} \quad \text{and} \quad h\left(\pm\frac{d}{2}\right) = h_0. \tag{14.2c}$$

The constant c_1 is determined by the constant volume condition (14.1a). Introducing the dimensionless variables

$$t = \frac{2x}{d}, \qquad H = \frac{2h}{d}, \tag{14.3}$$

we may rewrite (14.2) as

$$\frac{H''}{(1+H'^2)^{\frac{3}{2}}} - \frac{1}{H(1+H'^2)^{\frac{1}{2}}} - \varepsilon t + c_3 = 0, \tag{14.4a}$$

$$H(\pm 1) = \frac{2h_0}{d} \equiv H_0 \tag{14.4b}$$

where $c_3 = (c_1 - P_a)\, d/2\sigma$ is a constant,

$$\varepsilon = \frac{\rho g d^2}{4\sigma}$$

and

$$2H_0^2 = \int_{-1}^{1} H^2\, dt. \tag{14.4c}$$

The parameter $(\sigma/\rho g)^{\frac{1}{2}}$ is the capillary radius. For the fluid (STP) studied by JOSEPH, BEAVERS & FOSDICK (1973), $(\sigma/\rho g^{\frac{1}{2}} \sim (5.3\text{ cm})^{-1}$. Hence, for STP, $\varepsilon \ll 1$ when $d \ll 2/5.3$ cm. Small gaps of this size are typical in parallel plate viscometers.

When $\varepsilon = 0$, (14.4) determines a surface of constant mean curvature. The solution

$$H(t) \equiv H_0, \quad c_3 = \frac{1}{H_0} \tag{14.5}$$

is unique for almost all prescribed values of H_0 and d. Perturbing (14.5) with ε, we find

$$H''_{,1} + \frac{H_{,1}}{H_0^2} - t + c_{3,1} = 0, \tag{14.6a}$$

$$H_{,1}(\pm 1) = 0, \tag{14.6b}$$

$$\int_{-1}^{1} H_{,1}\, dt = 0 \tag{14.6c}$$

where

$$H_{,1} = \left.\frac{\partial H(t, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

The unique solution of problem (14.6c) is

$$H_{,1}(t) = H_0^2 \left\{ t - \frac{\sin t/H_0}{\sin 1/H_0} \right\}, \qquad c_{3,1} = 0. \tag{14.7}$$

For very large values of H_0 this becomes

$$H_{,1} = \tfrac{1}{6} t(t^2 - 1) \tag{14.8}$$

and

$$H(t) = H_0 + \frac{\varepsilon}{6} t(t^2 - 1) + O(\varepsilon^2). \tag{14.9}$$

In the analysis to follow we shall set $\varepsilon = 0$; this allows us to develop the solution of (14.1) as a perturbation (in ω) of the rest state in the convenient cylindrical domain $\mathscr{V}_0 (r < h_0)$. Equation (14.9) gives a simple expression for the static correction to the shape of the free surface.

15. The Perturbation Series

We now seek the solution of (14.1), with $g = 0$, as a power series in ω. It follows from the symmetry of the configuration that the free surface, the pressure distribution and the secondary motion ($V(r, z; \omega)$ and $U(r, z; \omega)$) all must be even functions of ω whereas the azimuthal component of velocity $W(r, z; \omega)$ must be an odd function of ω; thus, to leading order,

$$\begin{bmatrix} U \\ V \\ P - P_0 \\ h - h_0 \\ \omega W \end{bmatrix} = \begin{bmatrix} U^{\langle 2 \rangle} \\ V^{\langle 2 \rangle} \\ P^{\langle 2 \rangle} \\ h^{\langle 2 \rangle} \\ W^{\langle 1 \rangle} \end{bmatrix} \omega^2 + O(\omega^4) \tag{15.1}$$

where the superscript notation is as defined under (9.6) with differentiation with respect to ω understood.

The domain perturbation method to be used here is the one described immediately after (11.1) and, more fully, after (9.5). As in the die swell problem, the substantial derivative following the mapping onto a cylinder must be applied to equations on the free surface; and if $A(h(x), x) = 0$, then

$$A^{[n]} = \frac{1}{n!} \left(\frac{\partial}{\partial \omega} + h^{\langle 1 \rangle} \frac{\partial}{\partial r} \right)^n A(r, x)$$

with r set equal to h_0 after differentiation. The stresses are to be handled by the retardation expansions of Section 3. For later reference we note that

$$(\text{o})^{[2]}=(\text{o})^{\langle 2\rangle}+h^{\langle 1\rangle}\frac{\partial}{\partial r}(\text{o})^{\langle 1\rangle}+h^{\langle 2\rangle}\frac{\partial}{\partial r}(\text{o})^{\langle 0\rangle}+h^{\langle 1\rangle^2}\frac{1}{2}\frac{\partial^2}{\partial r^2}(\text{o})^{\langle 0\rangle}. \quad (15.2)$$

At first order we find that

$$\nabla P^{\langle 1\rangle}-\mu\nabla^2 U^{\langle 1\rangle}=0=\operatorname{div} U^{\langle 1\rangle} \quad \text{in } \mathscr{V}_0, \quad (15.3\text{a})$$

$$U^{\langle 1\rangle}\left(r, \pm\frac{d}{2}\right)=\pm e_\theta r, \quad (15.3\text{b})$$

$$V^{\langle 1\rangle}=S_{r\theta}^{\langle 1\rangle}=S_{xr}^{\langle 1\rangle}=0 \quad \text{on } r=h_0. \quad (15.3\text{c})$$

The solution of this problem is

$$U^{\langle 1\rangle}=e_\theta W^{\langle 1\rangle}=e_\theta\frac{2rx}{d} \quad (15.4\text{a})$$

and, using symmetry or by direct calculation using the normal stress condition arising from last two of equations (14.1d), we find that

$$S_{rr}^{\langle 1\rangle}=P^{\langle 1\rangle}=h^{\langle 1\rangle}=0. \quad (15.4\text{b})$$

For later use we evaluate the following quantities on the first-order solution:

$$u^{\langle 1\rangle}\cdot\nabla u^{\langle 1\rangle}=-e_r\frac{W^{\langle 1\rangle^2}}{r}=-e_r\frac{4rx^2}{d^2}, \quad (15.5\text{a})$$

$$A_1[u^{\langle 1\rangle}]^{*}=2\frac{r}{d}[e_\theta e_x+e_x e_\theta],$$

$$A_1\cdot A_1=4\frac{r^2}{d^2}[e_\theta e_\theta+e_x e_x],$$

$$u^{\langle 1\rangle}\cdot\nabla A_1=-\frac{4rx}{d^2}[e_r e_x+e_x e_r], \quad (15.5\text{b})$$

$$\nabla u^{\langle 1\rangle}=\frac{2}{d}[x(e_r e_\theta-e_\theta e_r)+r e_x e_\theta],$$

$$\nabla u^{\langle 1\rangle}\cdot A_1=\frac{4}{d^2}[xr e_r e_x+r^2 e_x e_x],$$

$$A_2=u\cdot\nabla A_1+\nabla u^{\langle 1\rangle}\cdot A_1+(\nabla u^{\langle 1\rangle}\cdot A_1)^T=\frac{8r^2}{d^2}e_x e_x, \quad (15.5\text{c})$$

$$S_2^{\langle 1\rangle}=\alpha_1 A_2+\alpha_2 A_1^2=\frac{4r^2}{d^2}[\alpha_2 e_\theta e_\theta+(2\alpha_1+\alpha_2)e_x e_x] \quad (15.5\text{d})$$

and

$$\nabla\cdot S_2^{\langle 1\rangle}=-\frac{4r}{d^2}\alpha_2 e_r. \quad (15.5\text{e})$$

* The bracket argument means the tensor A_1 is to be evaluated on the field $u^{\langle 1\rangle}$. We have not always used the explicit notation when the context makes the intent clear.

We note that $\operatorname{curl} \nabla \cdot S_2^{\langle 1 \rangle}=0$. It follows that at second order the forces in the interior of $\mathscr{V}_0$ are conservative, can be absorbed by the pressure, and do not induce motion. On the other hand, the centripetal acceleration field $W^{\langle 1 \rangle 2}/r$ is stratified vertically (depends on x) and, therefore, produces torques in azimuthal planes:

$$\operatorname{curl}(\boldsymbol{u}^{\langle 1 \rangle} \cdot \nabla \boldsymbol{u}^{\langle 1 \rangle}) = -8 \boldsymbol{e}_\theta r x/d^2.$$

These torques are responsible for the secondary motions when ω is small.

