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Repeated Supercritical Branching of Solutions Arising in the Variational Theory of Turbulence

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Contents

In the variational theory of statistically stationary turbulence one seeks bounds on the difference between the response of laminar and turbulent flow when the steady external forces driving the flow are specified. For example, the difference between the actual heat transported and the heat transported by conduction alone in a fluid layer heated from below is maximized when the temperature difference across the plates is specified; the difference between the mass flux in turbulent and laminar pipe flow is maximized when the pressure drop is specified; the difference between the torque in turbulent and laminar Couette flow between concentric cylinders is maximized when the angular velocity of the cylinders is given. To find the bounds we consider the maximum value of the response functional over a kinematically admissible class of fluctuation fields which includes at least all statistically stationary solutions of the governing problem.

The Euler equations for the response functionals are nonlinear and the solutions of these equations bifurcate repeatedly as the temperature difference, pressure drop or angular velocity is increased. In this paper we have developed and justified a bifurcation theory for the case of heat transported across a fluid-filled porous layer heated from below. We discuss the idea that the variational theory of turbulence is one kind of mathematical realization of the Landau-Hopf conjecture that transition to turbulence occurs through repeated branching of manifolds of quasi-periodic solutions.

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1. Introduction

It is generally believed that the increasing complexity of motions of a fluid which is observed as the Reynolds number is increased, and which is frequently called "turbulence", is a manifestation of the successive loss of stability of flows of less complicated structure to those with a more complicated structure. This process, which is frequently identified with the conjecture of L. D. LANDAU (1944) and E. HOPF (1948), is sometimes identified as the "transition to turbulence through repeated branching of solutions".

LANDAU and HOPF regard repeated branching as a process involving continuous bifurcation of manifolds of solutions with N frequencies into manifolds with $N+1$ frequencies. Here the attractive property of the stable solution is replaced with the attractive property of the manifold. For example, when the data is steady and the Reynolds (or Rayleigh) number is small, all solutions are attracted to the steady basic flow. For higher Reynolds numbers, the steady flow is unstable and stability is supposed now to be claimed by an attracting manifold of time-periodic motions differing from one another in phase alone. Arbitrary solutions of the initial value problem will be attracted to one or another of the members of the attracting set according to their initial values. At still higher Reynolds numbers, the manifold of periodic solutions loses its stability to a larger manifold of quasi-periodic solutions* with two frequencies of independently arbitrary phase. Now arbitrary solutions of the initial value problem are attracted to the manifold with two frequencies, and so on.

Aspects of this conjecture are in good agreement with observation and experiments. In particular, the notion of stable manifolds (or, more generally, sets) of solutions appears as an especially promising idea. The conjecture about quasi-periodic solutions can be made more precise (JOSEPH, 1973) but it appears to involve a difficulty associated with "small divisors".

The part of the Landau-Hopf conjecture which seems most in need of revision concerns their view that bifurcation is a process in which the norm of solutions on successively stable manifolds varies continuously with the Reynolds number. Of course, this continuity property does not prevail for subcritical bifurcations (see JOSEPH & SATTINGER (1972)). For these, as in the Poiseuille flow problems discussed by JOSEPH & CHEN (1974) and by JOSEPH (1974), the transition between stable solutions can be discontinuous. We have called this discontinuous transition a snap-through instability.

On the other hand, the Landau-Hopf conjecture appears to give a fairly accurate description of features of the transition involved in supercritical bifurcation. Since supercritical bifurcating solutions may be stable, continuous bifurcations are possible and lead to gradual evolution of successively more complicated flows.

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* Quasi-periodic functions are defined as the special class of almost periodic functions possessing only a finite basis of frequencies. In other words, we are studying oscillations containing finitely many (rationally independent) frequencies, $\omega_1, \omega_2, \dots \omega_n$. For example, the function $f(t)=\cos t \cos \pi t$ is a quasi-periodic function with frequencies $\omega_1=2\pi$ and $\omega_2=2$. The value $f(t)=1$ occurs when $t=0$ but not again; though $f(t)<1$ when $t\neq 0$, there is always $\tau(\varepsilon)>0$ such that $|f(\tau)-f(0)|<\varepsilon$ for preassigned $\varepsilon>0$.

Even in the supercritical case, however, the Landau-Hopf conjecture is not in strict accord with the facts. For example, a steady flow need not bifurcate into a time-periodic flow; instead, it may bifurcate into other more complicated steady flows. We have already noted that the view of "turbulence" as specifically quasi-periodic is also open to question.

It is very difficult to give a completely satisfactory account of these conjectures for the Navier-Stokes equations; the problems involved are just too tough. However, there is a rather striking correspondence between the Landau-Hopf conjectures and the bifurcating properties of solutions of the Euler equations which arise in the variational theory of turbulence. Our aim here is to develop a bifurcation theory for these Euler equations and to explain their significance in understanding supercritical bifurcations.

The objective of the variational theory of turbulence is to provide bounds on average properties of statistically stationary turbulent flows. The average properties are regarded as response functionals of the turbulent velocity field which can be defined for more general vector fields. The bounds are derived by determining the extremum of the functional among a class of vector fields which includes all statistically stationary solutions of the basic equations of motion.

HOWARD (1963) following earlier ideas of MALKUS (1954) was the first to use this approach when he derived upper bounds for the heat transported by convection in a fluid layer heated from below.

BUSSE (1969) made an important contribution toward the solution of the problem posed by HOWARD. He suggested that the extremalizing solutions should introduce smaller spatial scales as the intensity of the turbulence increases. He called these solutions multi-α solutions and studied them by boundary layer methods. The boundary layer analysis of the multi-α solutions rested on a number of unproven assumptions. These assumptions are most easily examined in the context of porous convection (BUSSE & JOSEPH 1972, hereafter called BJ) since this is possibly the simplest of natural configurations in which multi-α solutions occur. The analysis of porous convection allows one to characterize the multi-α solutions through "orthogonality" relations in which the wave numbers play the role of eigenvalues and to prove a number of results about the solution. The simplicity of the problem of porous convection allowed the numerical calculation of the 2-α solution by a Galerkin method. The analysis shows that the boundary layer solutions are fairly good approximations of the actual solution.

The boundary layer analysis, however, is misleading in certain very important details of the solution. In particular, the boundary layer solutions lead to definite "breaks" in the slope of the heat transport curve. The rigorous analysis of BJ, and the analysis of GUPTA & JOSEPH (1973, hereafter called GJ) indicates that the appearance of solutions with ever more wave numbers is a bifurcation phenomenon: for a certain range of Rayleigh numbers, the difference between the heat transported and the heat transported by conduction alone (the discrepancy) is maximized by a solution with N wave numbers. At a critical value of R a new solution with $N+1$ wave numbers differing infinitesimally from the N wave number solution becomes possible and maximizes the discrepancy. The bifurcation process implies that the bounding heat transport curve does not have "breaks" in slope but is a smooth curve having breaks in curvature. (The bounding heat transport curve is continuously differentiable with piecewise continuous second derivatives.)

We draw the reader's attention to the similarity of the bifurcation process just described and the Landau-Hopf conjecture about transition to turbulence through repeated supercritical branching of quasi-periodic solutions.