16. The Perturbation Problem at 2nd Order; a Cylindrical Edge Problem

The perturbation problem at second order is given by

$$\rho \boldsymbol{u}^{\langle 1 \rangle} \cdot \nabla \boldsymbol{u}^{\langle 1 \rangle} - \mu \nabla^2 \boldsymbol{u}^{\langle 2 \rangle} + \nabla P^{\langle 2 \rangle} - \nabla \cdot S_2^{\langle 1 \rangle} = 0 = \operatorname{div} \boldsymbol{u}^{\langle 2 \rangle} \text{ in } \mathscr{V}_0, \tag{16.1a}$$

and

$$\boldsymbol{u}^{\langle 2 \rangle}\left(r, \pm\frac{d}{2}\right) = 0 \tag{16.1b}$$

where $\boldsymbol{u}^{\langle 1 \rangle} \cdot \nabla \boldsymbol{u}^{\langle 1 \rangle}$ and $\nabla \cdot S_2^{\langle 1 \rangle}$ are given by (15.5a) and (15.5e). The computation of boundary conditions on the free surface is simplified by the fact that $h^{\langle 1 \rangle} \equiv 0$; then (15.2) may be written as

$$(\circ)^{[2]} = (\circ)^{\langle 2 \rangle} + h^{\langle 2 \rangle} \partial_r (\circ)^{\langle 0 \rangle} = (\circ)^{\langle 2 \rangle}$$

where the last equality follows from the fact that only constant fields arise on the rest solution

$$\partial_r (\circ)^{\langle 0 \rangle} = 0.$$

For the stresses at second order, we have

$$S^{\langle 2 \rangle} = S_1^{\langle 2 \rangle} + S_2^{\langle 1 \rangle} = \mu A_1 [\boldsymbol{u}^{\langle 2 \rangle}] + S_2^{\langle 1 \rangle}$$

where $S_2^{\langle 1 \rangle}$ is given by (15.5d). Equation (15.5d) shows that the rr, $r\theta$, rx and $x\theta$ components of $S_2^{\langle 1 \rangle}$ all vanish. Moreover,

$$J^{\langle 2 \rangle} = h^{\langle 2 \rangle \prime\prime} + h^{\langle 2 \rangle}/h_0^2.$$

On the surface $r = h_0$ we must satisfy:

$$V^{\langle 2 \rangle} = \mu\left(\frac{\partial V^{\langle 2 \rangle}}{\partial x} + \frac{\partial U^{\langle 2 \rangle}}{\partial r}\right) = 0,$$
$$2\mu \frac{\partial V^{\langle 2 \rangle}}{\partial r} - P^{\langle 2 \rangle} - \sigma[h^{\langle 2 \rangle \prime\prime} + h^{\langle 2 \rangle}/h_0^2] = h^{\langle 2 \rangle}\left(\pm\frac{d}{2}\right) = 0. \tag{16.1c}$$

In addition, the condition of constant volume (14.1a) requires that the mean value

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} h^{\langle 2 \rangle}(x)\, dx = 0. \tag{16.1d}$$

The azimuthal components of velocity ($W^{\langle 2 \rangle}$) and stress all vanish at even orders.

To solve the problem (16.1) we first reduce it to the cylindrical equivalent of an edge problem. The secondary flow is given by a stream function $\psi^{\langle 2\rangle}$

$$\mu(V^{\langle 2\rangle}, U^{\langle 2\rangle}) = \frac{1}{r}\left(\frac{\partial \psi^{\langle 2\rangle}}{\partial x}, -\frac{\partial \psi^{\langle 2\rangle}}{\partial r}\right). \tag{16.1e}$$

Introducing dimensionless variables,

$$(t, y) = \frac{2}{d}(x, r),$$

$$\Phi(t, y) = \frac{1}{480} t(t^2-1)^2 y^2 + \frac{4}{\rho d^5} \psi^{\langle 2\rangle}(x, r), \tag{16.1f}$$

$$p(t, y) = P^{\langle 2\rangle}/2\rho d^2,$$

$$\hbar(t) = \frac{2\sigma}{\rho d^4} h^{\langle 2\rangle}(x),$$

$$\gamma_i = \frac{\alpha_i}{\rho d^2},$$

we find that

$$-\nabla^2\left(\frac{1}{y}\frac{\partial \Phi}{\partial t}\right) + \frac{1}{y^3}\frac{\partial \Phi}{\partial t} + \frac{\partial p}{\partial y} + \left[\frac{\gamma_2}{2} - \frac{1}{40}\right] y = 0 \tag{16.2a}$$

and

$$\nabla^2\left(\frac{1}{y}\frac{\partial \Phi}{\partial y}\right) + \frac{\partial p}{\partial t} + \frac{t}{20} - \frac{t^3}{12} = 0 \tag{16.2b}$$

in $\mathscr{V}_0$,

$$\Phi = \frac{\partial \Phi}{\partial t} = 0 \tag{16.2c}$$

on $t = \pm 1$, and

$$\Phi - \frac{H_0^2}{480} t(t^2-1)^2 = H_0 \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial \Phi}{\partial y}\right) = 0, \tag{16.2d}$$

$$\hbar'' + \hbar/H_0^2 = -\frac{1}{240}(5t^4 - 6t^2 + 1) - p + 2\frac{\partial^2}{\partial y \, \partial t}(\Phi/y) \tag{16.2e}$$

on $y = H_0$. To conserve the mass of fluid between the disks, it is also necessary that

$$\int_{-1}^{1} \hbar(t)\, dt = 0. \tag{16.2f}$$

Elimination of p between (16.2a) and (16.2b) gives

$$\mathscr{L}^2 \psi = 0 \quad \text{in } \mathscr{V}_0 \tag{16.2g}$$

where

$$\mathscr{L} = \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial t^2}.$$

For later use we note that

$$-\nabla^2 \frac{1}{y}\frac{\partial \Phi}{\partial t} + \frac{1}{y^3}\frac{\partial \Phi}{\partial t} = -\frac{1}{y}\mathscr{L}\frac{\partial \Phi}{\partial t}. \tag{16.3}$$

17. Solution of the Cylindrical Edge Problem

The problem defined by (16.2c, d, g) is related to an edge problem in the classical theory of plane elasticity. In the elasticity theory one considers a semi-infinite strip clamped on the long side and loaded with stresses on the top. This classical problem is not generally well understood; for certain special conditions on the edge, however, the edge problem can be solved along the lines laid out in the excellent paper of SMITH (1952). Of course (16.2c, d, g) is not an edge problem of classical elasticity. Nevertheless, SMITH's work can be extended to (16.2c, d, g); this extension leads directly to eigenfunction expansions in the same set of Papkovich-Fadle functions as arise in the elasticity edge problem in the semi-infinite strip. The equivalence of the elasticity edge problem and (16.4) can be intuitively grasped by noting that (16.4) does define an edge problem in which the edge goes all the way around; the observer on the edge of a disk of large radius sees a flat edge in which the disk center is off at infinity.

Let $\Phi_c=\Phi+i\Phi_i$ where $\Phi(t, y)$ and $\Phi_i(t, y)$ are real-valued functions of t and y, for $-1\leqq t\leqq 1$, $0\leqq y\leqq H_0$, which satisfy

$$\mathscr{L}^2\Phi_c=0 \quad \text{in } \mathscr{V}_0 \tag{17.1a}$$

where

$$\frac{\partial^2}{\partial y^2}-\frac{1}{y}\frac{\partial}{\partial y}+\frac{\partial^2}{\partial t^2}=y\frac{\partial}{\partial y}\frac{1}{y}\frac{\partial}{\partial y}+\frac{\partial^2}{\partial t^2}$$

and

$$\Phi_c=\frac{\partial\Phi_c}{\partial t}=0 \quad \text{at } t=\pm 1. \tag{17.1b}$$

The real part of the second derivatives of Φ_c are prescribed on the edge of $\mathscr{V}_0$, $y=H_0$:

$$\begin{bmatrix} f_r \\ g_r \end{bmatrix}=\begin{bmatrix} H_0\dfrac{\partial}{\partial y}\dfrac{1}{y}\dfrac{\partial}{\partial y}\Phi \\ \dfrac{\partial^2\Phi}{\partial t^2} \end{bmatrix}=H_0^2\begin{bmatrix} 0 \\ \dfrac{1}{8}\left(\dfrac{t^3}{3}-\dfrac{t}{5}\right) \end{bmatrix}. \tag{17.1c}$$

The prescribed function $g_r(t)$ is required to satisfy compatibility conditions

$$0=\int_{-1}^{1} g_r(t)\,dt=\int_{-1}^{1} t\,g_r(t)\,dt. \tag{17.1c}$$

The solution of (17.1) can be found in a "Fourier" series of Papkovich-Fadle eigenfunctions. The odd functions with eigenvalues P_n are given by (17.9a); the even functions with eigenvalues S_n are given by (19.7a). The solution of (17.1) can be expressed in the odd functions alone because the given data (17.1c) is odd. When the given data is even, formulas (17.2) through (17.8) still hold with P_n replaced by S_n and with $\phi_1^{(n)}$, $\Phi_2^{(n)}$, $\psi_1^{(n)}$ and $\psi_2^{(n)}$ given by (17.9a) replaced by $\hat{\phi}_1^{(n)}$, $\hat{\Phi}_2^{(n)}$, $\hat{\psi}_1^{(n)}$ and $\hat{\psi}_2^{(n)}$ given by (19.7a).