To compare the two descriptions we must bear in mind that LANDAU and HOPF speak of solutions which are quasi-periodic in the time*. On the other hand, the variational problem for turbulence is time-independent. However, the bifurcation of solutions of the variational problem refers to solutions which can be described as quasi-periodic functions of the space variables in the horizontal plane. It is also important that the steady Euler equations cannot be "unstable" in a dynamic sense. The "stable" solutions are the ones that maximize the discrepancy for a fixed Rayleigh number (or minimize the Rayleigh number for a fixed value of the discrepancy).

Bearing in mind the differences between the variational problem and the actual problem of supercritical bifurcation of solutions of the Navier Stokes equations, we note that:

(1) Bifurcation of solutions arising in the variational problem is always supercritical (see Eq. (2.10)) and always leads to an increase in the value of the response functional.

(2) The solutions of the variational problem can be regarded as quasi-periodic functions of the variables in the horizontal plane (see the discussion following Eq. (2.9)).

(3) The maximizing solutions of the variational problem form a "stable" manifold whose dimensions increase by one at each new point of bifurcation.

The maximizing solutions of the variational theory of statistically stationary turbulence share many features with the branching solutions envisaged by LANDAU and HOPF. The variational problem may be regarded as modeling the process of transition to turbulence through repeated supercritical branching**.

2. Variational Problems for the Minimum Rayleigh Number for a Given Value of the Discrepancy in the Heat Transported

The variational theory of statistically stationary turbulent convection in a porous layer heated from below has been studied by BUSSE & JOSEPH (1972) and

* LANDAU believes that the bifurcation of quasi-periodic solutions of the time is associated with a similar bifurcation in the spatial structure of solutions. He says (LANDAU & LIFSCHITZ, 1959, p. 106) that a result of the bifurcation of periodic solutions "... is a quasi-periodic motion characterized by two different periods.

"In the course of the further increase of the Reynolds number new periods appear in succession, and the motion assumes an involved character typical of turbulence—so a turbulent motion is to a certain extent a quasi-periodical motion.

"... The range of Reynolds numbers between successive appearances of new frequencies diminishes rapidly in size. The new flows themselves are on a smaller and smaller scale. This means that the order of magnitude of the distances over which the velocity changes appreciably is the smaller, the later the flow in question appears."

** HOPF (1948, 1956) has given "model equations" which lead to quasi-periodic branching. HOPF's equations have very special properties, convenient for analysis, which are not shared by the Navier-Stokes equations.

GUPTA & JOSEPH (1973). The study of BJ has the advantage of simplicity; the problem treated by GJ is harder mathematically but is closer to the observed physics. We shall briefly review the statement of the two problems, emphasizing the mathematical rather than the physical aspects of the formulation.

The problem is to study the transport of heat in a horizontal layer heated from below. In dimensionless variables the domain of flow is the layer $0 \leqq z \leqq 1$.

It will be convenient and sufficient for our purpose to start the analysis with the dimensionless version of the Darcy-Oberbeck-Boussinesq (DOB) equations as set down by LAPWOOD (1948). We have

$$B(\hat{\mathbf{u}}_t + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}) + \nabla P - \mathbf{k} R (T - T_0) + \hat{\mathbf{u}} = 0 \tag{2.1a}$$

and

$$T_t + \hat{\mathbf{u}} \cdot \nabla T - \nabla^2 T = 0 \tag{2.1b}$$

where

$$\operatorname{div} \hat{\mathbf{u}} = 0, \qquad \hat{\mathbf{u}} = \mathbf{i}\hat{u} + \mathbf{j}\hat{v} + \mathbf{k}\hat{w}$$

and T_0 is a dimensionless reference temperature. The boundary conditions are

$$T = 1, \quad \hat{w} = 0 \quad \text{at } z = 0$$

and

$$T = 0, \quad \hat{w} = 0 \quad \text{at } z = 1. \tag{2.1c}$$

Since the Darcy constitutive assumption has replaced the Newtonian stress divergence $\nabla^2 \hat{\mathbf{u}}$ with a resistance proportional to a Darcy averaged velocity $(-\hat{\mathbf{u}})$ in the last term of (2.1a), we cannot impose boundary conditions on the tangential components of the velocity vector.

The parameters of the problem (2.1) are the Rayleigh number

$$R = \gamma g K d (T_2 - T_1) / \nu \kappa$$

and the Darcy-Prandtl number

$$B^{-1} = (\nu/\kappa)(d^2/K).$$

The constants γ, g, ν and K are the coefficient of thermal expansion, the acceleration due to gravity, the kinematic viscosity and the Darcy permeability coefficient, respectively. The thermal diffusivity κ is here defined as the ratio of thermal conductivity of the fluid-solid mixture to products of the specific heat and density of the fluid. The temperature difference across the layer is $T_2 - T_1$.

The physically appropriate value $B = 0$ follows from extraordinarily small values of the permeability coefficient in porous material: in sand, $K = 0\ (10^{-8})$ cm^2; in very porous fibre metals, $K = 0\ (10^{-4})$ cm^2. When $B = 0$ then

$$\nabla P - \mathbf{k} R (T - T_0) + \hat{\mathbf{u}} = 0 \tag{2.2}$$

is the appropriate form of the DOB equations independent of the form which is assumed for the nonlinear inertia in Darcy's law. The fact that $B \to 0$ for natural materials means that thermally-driven motion in porous material will ordinarily be very slow motion.

We define the horizontal average by an overbar

$$\bar{\circ} = \lim_{L\to\infty} \frac{1}{4L^2} \iint_{-L}^{L} \circ \, dx\, dy.$$

The overall average is designated by

$$\langle \circ \rangle = \int_0^1 \bar{\circ}\, dz = \langle \bar{\circ} \rangle;$$

for any constant C

$$\bar{C} = C, \quad \langle C \rangle = C.$$

The motion $(\hat{\boldsymbol{u}}, T)$ may be decomposed into a mean and fluctuating part:

$$(\hat{\boldsymbol{u}}, T) = (\bar{\boldsymbol{u}}(z) + \boldsymbol{u}(x, y, z, t), \bar{T}(z) + \theta(x, y, z, t)).$$

By *statistically stationary* motion we mean that horizontal averages do not depend on time and $\bar{\boldsymbol{u}}(z)=0$ ($\bar{w}=0$ may be deduced from the continuity equation; $\bar{u}=\bar{v}=0$ is assumed).

The Nusselt number is defined by

$$Nu = 1 + \mu/R \tag{2.3}$$

where

$$\mu = \langle \omega\theta \rangle = R(Nu - 1)$$

is the *discrepancy*.

The equation for the mean temperature is

$$\frac{d}{dz}\overline{w\theta} = \frac{d^2\bar{T}}{dz^2}. \tag{2.4}$$

The equations for the fluctuations are

$$B(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla P - kR\theta + \boldsymbol{u} = 0$$

and

$$w\frac{d\bar{T}}{dz} + \boldsymbol{u}\cdot\nabla\theta - \nabla^2\theta - \frac{d}{dz}\overline{w\theta} = 0.$$

The energy integrals for the fluctuations are

$$-R\langle w\theta \rangle + \langle |\boldsymbol{u}|^2 \rangle = 0 \tag{2.5}$$

and

$$\left\langle \theta w \frac{d\bar{T}}{dz} \right\rangle + \langle |\nabla\theta|^2 \rangle = 0. \tag{2.6}$$

The variational theory of turbulence is possible because equation (2.4) has a first integral which allows one to eliminate the mean motion from (2.6). Then (2.5) and (2.6) can be regarded as functionals defined on fluctuations alone.

The variational problems considered in BJ and GJ are:

(a) Given R, find upper bounds on the heat transported by turbulent convection.

(b) Given μ, find lower bounds on the temperature difference needed to drive turbulent convection with the given discrepancy. These two problems are equivalent formulations of a single problem.

The analysis of BJ leaves the Darcy-Prandtl number arbitrary and starts from the energy identities (2.5) and (2.6). The analysis of GJ assumes $B=0$; then one can replace the integral side constraint (2.5) with a differential side constraint which arises as the vertical component of the curl of the curl of (2.2). The variational problems may be formulated as follows:

Problem 1 (BJ). Statistically stationary convection with discrepancy μ cannot exist when $R < F(\mu)$ where

$$F(\mu) = \min_{\mathscr{H}} \mathscr{F}[\chi, \theta, \mu],$$

$$\mathscr{F}(\mu) = \frac{\langle |\nabla\theta|^2 \rangle}{\langle w\theta \rangle} + \mu \frac{\langle (\overline{w\theta} - \langle w\theta \rangle)^2 \rangle}{\langle w\theta \rangle^2}$$

and

$$\mathscr{H} = \{\chi, \theta \colon \langle \Delta_2\chi\Delta\chi \rangle = \langle w\theta \rangle, \quad (\chi, \theta)|_{z=0,1} = (0, 0), (\bar{\chi}, \bar{\theta}) = (0, 0),$$

where

$$\Delta_2 = \partial_{xx}^2 + \partial_{yy}^2 \quad \text{and} \quad w = -\Delta_2\chi.$$

Problem 2 (GJ). This is the same as problem 1 except that the minimum $\hat{F}$ of $\mathscr{F}$ is taken over the set H where

$$H = \{w, \theta \colon \Delta w - \Delta_2\theta = 0, \quad (w, \theta)|_{z=0,1} = (0, 0), (\bar{w}, \bar{\theta}) = (0, 0).$$

The admissibility conditions in H for problem 2 are more stringent than those in $\mathscr{H}$ for problem 1. In fact, we may obtain problem 1 from problem 2 by averaging the differential equation side constraint*. It follows that

$$\hat{F}(\mu) \geqq F(\mu). \tag{2.7}$$

To specify completely the sets $\mathscr{H}$ and H in the horizontally infinite layer, one must prescribe the behavior of admissible functions at infinity. The appropriate prescription is suggested by the Euler equations for problems 1 and 2. These equations are non-linear but they allow superposition of solutions which are proportional to eigenfunctions of the Laplacian in the (x, y) plane. It is therefore assumed that the minimizing functions are in the form

$$\begin{Bmatrix}\chi\\ \theta\end{Bmatrix} = \begin{Bmatrix}\chi^{(N)}\\ \theta^{(N)}\end{Bmatrix} = \sum_{n=1}^{N} \begin{Bmatrix}\chi_n^{(N)}(z)\\ \theta_n^{(N)}(z)\end{Bmatrix} g_n(x, y) \tag{2.8}$$

where the g_n are eigenfunctions of the horizontal Laplacian

$$\Delta_2 g_n + \alpha_n^2 g_n = 0, \qquad \overline{g_n g_m} = \delta_{nm}.$$

We then define

$$F_N(\mu) = \min_{\mathscr{H}(\mu)} \mathscr{F}[\chi^{(N)}, \theta^{(N)}, \mu]$$

* AUCHMUTY (1973) has shown that problems 1 and 2 are solvable if the fluid is bounded by lateral side walls on which the temperature or heat flux is prescribed.

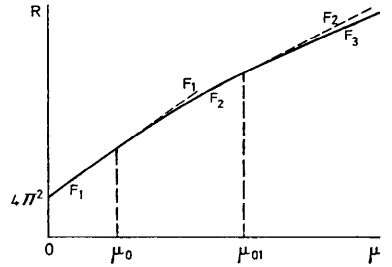

Fig. 1. Repeated branching of the bounding heat transport curve for statistically stationary turbulent convection. All the statistically stationary solutions lie above the bounding curve, and none lie below it. The bounding curve in (μ, F) coordinates is sketched here; the computed curve in (R, Nu) coordinates is graphed in Fig. 5.

and

$$F(\mu)=\min_N F_N(\mu). \tag{2.9}$$

The integer-valued function $N(\mu)$ defines the dimension of the manifold of minimizing functions for the given μ.

The assumption (2.8) implies that variational problems in the theory of turbulence are properly posed in the set of eigenfunctions of the Laplacian in the (x, y) plane. We may represent the eigenfunctions $g_n(x, y)$ of the Laplacian as the real (or imaginary) part of $g_n(x, y)=c_n\, e^{i(\beta_n x+\gamma_n y)}$, $\beta_n^2+\gamma_n^2=\alpha_n^2$. This is a quasi-periodic representation of the solution consistent with restricting the original problem to the space of almost periodic functions of x and y. A precise characterization of H and $\mathscr{H}$ in the fluid layer is an interesting and still open problem for analysis. We shall assume any development of the problem which reduces the original problem to (2.9); then we shall study (2.9).

It is important at this point to note that the variational theory of turbulence will not allow subcritical bifurcating solutions. A subcritical solution would allow the heat transported to increase as the temperature difference is decreased. However, from (2.4) we see that

$$dF/d\mu=\langle(\overline{w\theta}-\langle w\theta\rangle)^2\rangle/\langle w\theta\rangle^2>0 \tag{2.10}$$

and $F(\mu)$ is an increasing rather than decreasing function. The bifurcation of an $N+1$-α solution from an N-α solution necessarily leads to an increase in the value of F; clearly, $F_N(\mu)\geqq F_M(\mu)$ if $M>N$ since more functions are allowed to compete for the minimum of F_M.

Suppose that $(\mu_{N,N+1}, F(\mu_{N,N+1}))$ is a point of bifurcation. It was shown in BJ and GJ that

$$w_N^{(N+1)}(\mu)\to 0, \qquad \theta_N^{(N+1)}(\mu)\to 0$$

and

$$w_n^{(N+1)}(\mu)\to w_n^{(N)}(\mu), \qquad \theta_n^{(N+1)}(\mu)\to\theta_n^{(N)}(\mu)$$

as $\mu\downarrow\mu_{N,N+1}$. Then, following the argument presented in GJ, we find that

$$\frac{dF_{N+1}}{d\mu}=\frac{\left\langle\left(\sum_{n=1}^{N+1} w_n^{(N+1)}\theta_n^{(N+1)}+\langle w_n^{(N+1)}\theta_n^{(N+1)}\rangle\right)^2\right\rangle}{\left\langle\sum_{n=1}^{N+1} w_n^{(N-1)}\theta_n^{(N+1)}\right\rangle^2}\to\frac{dF_N}{d\mu} \tag{2.11}$$

as $\mu\downarrow\mu_{N+1}$. It follows that the slope of the bounding heat transport curve is continuous across a point of bifurcation.