The solution of (17.1) can be found from the series

$$\Phi_c=\sum_{n=1}\frac{c_n}{P_n^2}\phi_1^{(n)}(t)\,F(P_n y) \tag{17.2a}$$

and

$$\begin{bmatrix} f_r \\ g_r \end{bmatrix} = \frac{1}{2} \sum_{n=1} \left[c_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix} + \bar{c}_n \begin{bmatrix} \bar{\phi}_1^{(n)}(t) \\ \bar{\phi}_2^{(n)}(t) \end{bmatrix} \right]. \tag{17.2b}$$

Here c_n are constants given by (17.8),

$$F(P_n y) \equiv \frac{y}{H_0} \mathscr{I}_1(P_n y) = \frac{y}{H_0} \frac{I_1(P_n y)}{I_1(P_n H_0)},$$

where $I_1(P_n y)$ is the modified Bessel function of order 1 and $F(y)$ satisfies the differential equation

$$F'' - \frac{1}{y} F' - P^2 F = 0, \qquad F(H_0) = 1 \tag{17.3a}$$

which can be written as

$$\mathscr{L} F = P^2 F. \tag{17.3b}$$

From the basic differentiation formula for $I_1(y)$,

$$\frac{d}{dy}(y I_1(y)) = y I_0(y).$$

We note, for later use, that

$$\frac{d}{dy} F(P_n y) = \frac{P_n y}{H_0} \frac{I_0(P_n y)}{I_1(P_n H_0)} \equiv \frac{P_n y}{H_0} \mathscr{I}_0(P_n y) \tag{17.3c}$$

and

$$\frac{d}{dy}\left(\frac{F(P_n y)}{y}\right) = \frac{P_n}{H_0} \mathscr{I}_0(P_n y) - \frac{\mathscr{I}_1(P_n y)}{y H_0}. \tag{17.3d}$$

To determine the constants c_n, we introduce the vectors

$$f_r = \begin{bmatrix} f_r \\ g_r \end{bmatrix}, \quad \phi^{(n)} = \begin{bmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{bmatrix}, \quad \psi^{(n)} = \begin{bmatrix} \psi_1^{(n)} \\ \psi_2^{(n)} \end{bmatrix}$$

where $\phi^{(n)}$ and $\psi^{(n)}$ may be defined through the differential equations

$$\frac{d^2}{dt^2} \phi^{(n)} + P_n^2 A \cdot \phi^{(n)} = 0, \tag{17.4a}$$

$$\frac{d^2}{dt^2} \psi^{(m)} + P_m^2 A^T \cdot \psi^{(m)} = 0 \tag{17.4b}$$

where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

and the P_n^2 are chosen so that

$$\phi_1^{(n)} = \frac{d\phi_1^{(n)}}{dt} = \psi_2^{(n)} = \frac{d\psi_2^{(n)}}{dt} = 0 \quad \text{at } t = \pm 1. \tag{17.5a, b}$$

Using (17.4) and (17.5), we find that

$$\int_{-1}^{1} \psi^{(m)} \cdot A \cdot \phi^{(n)} \, dt = k_n \delta_{nm} \tag{17.6}$$

and, for all m and n,

$$\int_{-1}^{1} \psi^{(m)} \cdot A \cdot \bar{\phi}^{(n)} \, dt = 0. \tag{17.7}$$

Applying the biorthogonality conditions (17.6) and (17.7) to (17.2b), we find that

$$c_n = \frac{2}{k_n} \int_{-1}^{1} \psi^{(n)} \cdot A \cdot f_r \, dt = \frac{2}{k_n} \int_{-1}^{1} [\psi_1^{(n)}, \psi_2^{(n)}] \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} f_r \\ g_r \end{bmatrix} dt. \tag{17.8}$$

The Papkovich-Fadle eigenfunctions separate into even and odd functions. The odd functions are given by

$$\begin{aligned} \phi_1 &= P \cos P \sin Pt - Pt \sin P \cos Pt = \psi_2, \\ \phi_2 &= -(P\cos P - 2\sin P) \sin Pt + Pt \sin P \cos Pt, \\ \psi_1 &= (P\cos P + 2\sin P) \sin Pt - Pt \sin P \cos Pt, \\ &\sin 2P = 2P (P \neq 0), \; k = -4 \sin^4 P. \end{aligned} \tag{17.9a}$$

The first ten eigenvalues in the first quadrant of the complex P plane are given by HILLMAN & SALZER (1943); the first three eigenvalues are

$$\begin{aligned} P_1 &= 3.748838 + i\,1.384339, \\ P_2 &= 6.949980 + i\,1.676105, \\ P_3 &= 10.119259 + i\,1.858384. \end{aligned} \tag{17.9b}$$

For practical purposes the series which we shall give (see Table 1) converge in three terms so that three eigenvalues suffice for explicit representation of the practical solution. To establish mathematical convergence we shall use asymptotic forms for large values of n

$$2P_n \to (2n + \tfrac{1}{2})\pi + i \log(4n+1)\pi \tag{17.9c}$$

and for $t > 0$,

$$\sin P_n t = \tfrac{1}{2} [(4n+1)\pi]^{\frac{t}{2}} e^{-i(n+\frac{1}{4})\pi t} + O(n^{-\frac{t}{2}}). \tag{17.9d}$$

We are now ready to compute the coefficients c_n. We apply (17.8) to (17.1c) and use (17.4b) to find that

$$\begin{aligned} k_n c_n &= \frac{H_0}{4} \int_{-1}^{1} \left[\frac{t^3}{3} - \frac{t}{5}\right] (2\psi_2^{(n)} - \psi_1^{(n)}) \, dt \\ &= \frac{-H_0^2}{4P_n^2} \int_{-1}^{1} \left[\frac{t^3}{3} - \frac{t}{5}\right] \frac{d^2 \psi_2^{(n)}}{dt^2} \, dt = -\frac{H_0^2}{2P_n^2} \int_{-1}^{1} t \psi_2^{(n)} \, dt \\ &= -\frac{2H_0^2}{P_n^2} \tan^2 P_n, \end{aligned} \tag{17.10}$$

$$c_n = H_0^2 / 2P_n^2 \sin^2 P_n \cos^2 P_n = H_0^2 / 2P_n^4. \tag{17.11}$$

Table 1. *Convergence of the Papkovich-Fadle series*

$$\begin{bmatrix} f \\ g \end{bmatrix} = \frac{100}{4} \begin{bmatrix} 0 \\ \frac{t^3}{3} - \frac{t}{5} \end{bmatrix} \sim \mathrm{re} \sum_{l=1}^{N} \frac{100}{P_n^4} \begin{bmatrix} \phi_1^{(l)}(t) \\ \phi_2^{(l)}(t) \end{bmatrix}$$

t	$g(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
1	3.333	2.772	3.252	3.331	3.322	3.327
0.8	0.267	0.480	0.204	0.254	0.289	0.277
0.6	−1.200	−1.221	−1.115	−1.216	−1.199	−1.194
0.4	−1.467	−1.579	−1.485	−1.445	−1.478	−1.463
0.2	−0.933	−0.977	−0.930	−0.947	−0.940	−0.93
0	0	0	0	0	0	0
t	$f(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
1	0	0	0	0	0	0
0.8	0	0.2025	0.0953	−0.0102	−0.0295	−0.0065
0.6	0	0.2377	−0.0794	0.0269	−0.0046	−0.0046
0.4	0	0.0037	0.0419	−0.0258	0.0118	−0.0031
0.2	0	−0.1237	−0.0143	0.0144	0.0086	−0.0015
0	0	0	0	0	0	0

Hence,

$$\frac{1}{4} \begin{bmatrix} 0 \\ \frac{t^3}{3} - \frac{t}{5} \end{bmatrix} = \mathrm{re} \sum_{n=1} \frac{1}{P_n^4} \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix} \tag{17.12a}$$

and

$$\Phi_c = \frac{1}{2} H_0^2 \sum_{n=1} \frac{1}{P_n^6} \phi_1^{(n)}(t)\, F(P_n y). \tag{17.12b}$$

Numerical convergence of the partial sums

$$\mathrm{re} \sum_{n=1}^{N} \frac{1}{P_n^4} \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix} \to \frac{1}{4} \begin{bmatrix} 0 \\ \frac{t^3}{3} - \frac{t}{5} \end{bmatrix}$$

is demonstrated in Table 1. This type of convergence is representative of the series which are involved in the computations to follow. The given functions are adequately represented when $N=3$. Mathematical convergence of the series (17.12a) and (17.12b) follows from the asymptotic representations (17.9c) and (17.9d). These representations show that (17.12a) is dominated term by term by c/n^2 for a certain constant c. The series (17.12b) is dominated term by term by c/n^4. The solution (17.12b) is an edge solution of the Saint-Venant type; the modified Bessel functions $F(P_n y)$ decay exponentially from the edge (see 18.3). In the interior, $y<H_0$, the solution (17.12) rapidly approaches the function given by $N=1$. The stream function for the second-order problem consists in these

edge eddies and the big eddy given by (16.1f):

$$\frac{4}{\rho d^5}\psi^{\langle 2\rangle}(x,r)=\operatorname{re}\Phi_c(t,y)-\frac{1}{480}t(t^2-1)^2y^2. \qquad (17.13)$$

To complete the solution we turn next to equation (16.2e) which governs the free surface. We need expressions for the quantity $-P+2\partial^2(\Phi/y)/\partial t\,\partial y$. The pressure may be obtained from (16.2b) using (17.12b). We find that

$$\nabla^2\left(\frac{1}{y}\frac{\partial\Phi}{\partial y}\right)=\frac{1}{y}\frac{\partial}{\partial y}\mathscr{L}\Phi$$

and

$$\begin{aligned}P+\frac{t^2}{40}-\frac{t^4}{48}+\left(\frac{\gamma_2}{4}-\frac{1}{80}\right)y^2+c_4&=-\int\frac{1}{y}\frac{\partial}{\partial y}\mathscr{L}\Phi\,dt\\&=H_0\operatorname{re}\sum_{n=1}\frac{1}{P_n^4}\sin P_n\cos P_n t\,\mathscr{I}_0(P_n y)\end{aligned} \qquad (17.14)$$

where c_4 is a constant to be determined. We also find that

$$\begin{aligned}\partial^2_{ty}\left(\frac{\Phi}{y}\right)=\frac{H_0}{2}\operatorname{re}\sum_{n=1}\frac{\sin^2 P_n}{P_n^5}\left[P_n\mathscr{I}_0(P_n y)-\frac{\mathscr{I}_1(P_n y)}{y}\right]\\\cdot[t\cos P_n\sin P_n t-\cos P_n t\sin P_n].\end{aligned} \qquad (17.15)$$

We next use (17.14) and (17.15) evaluated at $y=H_0$ to rewrite (16.2e) as

$$\begin{aligned}\mathscr{h}''+\mathscr{h}/H_0^2=&-\frac{t^4}{24}+\frac{t^2}{20}+c_5\\&-\operatorname{re}\sum_{n=1}\frac{\sin P_n}{P_n^5}[P_n t\sin P_n t-\sin^2 P_n\cos P_n t]\\&+H_0\operatorname{re}\sum_{n=1}\frac{\sin P_n}{P_n^4}\mathscr{I}_0(P_n H_0)\\&\cdot[P_n t\sin P_n t-(1+\sin^2 P_n)\cos P_n t]\end{aligned} \qquad (17.16)$$

where

$$c_5=c_4+\frac{1}{240}+\left(\frac{\gamma_2}{4}-\frac{1}{80}\right)H_0^2.$$

Equation (17.16) is to be solved subject to the conditions that $\mathscr{h}(\pm 1)=0$. The constant c_5 is then determined by (16.2f). It is of interest and is very easy to solve (17.16). Our present purpose, however, is better served by considering the simpler forms which arise when H_0 is large; this limit is typical for the parallel plate viscometers which will be used to compare this theory with experiments.

18. The Secondary Motion and the Shape of the Free Surface

Equations (17.13b) and (17.16) simplify when H_0 is large. To simplify, we first note that the large argument expansion of the modified Bessel functions gives

$$I_l(P_n y)\sim e^{P_n y}/\sqrt{2\pi P_n y} \qquad 0<\arg P_n y<\tfrac{1}{2}\pi \quad \text{and all } l>0.$$

Table 2. *Free surface correction at 2nd order* (see Eqs.* (18.5), (19.4) *and* (20.2))

t	−1.0	−0.9	−0.8	−0.7	−0.6	−0.5	−0.4	−0.3	−0.2	−0.1	0
$10^3 h/H_0$	0	0.44	0.67	0.68	0.52	0.24	−0.09	−0.43	−0.71	−0.89	−0.958
$10^3 \hat{h}/H_0$	0	−1.95	−3.49	−4.54	−5.08	−5.12	−4.70	−3.89	−2.77	−1.44	0
$10^3 \tilde{h}/H_0$	0	−1.51	−2.82	−3.86	−4.56	−4.88	−4.79	−4.32	−3.48	−2.33	−0.958
t	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	
$10^3 h/H_0$	0	0.44	0.67	0.68	0.52	0.24	−0.09	−0.43	−0.71	−0.89	
$10^3 \hat{h}/H_0$	0	1.95	3.49	4.54	5.08	5.12	4.70	3.89	2.77	1.44	
$10^3 \tilde{h}/H_0$	0	2.39	4.10	5.22	5.60	5.31	4.61	3.46	2.06	0.55	

It follows that

$$\frac{y}{H_0}\frac{I_l(P_n y)}{I_1(P_n H_0)} \sim \sqrt{\frac{y}{H_0}}\, e^{-P_n(H_0-y)}. \tag{18.1}$$

Introducing $z=H_0-y\geqq 0$, we may further simplify (18.1):

$$\frac{y}{H_0}\frac{I_l(P_n y)}{I_1(P_n H_0)} \sim e^{-P_n z}. \tag{18.2}$$

Equation (18.2) differs from (18.1) by terms of order z/H_0 and the formulas of this section hold asymptotically for large $y \leqq H_0$ and $z/H_0 \ll 1$.

Equation (18.2) implies that

$$\Phi \sim \frac{1}{2} H_0^2 \,\mathrm{re} \sum_{n=1} \frac{1}{P_n^6} e^{-P_n z} \phi_1^{(n)}(t). \tag{18.3}$$

Asymptotically, as H_0 becomes large, the free surface formula (17.16) becomes

$$h'' \sim H_0 \,\mathrm{re} \sum_{n=1} \frac{\sin P_n}{P_n^4} [P_n t \sin P_n t - (1+\sin^2 P_n)\cos P_n t] + c_5. \tag{18.4}$$

Integrating (18.4), we find that the solution which vanishes at $t=\pm 1$ and has a zero mean value in the sense of (16.2f) is in the form

$$\begin{aligned} h \sim -H_0 \,\mathrm{re} \sum_{n=1} \frac{1}{P_n^5} \Big[& t \sin P_n \sin P_n t - \sin^2 P_n + \frac{3}{2}\frac{\sin^4 P_n}{\cos^2 P_n}(t^2-1) \\ & + \cos P_n \cos P_n t - \cos^2 P_n + \frac{3}{2}(1-\cos^2 P_n)(t^2-1) \Big] \\ = -H_0 \,\mathrm{re} \sum_{n=1} & \frac{1}{P_n^5} [t \sin P_n \sin P_n t + \cos P_n \cos P_n t - 1 + \tfrac{3}{2} \tan^2 P_n (t^2-1)] \end{aligned} \tag{18.5}$$

where

$$c_5 = -3H_0^2 \,\mathrm{re} \sum \frac{\tan^2 P_n}{P_n^5}.$$

The function h/H_0 is tabulated in Table 2 and is sketched in Fig. 9.

The level lines of the function $\Phi(t, y)$ in the reference domain $\mathscr{V}_0[y, t \mid 0<y\leqq H_0]$ are shown in Fig. 6. To express Φ in the deformed domain $\mathscr{V}_\omega$, we may use the

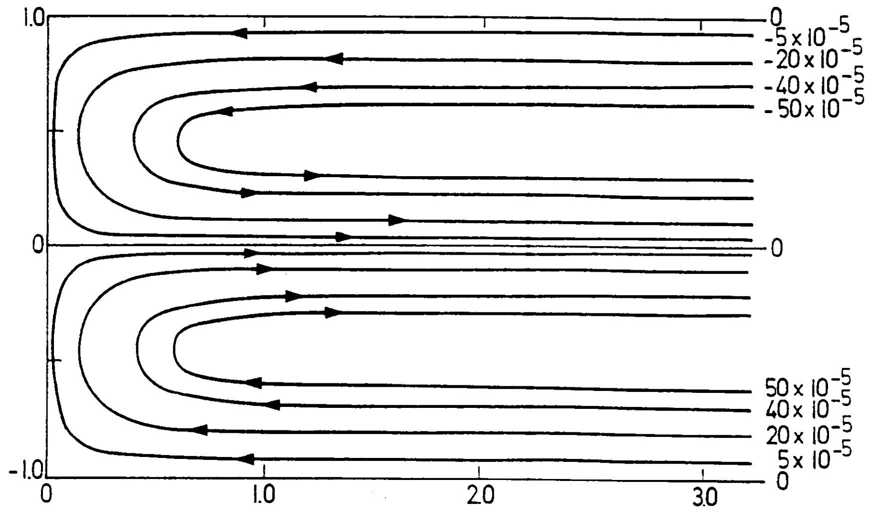

Fig. 5. Level lines of the stream function

$$[\Phi - y^2\, t(t^2-1)^2]/H_0^2$$

when the top and bottom disks have equal and opposite rotational speeds. The figure applies when H_0 is large and $y/H_0 \simeq 1$. The flow is very quickly dominated by the large eddy (the term proportional to y^2) and the cellular structure of the edge eddy Φ is not apparent.

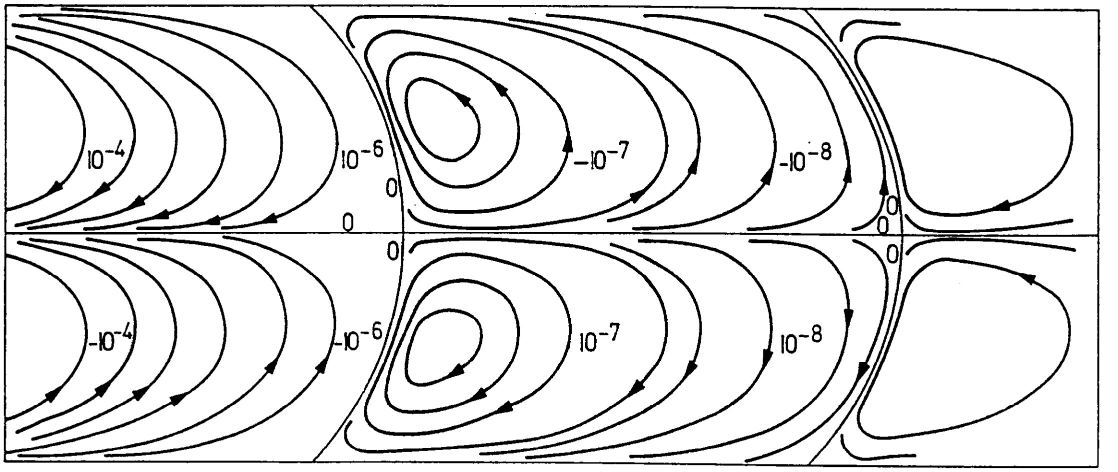

Fig. 6. Level lines of the edge eddies Φ/H_0^2 for when the top and bottom disks have equal and opposite rotational speeds.

prescription following (9.6) and replace $y(0<y\leqq H_0)$ with $y(0<y\leqq H(t))$; however, it is better to evaluate the series of total derivatives in the deformed domain by inverting the stretch mapping $(y, t)\to(y-H(t,\varepsilon), t)$.* The function $\Phi(y, t)$ describes a double array of edge eddies. The most persistent of these is associated with the eigenvalue P_1 whose real part is smallest. For $z>0$

$$\Phi \to \frac{1}{2} H_0^2 \,\mathrm{re}\left[\frac{1}{P_1^6} e^{-3.75z} e^{-i1.38z} \phi_1^{(1)}(t)\right]. \tag{18.6}$$

* This aspect of the theory of domain perturbations is discussed in the paper of JOSEPH & STURGES (1974).