For ease of exposition we shall develop and prove a bifurcation theory for problem 1. Extension of the formal part of this theory to all of the other problems which arise in the variational theory of turbulence is immediate.

3. The Variational Problem of BJ

It is shown in BJ that the equations (2.3, 4, 5) for the minimum temperature contrast necessary to transport heat at an assigned rate can be reduced to the following simpler problem:

$$F(\mu)=\min_{N=1,2,\ldots} F_N(\mu) \tag{3.1}$$

where

$$F_N(\mu)=\min_{\alpha_j}\min_{\theta_j}\mathscr{F}_N[\theta_j, \alpha_j, \mu] \tag{3.2}$$

and $\alpha_j>0$ and $\theta_j(0)=\theta_j(1)=0$. Here,

$$\mathscr{F}_N[\theta_j, \alpha_j, \mu]=\left\{\sum_{j=1}^N \mathscr{I}^2(\theta_j)+\mu\left\langle\left(\sum_{j=1}^N[\theta_j^2-\langle\theta_j^2\rangle]\right)^2\right\rangle\right\}\Big/\left\langle\sum_{j=1}^N\theta_j^2\right\rangle^2,$$

$$\mathscr{I}(\theta_j)=-\langle\theta_j\mathscr{L}_j\theta_j\rangle=\langle\theta_j'^2/\alpha_j+\alpha_j\theta_j^2\rangle$$

and

$$\mathscr{L}_j\theta_j=\theta_j''/\alpha_j-\alpha_j\theta_j.$$

The Euler equations for the minimum problem (3.2) over functions θ_j are

$$\left(\sum_{j=1}^N\mathscr{I}_j(\theta_j)\right)\mathscr{L}_n\theta_n+\sum_{j=1}^N\{G\langle\theta_j^2\rangle-\mu\theta_j^2\}\,\theta_n=0 \tag{3.3a}$$

where

$$G=F+\mu$$

and

$$\theta_n(0)=\theta_n(1)=0. \tag{3.3b}$$

The Euler equations for the minimum problem (3.2) over wave numbers α_j are

$$\alpha_j^2\langle\theta_j^2\rangle=\langle\theta_j'^2\rangle. \tag{3.3c}$$

It may be assumed that all N of the wave numbers α_j are different. If $\alpha_l=\alpha_m$, then $\theta_l=\theta_m$ (see BJ). Solutions belonging to different values of α satisfy the following orthogonality relation:

$$\langle\theta_l'\theta_m'\rangle-\alpha_l\alpha_m\langle\theta_l\theta_m\rangle=0. \tag{3.3d}$$

Solutions $\theta_l(z)$ of (3.3a, b) are either symmetric or antisymmetric with respect to $z=\frac{1}{2}$.

In section 4 of BJ it is shown that when

$$\mu=0: N(\mu)=1, \quad \theta_1=\sin \pi z, \quad \alpha_1=\pi, \quad F(\mu)=4\pi^2, \text{ and } \frac{dF(\mu)}{d\mu}=\frac{1}{2}. \tag{3.4}$$

These values are the best possible since they are attained by roll solutions of the governing Darcy-Oberbeck-Boussinesq equations (2.1) with $B=0$.

For a certain range of values $\mu>0$, it appears that the minimizing field is attained for $N=1$, and $\theta_1(z)$ and α_1 can be given explicitly in terms of elliptic functions and integrals. These solutions are analytic functions of the parameter μ.

At the first point of bifurcation $(\mu_0, F(\mu_0))$ a two-α solution, defined by equations (3.3) with $N=2$, becomes possible. At the point of bifurcation,

$$\begin{aligned}(\mu_0, F(\mu_0))&=(318.506391, 113.115269),\\ Nu&=3.815768,\\ \alpha_1^2(\mu_0)&\equiv\alpha_0^2\cong 16.640702,\\ \alpha_2^2(\mu_0)&\equiv\beta_0^2\cong 127.482213,\end{aligned} \tag{3.5}$$

and with

$$\langle\theta_1^2\rangle=1 \quad \text{and} \quad \psi_0(z)=\frac{\theta_1(z)}{\theta_1(\frac{1}{2})},$$

we find

$$\langle\psi_0^2\rangle=\frac{1}{\theta_1^2(\frac{1}{2})}\cong 0.7998803.$$

The numerical analysis given in BJ and the perturbation analysis to be given here (see 9.17) show that the two-α solution is minimizing for a certain interval of $\mu\geqq\mu_0$. The amplitude $\theta_2\to 0$ as $\mu\downarrow\mu_0$. To find μ_0 and $F(\mu_0)$, BJ treated the "stability" problem defined by

$$\begin{aligned}&(3.3\text{a, b}) && \text{with } N=1,\\ &(3.3\text{c}) && \text{with } j=1,2,\\ &(3.3\text{d}) && \text{with } l=1, \quad m=2\end{aligned} \tag{3.6a}$$

and the linearized equation for small θ_2

$$\mathscr{J}_1(\theta_1)\mathscr{L}_2\theta_2+\{G\langle\theta_1^2\rangle-\mu\theta_1^2\}\theta_2=0, \quad \theta_2(0)=\theta_2(\tfrac{1}{2})=0. \tag{3.6b}$$

In writing (3.6b) we changed the given boundary conditions to antisymmetric boundary conditions. These conditions are consistent with the result (of BJ) that all solutions are either symmetric or antisymmetric with respect to $z=\frac{1}{2}$ and they reduce (3.3d) to an identity. Numerical analysis of the two-α problem showed that θ_2 was antisymmetric with respect to $z=\frac{1}{2}$.

The graphs of $\psi_0(z)$ and $\theta_2\sim\phi_0(z)$ are shown in Figs. 2a and b. In Fig. 3 we have shown how the minimizing value β_0 is selected from among the eigenvalues β of (3.6b).

To motivate our bifurcation theory for problem (3.3), it will be useful to review results of the numerical analysis of problem 2 considered by GJ.

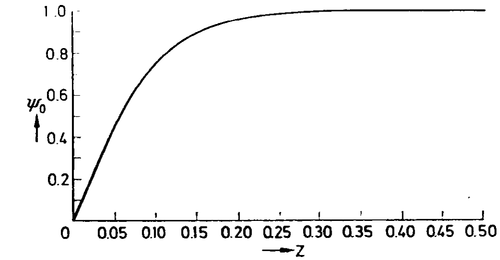

Fig. 2a. Graph of the function $\psi_0(z)=\theta_1(z)/\theta_1(\frac{1}{2})$ satisfying (3.3) when $N=1$.