This function oscillates as it decays; the period of its oscillation, the wave length of a cell pair, is $T=z_T\simeq 2\pi/1.38$. This is a period of about $2\frac{1}{4}$ times the plate separation. The field (18.6) is therefore composed of concentric cells with a radial spacing of the order of the plate separation.

In the flow, the edge eddies, described by the function Φ, are dominated by the big eddy at the end of (18.7)

$$\frac{4\psi^2}{\rho d^5}=\Phi(t,y)-\frac{y^2 t(t^2-1)^2}{480}. \tag{18.7}$$

The big eddy is not an edge eddy; it goes all the way to $y=0$ and is driven by torques associated with vertical stratification of centripetal acceleration.

The level lines of the function $4\psi^{\langle 2\rangle}/\rho H_0^2 d^5$ in the region where $y\leqq H_0$ is large enough so that we may replace y^2/H_0^2 by 1 are sketched in Fig. 5.

19. The Secondary Motion and the Shape of the Free Surface when the Disk Speeds are Arbitrary

The problem solved in Sections 17 and 18 is very slightly simpler than the general problem to which we now turn. In this configuration we allow arbitrary speeds for the two plates; the problem splits into two parts: the part which we just solved and another part which involves the even Papkovich-Fadle functions (19.7a). In the general problem the difference between the angular velocity of the top and bottom disk is 2ω; the bottom disk rotates with an angular velocity of $2\kappa\,\omega/(1-\kappa)$ and the top disk rotates with an angular velocity of $2\omega/(1-\kappa)$. We designate the dependent variables in the general problem with a tilde overbar; then (14.1a, b, d) is written in the variables $(\tilde{U}, \tilde{P}, \tilde{S}, \tilde{h}, \tilde{\phi}, \tilde{J})$ and (14.1c) is replaced by

$$\tilde{U}(r,d/2;\omega)=e_\theta 2\omega r/(1-\kappa),\qquad \tilde{U}(r,-d/2;\omega)=e_\theta 2\kappa\omega r/(1-\kappa) \tag{19.1}$$

where κ is any preassigned real number. As in (15.1) the secondary motion, pressure, and free surface are even functions of ω and the azimuthal velocity is an odd function of ω. We shall find a solution in the form

$$\begin{bmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{P}-P_0\\ \tilde{h}-h_0\\ \omega\tilde{W}\end{bmatrix}=\begin{bmatrix}U^{\langle 2\rangle}\\ V^{\langle 2\rangle}\\ P^{\langle 2\rangle}\\ h^{\langle 2\rangle}\\ W^{\langle 1\rangle}\end{bmatrix}\omega^2+\frac{1+\kappa}{1-\kappa}\begin{bmatrix}\hat{U}^{\langle 2\rangle}\\ \hat{V}^{\langle 2\rangle}\\ \hat{P}^{\langle 2\rangle}\\ \hat{h}^{\langle 2\rangle}\\ r\end{bmatrix}\omega^2+O(\omega^4) \tag{19.2}$$

where $U^{\langle 2\rangle}$, $V^{\langle 2\rangle}$, *etc.* are the solutions of (14.1a, b, c, d)

$$P_0=P_a+\frac{2\sigma}{dH_0}$$

and the roof variables are to be determined.

At first order $\hat{U}^{\langle 1\rangle}=\tilde{V}^{\langle 1\rangle}=\tilde{h}^{\langle 1\rangle}=0$ and

$$\tilde{W}^{\langle 1\rangle}=W^{\langle 1\rangle}+\tau r;\qquad \tau\equiv\frac{1+\kappa}{1-\kappa},\qquad W^{\langle 1\rangle}=2rx/d \tag{19.3}$$

satisfies the prescribed condition (19.1) and the differential equations (15.3a). The tilde overbar stresses are independent of the rigid rotation τr and the expressions (15.2) hold for tilde overbar stresses. However, equation (15.5a) becomes

$$\begin{aligned}\tilde{\boldsymbol{u}}^{\langle 1\rangle}\cdot\nabla\tilde{\boldsymbol{u}}^{\langle 1\rangle}&=-\boldsymbol{e}_r\left[\frac{4rx^2}{d^2}\right]-\boldsymbol{e}_r\tau\left[\frac{4xr}{d}+\tau r\right]\\&=\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla\boldsymbol{u}^{\langle 1\rangle}-\boldsymbol{e}_r\tau\left[\frac{4xr}{d}+\tau r\right].\end{aligned}\tag{19.4}$$

It follows that the second-order problem is the same as (16.1) except for the new terms which arise from the previously absent two terms on the right side of (19.4). All three terms arise as centripetal accelerations, but only the first two terms are stratified vertically (depend on x) and produce torques leading to secondary motion. The term which is independent of x can be absorbed in the pressure. The new term $4x\,r/d$ has a larger absolute value than the term $4x^2\,r/d^2$ and it induces a larger torque, more intense secondary motion, and greater changes in the shape of the free surface.

The equations which govern the tilde overbar variables at second order are (16.1a, b, c, d, e). This problem may then be split into parts as in (19.2); the first part of the problem is the one solved in Section 17. The second part is generated by the second two terms of (19.4) and is designated with a roof overbar. The variables $(\hat{\boldsymbol{u}}^{\langle 2\rangle}, \hat{P}^{\langle 2\rangle}, \hat{h}^{\langle 2\rangle})$ are governed by (16.1a, b, c, d, e) with

$$\hat{S}_2^{\langle 1\rangle}=0 \quad\text{and}\quad \hat{\boldsymbol{u}}^{\langle 1\rangle}\cdot\nabla\hat{\boldsymbol{u}}^{\langle 1\rangle}=-\boldsymbol{e}_r\left(\frac{4xr}{d}+\tau r\right).$$

In dimensionless variables

$$\begin{aligned}(t,y)&=\frac{2}{d}(x,r),\\ \hat{\Phi}(t,y)&=\frac{y^2}{192}(t^2-1)^2+\frac{4}{\rho d^5}\hat{\psi}^{\langle 2\rangle}(x,r),\\ \hat{P}(t,y)&=\frac{\hat{p}^{\langle 2\rangle}}{2\rho d^2},\\ \hat{\mathscr{h}}(t)&=\frac{2\sigma}{\rho d^4}\hat{h}^{\langle 2\rangle};\end{aligned}\tag{19.5}$$

we find that in $\mathscr{V}_0$

$$-\nabla^2\left(\frac{1}{y}\frac{\partial\hat{\Phi}}{\partial t}\right)+\frac{1}{y^3}\frac{\partial\hat{\Phi}}{\partial t}+\frac{\partial\hat{P}}{\partial y}=\tau\frac{y}{8},\tag{19.6a}$$

$$\nabla^2\left(\frac{1}{y}\frac{\partial\hat{\Phi}}{\partial y}\right)+\frac{\partial\hat{P}}{\partial t}-\left(\frac{t^2}{8}-\frac{t}{24}\right)=0,\tag{19.6b}$$

$$\mathscr{L}^2\hat{\Phi}=0\tag{19.6c}$$

and

$$\hat{\Phi}=\frac{\partial\hat{\Phi}}{\partial t}=0\quad\text{on } t=\pm1.\tag{19.6d}$$

On the edge $y=H_0$, $\hat{\Phi}=\frac{H_0^2}{192}(t^2-1)^2$ and the edge data is

$$\begin{bmatrix} f_r \\ g_r \end{bmatrix}=\begin{bmatrix} H_0 \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial \hat{\Phi}}{\partial y}\right) \\ \partial^2 \hat{\Phi}/\partial t^2 \end{bmatrix}=\frac{H_0^2}{16}\begin{bmatrix} 0 \\ t^2-\frac{1}{3} \end{bmatrix}. \tag{19.6e}$$

The free surface equation is

$$\hat{h}''+\frac{\hat{h}}{H_0^2}=-\frac{1}{96}\frac{d}{dt}(t^2-1)^2-P+2\frac{\partial^2(\hat{\Phi}/y)}{\partial t\,\partial y} \tag{19.6f}$$

where the last two terms of (19.6g) are evaluated at $y=H_0$ and

$$\hat{h}(\pm 1)=0 \quad \text{and} \quad \int_{-1}^{1} \hat{h}(t)=0. \tag{19.6g}$$