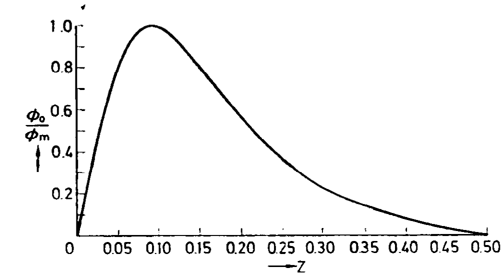

Fig. 2b. Graph of the function $\theta_2(z)/\theta_{2m}=\phi_0(z)/\phi_m$ satisfying (3.6b). Here θ_{2m} and ϕ_m are the maximum values of $\theta_2(z)$ and $\phi_0(z)$ on $0<z<\frac{1}{2}$.

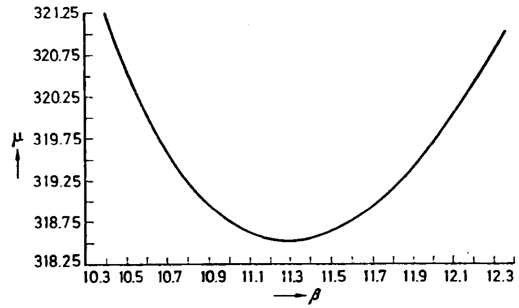

Fig. 3. Values of $\beta(\mu)$ for which solutions of (3.6b) exist when $\theta_1(\mu)$, $\alpha(\mu)$ and $G(\mu)$ are evaluated on the single-α solution.

4. The Variational Problem of GJ

Though the problem treated in GJ lacks some of the simplifying properties of symmetry enjoyed by problem 1, the qualitative structure of both problems is

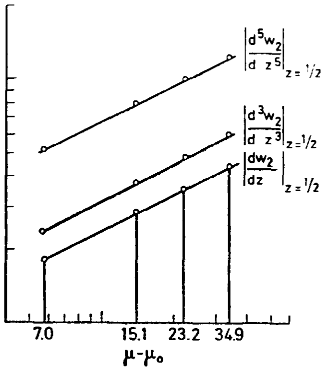

Fig. 4. The variation of the two-α solution of GJ with μ near the point of bifurcation $(\mu, \hat{F})= (\mu_0, F_2) \cong (957.5, 221.5)$. The points are taken from the first four entries of Table 3 of GJ. Suppose that $|dw_2/dz|_{z=\frac{1}{2}} = \sqrt{\mu-\mu_0} f(\mu)$ is slowly varying. Then $\log|dw_2/dz| = \frac{1}{2}\log(\mu-\mu_0) + \log f(\mu)$ should appear as a straight line of slope $\frac{1}{2}$ in a log-log plot, as above.

identical. In fact, statement (3.4) holds for problem 2 of GJ as well as for problem 1. Replacing (3.5), we find from numerical analysis that when

$$0 \leqq \mu \leqq \mu_0 \cong 957.5: \quad N(\mu)=1, \quad 4\pi^2 < \hat{F}(\mu) < F(\mu_0) \cong 221.5.$$

The bifurcation of a single-α solution into a two-α solution is treated, as in BJ, by a method of linearization. For an interval $\mu > \mu_0$, the two-α solution $(N(\mu)=2)$ minimizes $\mathscr{F}$. The two-α solution was computed numerically.

Two facts of importance come from the numerical analysis of GJ:

(1) The value and slope of the minimizing functional

$$\hat{F}(\mu), \quad \frac{d\hat{F}(\mu)}{d\mu}$$

are continuous at the point of bifurcation

$$(\mu, \hat{F}(\mu)) = (\mu_0, \hat{F}(\mu_0)).$$

(2) The two-α solutions are the collection of minimizing functions $w_1, w_2, \theta_1, \theta_2$ and wave numbers α_1 and α_2. The perturbation theory which is given in section 5 indicated that $w_2/\sqrt{\mu-\mu_0}$ and $\theta_2/\sqrt{\mu-\mu_0}$ tended to finite limiting values as $\mu \downarrow \mu_0$. A review of the numerical results of GJ confirms that this scaling is correct (see Fig. 4). This scaling is the first step in the construction of the bifurcation theory.

(3) The bounding heat transport curve given in the form $Nu(R)$ in Fig. 5 is in good agreement with the experiments of BURRETTA and BERMAN and others, particularly with respect to the position of the first point of bifurcation. The bounding curve nearly coincides with the heat transport curve computed from a two-dimensional, steady solution of (2.1) shown as a dashed line in Fig. 5. This

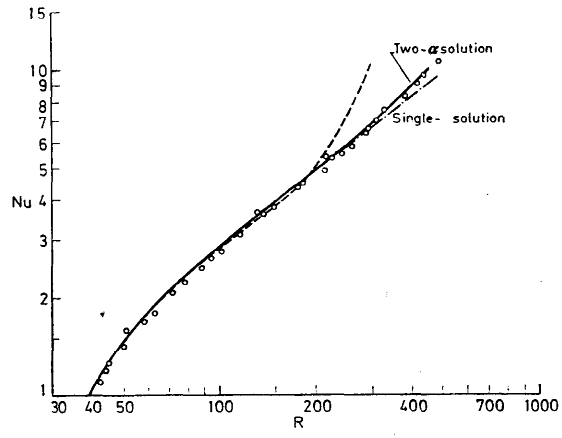

Fig. 5. Heat transport in a porous layer. The black line is the solution of problem 2 and (2.9) (see GJ). The black boxes are from experiments. The dashed line is from a two-dimensional perturbation analysis of (2.1) with $B=0$, carried out to terms of order 6 (PALM, WEBER, & KVERNVOLD (1972)).

solution was constructed by PALM, WEBER & KVERNVOLD (1972) as a perturbation series carried out to terms of order 6. The exact two-dimensional solution must lie below the bound. Therefore, Fig. 5 shows that when $R > 180$, the exact two-dimensional solution is not accurately represented by its partial sum through terms of order 6.

5. The Amplitude Ratio and the Bifurcating Two-α Solution

We shall seek the solution of (3.3a, b, c, d) when $N=2$ and $|\mu-\mu_0|$ is small as a power series in the amplitude ratio, $\mu-\mu_0$. The perturbation series for the two-α solution is not standard; to understand it better it is useful to compare it to the Poincaré-Lindstedt perturbation method for steady solutions of the Navier-Stokes equations. In the Poincaré-Lindstedt theory, one constructs steady motions $\varepsilon\boldsymbol{u}(\boldsymbol{x}, \varepsilon)$ which arise from the instability and bifurcation of a basic steady flow at a critical Reynolds number $R=R_c$. The amplitude parameter ε can be specified in many ways; for example, ε^2 can be taken as the L^2 norm of $\varepsilon\boldsymbol{u}$ or as the projection $\varepsilon = \langle \varepsilon\boldsymbol{u}(\boldsymbol{x}, \varepsilon) \cdot \boldsymbol{u}(\boldsymbol{x}, 0)\rangle$. In all cases $\boldsymbol{u}(\boldsymbol{x}, \varepsilon)$ satisfies the following equation:

$$\boldsymbol{U} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{U} + \varepsilon \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \frac{1}{R} \Delta \boldsymbol{u} + \nabla P = 0. \tag{5.1}$$

All the solutions in a neighborhood of $\varepsilon=0$ can be found in the form of power series for $\boldsymbol{u}(\boldsymbol{x}, \varepsilon)$, $p(\boldsymbol{x}, \varepsilon)$ and $R(\varepsilon)$. The crucial point is that (5.1) is not a homogeneous problem.