The edge problem is solved as in Section 17. Since the data (19.6e) is even we shall need the even Papkovich-Fadle functions:

$$\begin{aligned} \hat{\phi}_1 &= S\sin S\cos St - St\cos S\sin St=\hat{\psi}_2, \\ \hat{\phi}_2 &= -(S\sin S+2\cos S)\cos St+St\cos S\sin St, \\ \hat{\psi}_1 &= S\sin S\cos St-St\cos S\sin St, \\ \sin 2S &= -2S\,(S\neq 0), \qquad k=-4\cos^4 S. \end{aligned} \tag{19.7a}$$

The first ten eigenvalues in the first quadrant of the complex S plane have been given by Robbins & Smith (1943). The first three of these eigenvalues are

$$\begin{aligned} S_1 &= 2.106196+i\,1.125365, \\ S_2 &= 5.356269+i\,1.551575, \\ S_3 &= 8.536683+i\,1.775544. \end{aligned}$$

For practical purposes the series given in this paper have converged after three terms (see Table 3). Mathematical convergence, demonstrated below, requires that we give the large n forms of (19.7a):

$$2S_n=(2n+\tfrac{3}{2})\pi+i\log(4n+3)\pi$$

and for $t>0$

$$\sin S_n t=\frac{i}{2}\left[(4n+3)\pi\right]^{\frac{t}{2}}e^{-i(n+\frac{3}{4})\pi t}+O\!\left(n^{-\frac{t}{2}}\right). \tag{19.8}$$

The coefficients c_n for the series solution (17.2) may be computed from (17.8):

$$\begin{aligned} k_n c_n &= \frac{H_0^2}{8}\int_{-1}^{1}(t^2-\tfrac{1}{3})(2\hat{\psi}_2^{(n)}-\hat{\psi}_1^{(n)})\,dt \\ &= -\frac{H_0^2}{4S_n^2}\int_{-1}^{1}\hat{\psi}_2^{(n)}\,dt=-\frac{H_0^2}{S_n^2}, \end{aligned}$$

and

$$c_n=\frac{H_0^2}{4S_n^2}\cos^4 S_n. \tag{19.9}$$

Table 3. *Convergence of the Papkovich-Fadle series*

$$\begin{bmatrix} f \\ g \end{bmatrix} \equiv \frac{1000}{32} \begin{bmatrix} 0 \\ t^2-\frac{1}{3} \end{bmatrix} \sim \mathrm{re} \sum_{l=1}^{N} \frac{1000}{8S_l^2 \cos^4 S_l} \begin{bmatrix} \hat{\phi}_1^{(l)}(t) \\ \hat{\phi}_2^{(l)}(t) \end{bmatrix}$$

t	$g(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
1	2.083	1.948	2.070	2.079	2.081	2.083
0.8	0.958	0.999	0.950	0.955	0.961	0.960
0.6	0.083	0.110	0.863	0.836	0.818	0.848
0.4	−0.541	−0.548	−0.588	−0.540	−0.543	−0.541
0.2	−0.916	−0.933	−0.922	−0.919	−0.916	−0.916
0	−1.041	−1.057	−1.037	−1.039	−1.040	−1.104
t	$f(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
1	0	0	0	0	0	0
0.8	0	2.986	1.434	0.050	−0.323	−0.128
0.6	0	5.105	−0.8323	0.1061	0.0864	−0.1202
0.4	0	2.916	−0.0656	−0.1830	0.1582	−0.0944
0.2	0	−1.086	0.5208	0.2174	−0.0278	−0.0718
0	0	−3.003	−0.6155	−0.2265	−0.1105	−0.0631

Hence

$$\frac{1}{4} \begin{bmatrix} 0 \\ t^2-\frac{1}{3} \end{bmatrix} = \mathrm{re} \sum_{n=1} \frac{1}{S_n^2 \cos^4 S_n} \begin{bmatrix} \hat{\phi}_1^{(n)} \\ \hat{\phi}_2^{(n)} \end{bmatrix}, \tag{19.10}$$

$$\hat{\Phi}_c = \frac{1}{4} H_0^2 \sum_{n=1} \frac{1}{S_n^4 \cos^4 S_n} \hat{\phi}_1^{(n)} F(S_n y). \tag{19.11}$$

The pressure is given by

$$\hat{P} = \frac{1}{24}(t^3-t) + \frac{y^2}{16}\tau + c_6 \frac{H_0}{2} \mathrm{re} \sum_{n=1} \frac{\sin S_n t}{S_n^2 \cos^3 S_n} \mathscr{I}_0(S_n y) \tag{19.12}$$

where c_6 is to be determined. To find the shape of the free surface, we combine (19.12) and the expression

$$2\frac{\partial^2(\hat{\Phi}_c/y)}{\partial t\,\partial y} = \frac{1}{2} H_0 \sum \frac{1}{S_n^2 \cos^2 S_n} \left[\mathscr{I}_0(S_n y) - \frac{\mathscr{I}_1(S_n y)}{S_n y}\right]$$
$$\cdot\,[t \sin S \cos S_n t - \cos S_n \cos S_n t]$$

with (19.6g). We find that on $y=H_0$,

$$\hat{h}'' + \frac{\hat{h}}{H_0^2} = -\frac{1}{12}(t^3-t) - \frac{H_0^2}{12}\tau + c_6$$
$$+\frac{1}{2}\,\mathrm{re}\sum_{n=1} \frac{1}{S_n^3 \cos^3 S_n}[S_n t \cos S_n t + \cos^2 S_n \sin S_n t] \tag{19.13}$$
$$-\frac{H_0}{2}\,\mathrm{re}\sum_{n=1} \frac{\mathscr{I}_0(S_n H_0)}{S_n^3 \cos^3 S_n}[S_n t \cos S_n t + (1+\cos^2 S_n) \sin S_n t].$$

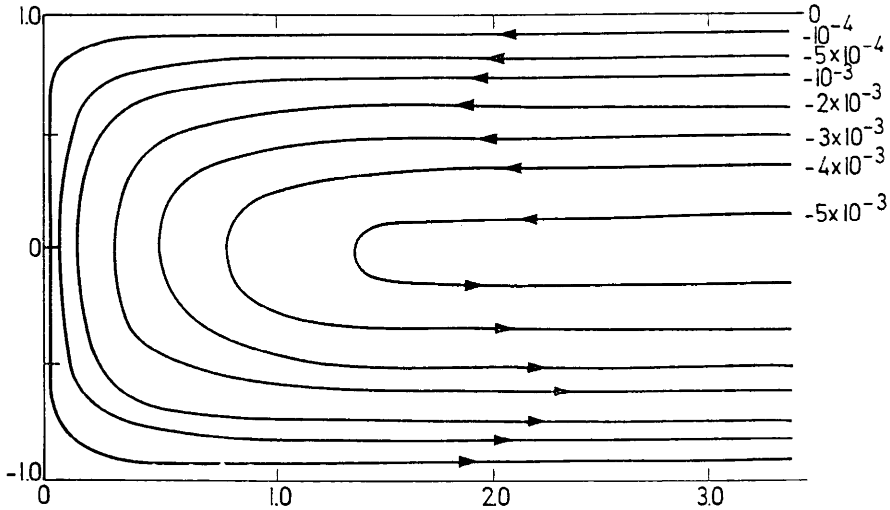

Fig. 7. Level lines of the stream function

$$[\hat{\Phi}-y^2(t^2-1)/192]/H_0^2.$$

The stream function for configuration in which the bottom plate is stationary whilst the top plate rotates with a speed of 2ω is this one plus the stream function of Fig. 5. The flow is very quickly dominated by the large eddy $(y^2(t^2-1)/192)$ and the cellular structure of the edge eddy is not apparent.

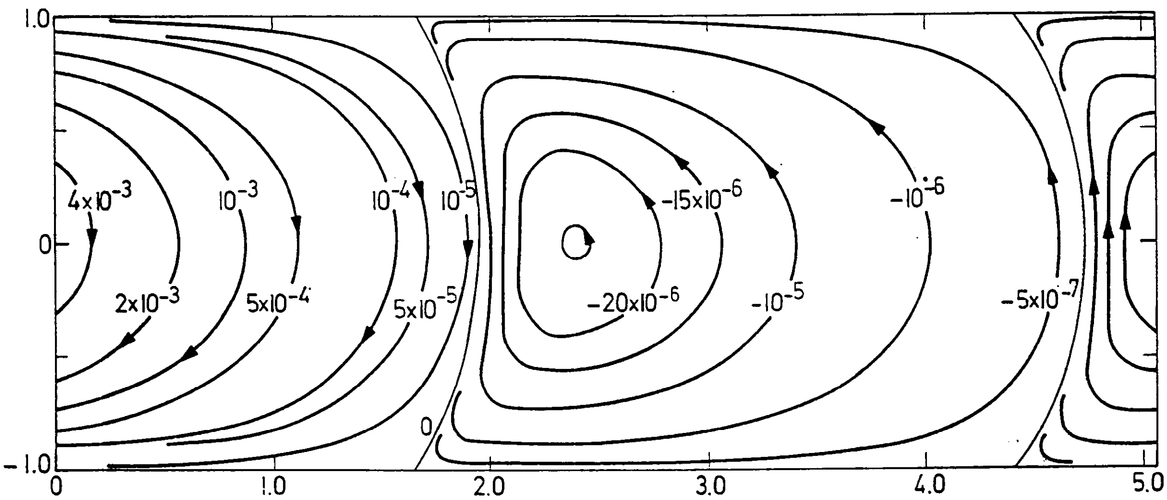

Fig. 8. Level lines of the edge eddies $\hat{\Phi}/H_0^2$.

The solution of (19.13) which satisfies the condition (19.6g) has

$$c_6 = -\frac{\tau H_0^2}{16}.$$

We shall not give the solution of (19.13). As in Section 17, it is best to consider the simpler forms which arise when H_0 is large.

When H_0 is large, we have

$$\hat{\Phi}_c \sim \frac{1}{4} H_0^2 \sum \frac{1}{S_n^4 \cos^4 S_n} \hat{\phi}_1^{(n)}(t)\, e^{-S_n z}, \tag{19.14a}$$

$$\hat{P} \sim \frac{\tau}{16}(y^2 - H_0^2) + \frac{H_0}{2} \operatorname{re} \sum_{n=1} \frac{\sin S_n t}{S_n^2 \cos^3 S_n} e^{-S_n z}, \tag{19.14b}$$

$$\hat{h}'' \sim -\frac{H_0}{2} \operatorname{re} \sum_{n=1} \frac{\cos^{-3} S_n}{S_n^2} [S_n t \cos S_n t + (1 + \cos^2 S_n) \sin S_n t],$$

$$\hat{h} \sim \frac{H_0}{2} \operatorname{re} \sum \frac{1}{S_n^3 \cos^4 S_n} [t \cos S_n \cos S_n t + \sin S_n \sin S_n t - t]. \tag{19.14c}$$

The function $\hat{h}/H_0$ is tabulated in Table 2.

The level lines of the function $\hat{\Phi}$ in the reference domain $\mathscr{V}_0$ are plotted in Fig. 8. The function $\hat{\Phi}(y, t)$ describes a single array of edge eddies. The most persistent of these is associated with the eigenvalue S_1 whose real part is smallest. For $z > 0$,

$$\Phi \to \frac{1}{4} H_0^2 \operatorname{re} \left[\frac{1}{S_1^4 \cos^4 S_1} \hat{\phi}_1^{(1)}(t) e^{-2.106 z} e^{-i 1.125 z} \right]. \tag{19.15}$$

This function oscillates as it decays; the period of its oscillation, the wave length of a cell pair, is $T = z_T \simeq 2\pi/1.125365$. The field (19.15), like (18.6) describes a sequence of concentric cells with a radial spacing of the order of the plate separation.

20. Summary and Concluding Remarks

Most torsion flow viscometers have one stationary boundary. If gravity is neglected, the "up" direction is arbitrary and, without losing generality, we may put the bottom boundary to rest; then $\kappa = 0$ and top plate rotates with an angular velocity of 2ω. To designate solutions for which $\kappa = 0$, we again use tilde overbar functions:

$$(\tilde{\Phi}, \tilde{P}, \tilde{h}) = (\Phi + \hat{\Phi}, P + \hat{P}, h + \hat{h})$$

where (Φ, P, h) are the expressions derived in Sections 17 and 18, and $(\hat{\Phi}, \hat{P}, \hat{h})$ are the expressions derived in Section 19. The interior fields in $\mathscr{V}_0$ are given by

$$\begin{bmatrix} 4\psi/\rho d^5 \\ \left(P - P_a - \dfrac{2\sigma}{dH_0}\right) \Big/ 2\rho d^2 \\ 2\omega W/d \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}(t, y) - \dfrac{y^2 t (t^2-1)^2}{480} - \dfrac{y^2 (t^2-1)^2}{192} \\ P(t, y) \\ (ty + y) \end{bmatrix} \omega^2 + O(\omega^4). \tag{20.1}$$

To express these fields in the deformed domain, we use the prescription specified following (18.5). The function $H(t)$ is given by

$$\sigma(H(t) - H_0)/\rho d^3 = \tilde{h}(t)\omega^2 + O(\omega^4). \tag{20.2}$$

The free surfaces h and $\tilde{h}$ and the accompanying streamlines in the deformed domains are sketched in Fig. 9.

To correct the pressure and deflection of the free surface for the effects of gravity, replace $2\sigma/dH_0$ with $2\sigma(-\varepsilon t + 1/H_0)/d$ in the pressure formula and replace H_0 in (20.2) with $H_0 + \frac{\varepsilon}{6} t(t^2 - 1)$. The resulting equations are a formal

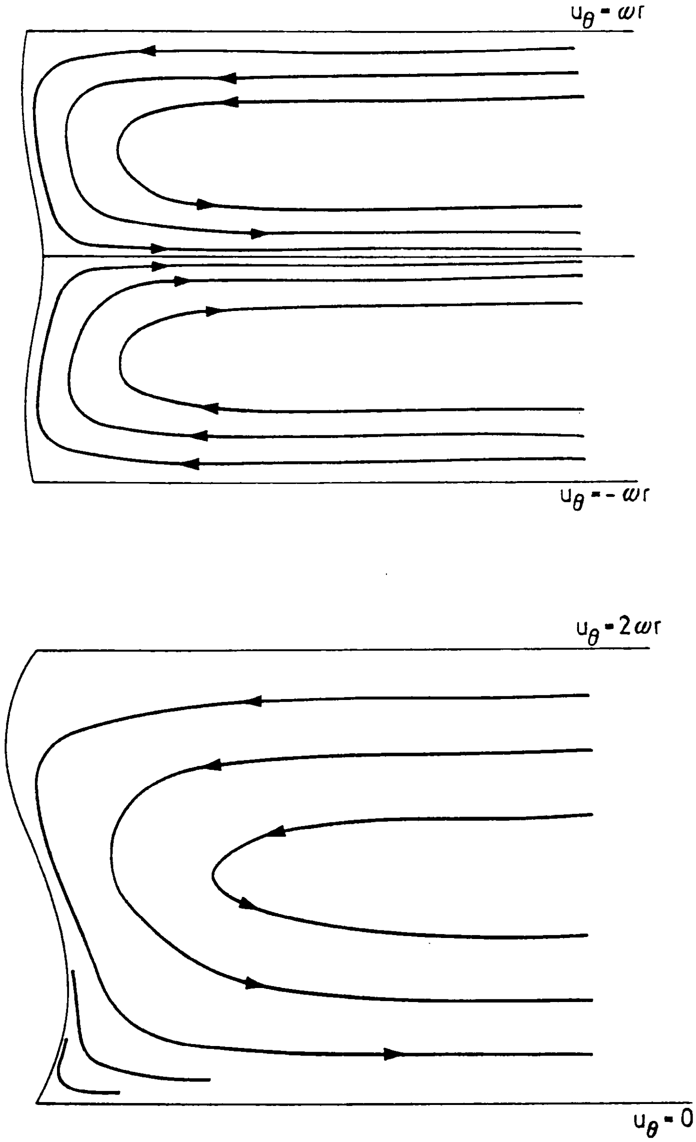

Fig. 9. Sketch of the free surface and streamline patterns when the bottom and top disks rotate with angular velocity $(-\omega, \omega)$ and $(0, 2\omega)$ respectively. The effect of gravity (not shown) would be to make the free surface bulge out more near the bottom disk.

solution of the original problem with a presumed error of $O(\omega^2 \varepsilon)$ where $\varepsilon = \rho g d^2/4\sigma$.

It is of interest to examine the way in which the Rivlin-Ericksen constants enter into the solution through the second order. The motion, given by $\psi^{\langle 2\rangle}$ and $W^{\langle 1\rangle}$, is independent of the Rivlin-Ericksen constants and is the same for all simple fluids. The pressure (17.14) depends weakly on γ_2; since this part of the pressure varies only with y and not on t, it does not affect the shape of the free surface. The principal effect of the Rivlin-Ericksen constants, which characterize the fluid at order ω^2, is on the distribution of the normal stresses acting on planes parallel to the disks. These stresses are given by

$$T_{xx}(x, r) = -P_0 + \left[-P^{\langle 2\rangle} + (S_2^{\langle 1\rangle})_{xx} + 2\mu \frac{\partial U^{\langle 2\rangle}}{\partial x}\right] \omega^2 + O(\omega^4).$$

Using (15.5d), (16.1e) and (20.1), we find that

$$T_{xx}(x,r) = -P_a - \frac{2\sigma}{dH} + 2\rho d^2\left[-P - \hat{P} + (2\gamma_1+\gamma_2)\frac{y^2}{2} + \frac{1}{120}\frac{d}{dt}t(t^2-1)^2 + \frac{1}{48}\frac{d}{dt}(t^2-1)^2 - \frac{2}{y}\frac{\partial^2(\Phi+\hat{\Phi})}{\partial y\,\partial t}\right]\omega^2 + O(\omega^4).$$

Introducing the functions P, $\hat{P}$, Φ and $\hat{\Phi}$, we find that on $\mathscr{V}_0$,

$$\begin{aligned} T_{xx} - P_a - \frac{2\sigma}{dH_0} = 2\rho d^2 \Bigg\{ & \frac{1}{240}(t^2-1)(5t^2-1) + \frac{1}{24}(t^3-t) \\ & + \left(\gamma_1 + \frac{3}{4}\gamma_2 - \frac{3}{40}\right)y^2 = \left(\frac{\gamma_2}{4} - \frac{3}{40}\right) + c_5 \\ & + H_0 \operatorname{re}\sum_{n=1}\left[\frac{1}{P_n^4}\sin P_n \cos P_n t\, \mathscr{I}_0(P_n y) + \frac{1}{2}\frac{\sin S_n t}{S_n^2 \cos^3 S_n}\mathscr{I}_0(S_n y)\right] \\ & + \frac{H_0}{2}\operatorname{re}\sum_{n=1}\left[\frac{\mathscr{I}_0(P_n y)}{P_n^5}\frac{d\phi_1^{(n)}}{dt} + \frac{\mathscr{I}_0(S_n y)}{2S_n^3\cos^4 S_n}\frac{d\hat{\phi}_1^{(n)}}{dt}\right]\Bigg\}\omega^2 + O(\omega^4) \end{aligned} \tag{20.3}$$

where c_5 is the constant, independent of γ_1 and γ_2, which is determined in the way described below (17.16). Since $\mathscr{I}_0(S_n y)$ and $\mathscr{I}_0(P_n y)$ decay exponentially as y is decreased from H_0, the variation in the distribution of normal stresses on the plate ($t = \pm 1$) is dominated by the term

$$\left(\gamma_1 + \frac{3}{4}\gamma_2 - \frac{3}{40}\right)y^2. \tag{20.4}$$

This distribution of normal stresses differs from one which is based on the assumption of negligibility of inertia by the term $3y^2/40$ which arises from inertial effects. This term has an important effect when

$$(40\gamma_1 + 30\gamma_2)/3 = (40\alpha_1 + 30\alpha_2)/3\rho d^2 \tag{20.5}$$

is small. When this ratio is smaller than unity, the pressures normal to the disks will increase, rather than decrease, with distance from the center of the disk.