The perturbation problem for the two-α solution is different. Since the governing problem (3.3) is homogeneous, the solution cannot depend on the amplitude as in (5.1). In the two-α problem, the role of the amplitude in the Poincaré-Lindstedt expansions is taken up by the *amplitude ratio*

$$\frac{\langle\theta_2^2\rangle}{\langle\theta_1^2\rangle}=\varepsilon b(\varepsilon), \qquad \varepsilon=\mu-\mu_0. \tag{5.2a, b}$$

Here $b(\varepsilon)$ is to be determined and $b(0)=b_0\neq 0$. Without loss of generality we may set

$$\theta_1=C\psi, \qquad \theta_2=C(\varepsilon b)^{\frac{1}{2}}\phi. \tag{5.3}$$

Then (5.2) implies that

$$\langle\psi^2\rangle=\langle\phi^2\rangle. \tag{5.4}$$

At the same time, since the two-α problem is homogeneous, we may require that all solutions have a unit norm

$$1=\langle\theta_1^2+\theta_2^2\rangle=C^2\langle\psi^2+\varepsilon b\phi^2\rangle. \tag{5.5}$$

Since (5.5) is to hold when $\varepsilon=0$ and $\psi=\psi_0$, $C^2=1/\langle\psi_0^2\rangle$ and we may use (5.5) to form an alternate expression for the amplitude ratio

$$\varepsilon b=\langle\psi_0^2-\psi^2\rangle/\langle\phi^2\rangle. \tag{5.6}$$

The two-α problem, written in the new variables (5.3), has the following form:

$$\mathscr{M}\psi=0, \qquad \psi(0)=\psi'(\tfrac{1}{2})=0, \tag{5.7a, b}$$

$$\mathscr{N}\phi=0, \qquad \phi(0)=\phi(\tfrac{1}{2})=0, \tag{5.8a, b}$$

where

$$\mathscr{M}=\mathscr{k}(\psi,\phi)\mathscr{L}_1+\Phi(\psi,\phi) \quad\text{and}\quad \mathscr{N}=\mathscr{k}(\psi,\phi)\mathscr{L}_2+\Phi(\psi,\phi),$$

$$\mathscr{k}(\psi,\phi)=\mathscr{I}(\psi)+\langle\psi_0^2-\psi^2\rangle g(\phi)/\langle\phi^2\rangle,$$

$$g(\phi)=-\langle\phi\mathscr{L}_2\phi\rangle,$$

and

$$\Phi(\psi,\phi)=G\langle\psi_0^2\rangle-(\varepsilon+\mu_0)(\psi^2+\langle\psi_0^2-\psi^2\rangle\phi^2/\langle\phi^2\rangle)$$

where $G\equiv F+\mu=F+\mu_0+\varepsilon$. In this formulation, as in the work of BJ which was cited following equation (3.6b), we have replaced the full-channel boundary conditions (3.3b) with the half-channel boundary conditions (5.7b) and (5.8b). These conditions together with the governing differential equations are enough to guarantee that $\psi(z)$ and $\phi(z)$ are, respectively, symmetric and antisymmetric with respect to $z=\frac{1}{2}$ and reduce (3.3d) to an identity.

In sum, we must solve equations (5.7) and (5.8) with ϕ and ψ related by (5.4). The amplitude ratio may be computed from (5.6) when ψ is known.

When $\varepsilon=0$ the two-α problem coalesces with the single-α problem at the point of bifurcation. This single-α problem was solved in BJ and the values of μ_0, $G_0=F(\mu_0)+\mu_0$, α_0 and β_0 are given by (3.5). At the point of bifurcation, $\psi(z)=\psi_0(z)$, $\phi(z)=\phi_0(z)$. These functions satisfy the problems

$$0=\mathscr{M}_0\psi_0=\psi_0(0)=\psi_0'(\tfrac{1}{2}) \tag{5.9a}$$

and

$$0=\mathscr{N}_0\phi_0=\phi_0(0)=\phi_0(\tfrac{1}{2}). \tag{5.9b}$$

From these equations we obtain

$$-\mathscr{I}_0^2+G_0\langle\psi_0^2\rangle^2-\mu_0\langle\psi_0^4\rangle=0 \tag{5.10a}$$

and

$$-\mathscr{I}_0 g_0+G_0\langle\psi_0^2\rangle\langle\phi_0^2\rangle-\mu_0\langle\psi_0^2\phi_0^2\rangle=0. \tag{5.10b}$$

We are going to construct a two-α solution which bifurcates from this zero$^{\text{th}}$-order problem as a power series in ε. It will be easiest at first to develop the theory when the wave numbers α and β are fixed at their values α_0 and β_0 at the point of bifurcation. In section 9 we shall consider the perturbation problem when the wave numbers vary. We now seek solutions in the form

$$\begin{Bmatrix}\psi(z;\varepsilon)\\ \phi(z;\varepsilon)\\ G(\varepsilon)\end{Bmatrix}=\sum_{n=0}\begin{Bmatrix}\psi_n(z)\\ \phi_n(z)\\ G_n\end{Bmatrix}\varepsilon^n, \tag{5.11}$$

where, apart from factorials, the quantities with subscript n are Taylor coefficients; *e.g.*,

$$G_n=\frac{1}{n!}\partial_\varepsilon^n G|_{\varepsilon=0}.$$

6. Boundary-Value Problems for the Taylor Coefficients

The Taylor coefficients at order n must satisfy the following system of equations ($n\geq 1$):

$$0=\sum_{l+\nu=n}\mathscr{M}_\nu\psi_l=\sum_{l+\nu=n}\mathscr{N}_\nu\phi_l=\sum_{l+\nu=n}(\langle\psi_\nu\psi_l\rangle-\langle\phi_\nu\phi_l\rangle) \tag{6.1a, b, c}$$

and

$$0=\phi_n(0)=\phi_n(\tfrac{1}{2})=\psi_n(0)=\psi_n'(\tfrac{1}{2}). \tag{6.2}$$

When $n=1$ we find that (6.1) reduces to

$$\mathscr{M}_0\psi_1+\mathscr{M}_1\psi_0=L_0\psi_1+2I(\psi_1)\psi_0+G_1\langle\psi_0^2\rangle\psi_0-\psi_0^3=0 \tag{6.3}$$

where

$$L_0=\mathscr{M}_0-2\mu_0\psi_0^2$$

and $I(\psi_1)$ is an integral-differential operator linear in ψ_1,

$$I(\psi_1)=-\langle\psi_1\mathscr{L}_1\psi_0\rangle\mathscr{L}_1+\frac{\langle\psi_0\psi_1\rangle}{\langle\phi_0^2\rangle}[\langle\phi_0\mathscr{L}_2\phi_0\rangle\mathscr{L}_1+\mu\phi_0^2].$$

Moreover,

$$\mathscr{N}_0\phi_1+\mathscr{N}_1\phi_0=\mathscr{N}_0\phi_1+2\mathscr{K}(\psi_1)\phi_0+G_1\langle\psi_0^2\rangle\phi_0-\psi_0^2\phi_0=0 \tag{6.4}$$

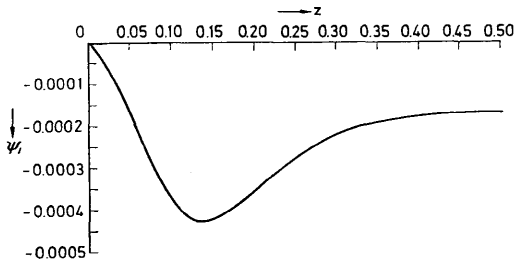

Fig. 6. Graph of the function ψ_1 satisfying (6.3)

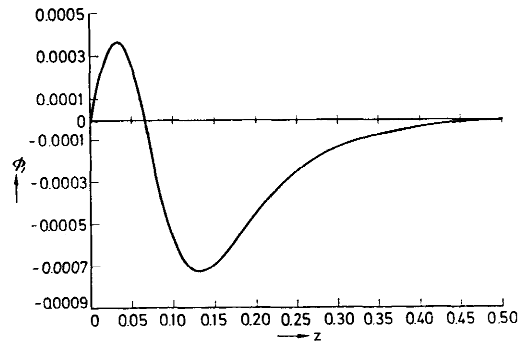

Fig. 7. Graph of the function ϕ_1 satisfying (6.4)

where

$$\mathscr{K}(\psi_1) = -\langle \psi_1 \mathscr{L}_1 \psi_0 \rangle \mathscr{L}_2 + \frac{\langle \psi_0 \psi_1 \rangle}{\langle \phi_0^2 \rangle} [\langle \phi_0 \mathscr{L}_2 \phi_0 \rangle \mathscr{L}_2 + \mu_0 \phi_0^2] - \mu_0 \psi_0 \psi_1.$$

The functions ϕ_1 and ψ_1 are related through the normalizing condition (5.4) which leads to

$$\langle \psi_1 \psi_0 \rangle = \langle \phi_1 \phi_0 \rangle. \tag{6.5}$$

Graphs of the functions ψ_1 and ϕ_1 satisfying (6.1)–(6.5) are shown in Figs. 6 and 7*.

Higher-order problems for ψ_n are in the form

$$L_0 \psi_n + 2I(\psi_n)\psi_0 + G_n \langle \psi_0^2 \rangle \psi_0 + A_n = 0 \tag{6.6}$$

* These functions were computed numerically by W. J. SUN using a standard shooting method and the procedures required by Lemma 4.

where A_n depends only on lower-order terms. Higher-order problems for ϕ_n are in the form

$$\mathscr{N}_0 \phi_n + 2\mathscr{K}(\psi_n)\phi_0 + B_n = 0 \tag{6.7}$$

where B_n depends only on lower-order terms.

We show next that all of these perturbation problems for ψ_n and ϕ_n are uniquely solvable.

7. Necessary and Sufficient Conditions for Solvability

The multi-α solutions are a new type of mathematical construction. The perturbation theory for these solutions is not standard. The perturbation theory is based on the following four lemmas. These lemmas guarantee that the perturbation problems defined in section 6 are solvable.

Lemma 1. *A necessary condition for the solvability of* (6.1 a, c) *and* (6.2) *is that*

$$\sum_{\substack{l+\nu=n \\ \nu \neq 0}} \langle \psi_0 \mathscr{M}_\nu \psi_l \rangle = 0. \tag{7.1a}$$

A necessary and sufficient condition for the solvability of (6.1 b) *and* (6.2) *is that*

$$\sum_{\substack{l+\nu=n \\ \nu \neq 0}} \langle \phi_0 \mathscr{N}_\nu \phi_l \rangle = 0. \tag{7.1b}$$

If $\sum_{l+\nu=n} \mathscr{N}_\nu \phi_l \in C^m[0, \frac{1}{2}]$, *then* $\phi_n \in C^{m+2}[0, \frac{1}{2}]$.

Proof. Solutions of (6.1) and (6.2) must satisfy (7.1 a, b) because

$$\langle \psi_0 \mathscr{M}_0 \psi_n \rangle = \langle \psi_n \mathscr{M}_0 \psi_0 \rangle = 0 \tag{7.2}$$

and

$$\langle \phi_0 \mathscr{N}_0 \phi_n \rangle = \langle \phi_n \mathscr{N}_0 \phi_0 \rangle = 0. \tag{7.3}$$

$\mathscr{N}_0$ is a linear operator of the Sturm-Liouville type. The eigenvalue zero is a simple eigenvalue of $\mathscr{N}_0$ ($\mathscr{N}_0 \phi_0 = 0$ has only one solution). Fredholm solvability theory applies to $\mathscr{N}_0$ and guarantees bounded invertibility of (6.1 b) and (6.2). Since $\mathscr{N}_0$ is a Sturm-Liouville operator, there exists a Green function integral inverse $\mathscr{N}_0^{-1}$ which maps elements of $C^m[0, \frac{1}{2}]$ in the complement of the null space of $\mathscr{N}_0$ into the space $C^{m+2}[0, \frac{1}{2}]$.

Lemma 2. *Suppose* ψ_l, ϕ_l *and* G_l *are all known when* $l < n$. *Then* (7.1 a) *may be solved for* G_n.

In fact (7.1 a) arises as an orthogonality condition for (6.1 a) which also may be written as $\langle \psi_0 (6.6) \rangle = 0$; from this we find that

$$G_n \langle \psi_0^2 \rangle^2 = -\langle A_n \psi_0 \rangle.$$

Lemma 3. *The boundary-value problem*

$$L_0 \Gamma = f(z), \quad \Gamma(0) = \Gamma'(\tfrac{1}{2}) = 0 \tag{7.4}$$

is uniquely solvable for every $f(\circ)\in C[0,\frac{1}{2}]$. *Morever,*

$$\Gamma(\circ)=\mathscr{G}f\in C^2[0,\tfrac{1}{2}]$$

where $\mathscr{G}$ *is the Green function operator for* (7.4).