The unexpected effect of inertia given by (20.5) arises both from the rotational property of the centripetal acceleration and from the existence of an edge.* Since

* Several numerical studies of problems which are related to torsional flow are in broad agreement with some of the conclusions of this study. In the first place we mention the experimental numerical study by GRIFFITHS, JONES & WALTERS (1969) of the flow caused by the slow rotation of a *finite* disk rotating about a vertical axis of symmetry in an elastic-viscous liquid, the liquid being otherwise confined. GRIFFITHS, JONES & WALTERS note that, "... theory based on the rotation of an *infinite* disk predicts no flow reversal—whereas an experimental investigation clearly indicates a reversal of flow throughout the liquid." As a result of their study they conclude that, "... edge effects can affect the flow characteristics throughout a flow field and not just in regions 'near the edges'." MCCOY & DENN (1970) have given results of a numerical study of torsion flow of Newtonian fluid. They do not allow the free surface to deflect. They also find a big inertial eddy extending from the edge to the center of disk.

both effects are neglected in standard treatments of torsion flow, we are going to demonstrate anew, with a direct and simple derivation, the way in which these important physical effects come into the dynamics of torsion flow.

We start by writing (16.1a) in component form using the stream function defined by (16.1e) and the first-order solution

$$U^{\langle 1\rangle}=e_\theta\left[\frac{2rx}{d}+r\right]. \tag{20.6}$$

Then

$$-\nabla^2\left(\frac{1}{r}\frac{\partial\psi^{\langle 2\rangle}}{\partial x}\right)-\frac{1}{r^3}\frac{\partial\psi^{\langle 2\rangle}}{\partial x}+\frac{\partial P^{\langle 2\rangle}}{\partial r}=\rho\left[\frac{4rx^2}{d^2}+\frac{4rx}{d}+r\right]-\frac{4r}{d^2}\alpha^2$$

and

$$\nabla^2\frac{1}{r}\frac{\partial\psi^{\langle 2\rangle}}{\partial r}+\frac{\partial P^{\langle 2\rangle}}{\partial x}=0$$

in $\mathscr{V}_0$ and $\psi^{\langle 2\rangle}=\partial\psi^{\langle 2\rangle}/\partial x=0$ on $x=\pm d/2$. The torques are associated with the x variation in the inhomogeneous terms; these have a nonvanishing curl. We move the rotational inhomogeneous terms in the momentum equations to the edge, leaving behind only a potential field which can be equilibrated by a pressure, by the following change of variables:

$$\psi^{\langle 2\rangle}=\psi-\frac{\rho}{15d^2}x\left(x^2-\frac{d^2}{4}\right)r^2-\frac{\rho}{6}\left(x^2-\frac{d^2}{4}\right)r^2.$$

Such a change of variables would be hard to justify on an unbounded domain. We find, using (16.3), that

$$\begin{aligned}-\frac{1}{r}\mathscr{L}\frac{\partial\psi}{\partial x}+\frac{\partial P^{\langle 2\rangle}}{\partial r}+\left[\frac{4\alpha_2}{d^2}-\frac{6}{5}\rho\right]r=0,\\ \nabla^2\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial P^{\langle 2\rangle}}{\partial x}-\frac{8\rho}{3d^2}x^3+\frac{2\rho}{5}x-\frac{4\rho}{d}x^2+\frac{\rho d}{3}=0\end{aligned} \tag{20.7}$$

in $\mathscr{V}_0$ and $\psi=\partial\psi/\partial x=0$ on $x=\pm d/2$. This is an edge problem because it is driven by inhomogeneous terms at the edge $r=H_0$. The inhomogeneous terms in (20.7) are irrotational and they may be equilibrated by a pressure; thus

$$P^{\langle 2\rangle}=-\frac{2\alpha_2}{d^2}r^2+\frac{3}{5}\rho r^2+\frac{2\rho x^4}{3d^2}-\frac{\rho}{5}x^2+\frac{4\rho x^3}{3d}-\frac{\rho x d}{3}+\pi(x,r) \tag{20.8}$$

and

$$-\frac{1}{r}\mathscr{L}\frac{\partial\psi}{\partial x}+\frac{\partial\pi}{\partial r}=0,\qquad \nabla^2\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial\pi}{\partial x}=0.$$

The functions $\psi(x,r)$, $\pi(x,r)$ are edge functions, expressible in Papkovich-Fadle series which decay rapidly with distance from the edge. At $x=\pm d/2$ the terms which depend on x alone are constants and the main interior variation of the pressure (20.8) is given by the first two terms of (20.8). The normal stress is then

(see 15.5d)

$$T_{xx}^{\langle 2\rangle} = -P^{\langle 2\rangle} + (S_2^1)_{xx} = r^2\left[\frac{8\alpha_1}{d^2} + \frac{6\alpha_2}{d^2} + \frac{3\rho}{5}\right] \tag{20.9}$$

$$+\text{constants and edge terms.}$$

In STP at about room temperature (see JOSEPH, BEAVERS & FOSDICK, 1973)

$$2\alpha_1 = -3.95 \text{ gm/cm}, \quad \alpha_2 = 4.7 \text{ gm/cm}, \quad \rho = 0.89 \text{ gm/cm}^3$$

and the coefficient of r^2 in (20.9) vanishes when

$$d^2 = \frac{40}{3\rho}\alpha_1 + \frac{10\alpha_2}{\rho} \simeq 23 \text{ cm}^2.$$

All the results of the analysis given here are restricted to small values of ω. In seeking experimental verification of the analysis care should be taken to insure the condition of constant volume as expressed by (14.1a). When $\varepsilon = \rho g d^2/4\sigma$ is small, the deviation of the free surface with ε from a right circular cylinder is given by the cubic polynomial (14.9). This polynomial serves as a correction to the free surface for cases in which $\varepsilon \neq 0$; (1) the shape of the free surface depends strongly on the distribution of plate speeds even when the difference of the plate speeds is fixed (at 2ω).* This difference is exhibited in Table 2. (2) The free surface is proportional to H_0 when H_0 is large. (3) The streamline pattern of the secondary motion and the shape of the free surface are the same in all fluids of equal density and surface tension. (4) The speed of the secondary motion is proportional to the fluidity* (see Equation 16.1e). (5) The secondary motion consists of one big eddy going all the way to the center superimposed on smaller edge eddies which die away quickly. When the angular velocities of the plates at $x = (-d/2, d/2)$ are $(-\omega, \omega)$, the edge eddies are odd functions of x and there is a double row of edge eddies disposed around the symmetry line $x = 0$ (Fig. 6). When at $x = (-d/2, d/2)$ the speeds are $(0, 2\omega)$, the motion is not symmetrical (Fig. 9) but the symmetric part of it (Figs. 7, 8) dominates the flow. In both cases the persistent edge eddy is about one plate separation in length, and in both cases the edge eddies are dominated by big eddies extending from the edge to the center of the disk viscometer. (6) The distribution of normal stresses follows the law (20.3).

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* This observation and the free surface computation may explain the fracturing of the surface in cone and plate and parallel plate viscometers. In the viscometers the free surface is strongly sucked into the center of the viscometer when the difference in angular velocity is large. It is generally supposed that this phenomenon is "viscoelastic." Rheologists who observe the fracturing know that they must always use very viscous liquids. I am suggesting that fracturing is driven by the torques associated with the stratification of *inertia*. The viscosity enters because only in the very viscous liquids will the speed of the secondary motions be slow enough to keep the free surface from rupturing.

the subject in 1972 and later developed into his essay (1974). The point of view of this paper takes much from that essay and also from the point of view expressed by PIPKIN & TANNER in their recent survey of the subject. I was stimulated to analyze the free surface at the edge of a torsion flow viscometer by a description of fracturing given in a lecture by Professor C. MACOSKO. I am grateful to Professor MACOSKO for freely sharing with me his knowledge of this flow and of the actual response of the working instrument. I am indebted to J. YOO for writing all the computer programs used in the computations and for carrying these computations to completion. I want also to thank YOO and LEROY STURGES for checking the derivation of the equations of this paper. Our work was supported, in part, by a grant from the U.S. National Science Foundation (GK 12500).

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