Proof. L_0 is a Sturm-Liouville operator. The asserted properties follow directly if there is no solution of (7.4) when $f=0$. Suppose the contrary; there is a solution $\Gamma=H\in C^2[0,\frac{1}{2}]$ when $f=0$. Then we find that

$$\begin{aligned} G(\mu_0)&=\left[\mathscr{J}(\psi_0)\left\langle\frac{H'^2}{\alpha}+\alpha H^2\right\rangle+3\mu\langle H^2\psi_0^2\rangle\right]\Big/\langle H^2\rangle\langle\psi_0^2\rangle \\ &>\left[\mathscr{J}(\psi_0)\left\langle\frac{H'^2}{\alpha}+\alpha H^2\right\rangle+\mu\langle H^2\psi_0^2\rangle\right]\Big/\langle H^2\rangle\langle\psi_0^2\rangle \\ &\equiv\lambda[H]\geqq\min_H\lambda[H]=\tilde{\lambda}=\lambda[\tilde{H}] \end{aligned} \tag{7.5}$$

where $\lambda[H]$ is a homogeneous functional of degree zero defined for functions $H\in C^1[0,\frac{1}{2}]$ satisfying $H(0)=0$. The values μ_0 and α and the fixed function $\psi_0(z)>0$ for $z\in(0,\frac{1}{2}]$ are given by the single-α solution at the point of bifurcation. It is well known that the minimizing element $\tilde{H}(z)$ is a smooth function. It must be of one sign on $(0,\frac{1}{2})$ for it is clear that if $\tilde{H}$ is minimizing, then $|\tilde{H}|$ is also minimizing. But if $\tilde{H}$ changes sign, $|\tilde{H}|$ could not be continuously differentiable. The possibility of a flat tangent where $H=0$ can be eliminated by the uniqueness theorem for the initial value problem with starting values at the node.

The Euler problem for the minimum value $\tilde{\lambda}$ of λ is

$$\left\langle\frac{\psi_0'^2}{\alpha}+\alpha\psi_0^2\right\rangle\left(\frac{\tilde{H}''}{\alpha}-\alpha\tilde{H}\right)+\tilde{\lambda}\langle\psi_0^2\rangle\tilde{H}-\mu_0\psi_0^2\tilde{H}=0 \tag{7.6}$$

and

$$\tilde{H}(0)=\tilde{H}'(\tfrac{1}{2})=0.$$

We shall show that there is no solution of one sign $\tilde{H}(z)=H(z)$ with $\tilde{\lambda}<G(\mu_0)$. Consider the single-α problem

$$\left\langle\frac{\psi_0'^2}{\alpha}+\alpha\psi_0^2\right\rangle\left(\frac{\psi_0''}{\alpha}-\alpha\psi_0\right)+G(\mu_0)\langle\psi_0^2\rangle\psi_0-\mu_0\psi_0^3=0 \tag{7.7}$$

and

$$\psi_0(0)=\psi_0'(\tfrac{1}{2})=0.$$

Comparing (7.6) and (7.7), we find that

$$[\tilde{\lambda}-G(\mu_0)]\langle\psi_0\tilde{H}\rangle=0,$$

and since $\langle\psi_0\tilde{H}\rangle\neq 0$, we must have $\tilde{\lambda}=G(\mu_0)$ contradicting (7.5). It follows that the starting assumption that there is a solution $\Gamma(z)$ of (7.4) when $f(z)\equiv 0$ is false, and (7.4) is uniquely invertible for inhomogeneous terms $f(z)$.

We next face a somewhat different problem than that treated in Lemma 3. We must consider the solvability of problems like (6.3); that is

$$L_0F+2I(F)\psi_0+J=0,\qquad F(0)=F'(\tfrac{1}{2})=0 \tag{7.8}$$

where

$$\langle J\psi_0\rangle=0.$$

Lemma 3, concerned with solvability, guarantees that there are solutions of (7.8); since the term $2I(F)\psi_0$ depends on the solution, the solutions of (7.8) need not be (and are not) unique.

To study the non-uniqueness of solutions of (7.8) we shall first reduce (7.8) to an equivalent problem,

$$L_0Y-2\langle Y\mathscr{L}_1\psi_0\rangle\mathscr{L}_1\psi_0+2G_0\langle Y\psi_0\rangle\psi_0+g=0 \tag{7.9a}$$

and

$$Y(0)=Y'(\tfrac{1}{2})=\langle g\psi_0\rangle=0. \tag{7.9b, c, d}$$

To effect this reduction, we first write that

$$\begin{aligned} I(F)=&-\langle F\mathscr{L}_1\psi_0\rangle\ {}_1+G_0\langle F\psi_0\rangle \\ &+\frac{\langle F\psi_0\rangle}{\langle\phi_0^2\rangle}[\langle\phi_0\mathscr{L}_2\phi_0\rangle\mathscr{L}_1-G_0\langle\phi_0^2\rangle+\mu_0\phi_0^2]; \end{aligned}$$

then, with

$$F=q+\frac{\langle F\psi_0\rangle}{\langle\phi_0^2\rangle}Q, \tag{7.10}$$

we find that

$$L_0q-2\langle q\mathscr{L}_1\psi_0\rangle\mathscr{L}_1\psi_0+2G_0\langle q\psi_0\rangle\psi_0+g=0 \tag{7.11}$$

and

$$\begin{aligned} &L_0Q-2\langle Q\mathscr{L}_1\psi_0\rangle\mathscr{L}_1\psi_0+2G_0\langle Q\psi_0\rangle\psi_0 \\ &\quad+2[\langle\phi_0\mathscr{L}_2\phi_0\rangle\mathscr{L}_1\psi_0-G_0\langle\phi_0^2\rangle\psi_0+\mu_0\phi_0^2\psi_0]=0 \end{aligned} \tag{7.12}$$

where q and Q satisfy the boundary conditions. Since, by (5.10b),

$$\langle\phi_0\mathscr{L}_2\phi_0\rangle\langle\psi_0\mathscr{L}_1\psi_0\rangle-G_0\langle\phi_0^2\rangle\langle\psi_0^2\rangle+\mu_0\langle\phi_0^2\psi_0^2\rangle=0,$$

both (7.11) and (7.12) are in the form (7.9). When q and Q are known, we may solve (7.10) for $\langle F\psi_0\rangle$.

Lemma 4. *There exists a one-parameter family of solutions to the differential-integral boundary-value problem* (7.9). *These solutions are in the form*

$$Y=C_1Y_1+C_2Y_2+Y_3. \tag{7.13}$$

Here, the functions Y_1, Y_2 *and* Y_3 *are determined uniquely as the solutions of the boundary-value problems*

$$\begin{aligned} L_0Y_1+\psi_0&=0, \\ L_0Y_2+\mathscr{L}_1\psi_0&=0, \\ L_0Y_3+J&=0, \end{aligned} \tag{7.14a, b, c}$$

and $Y_i(0)=Y_i'(\frac{1}{2})=0$ $(i=1, 2, 3)$, *and the constants* C_1 *and* C_2 *are linearly related;* i.e.,

$$C_1[\langle Y_1\psi_0\rangle-\tfrac{1}{2}G_0]+C_2\langle Y_2\psi_0\rangle+\langle Y_3\psi_0\rangle=0. \tag{7.15}$$

Proof. Equations (7.14) may be derived from (7.9) using (7.13) along with the two additional relations

$$C_1=2G_0\langle Y\psi_0\rangle \tag{7.16a}$$

and

$$C_2=-2\langle Y\mathscr{L}_1\psi_0\rangle. \tag{7.16b}$$

Equation (7.16a) together with (7.13) gives (7.15). It remains to show that the linear algebraic equation in C_1 and C_2 which arises from (7.16b), and (7.13) does not determine C_1 and C_2 uniquely; that is, (7.15) and the expression

$$C_1\langle Y_1\mathscr{L}_1\psi_0\rangle+C_2[\langle Y_2\mathscr{L}_1\psi_0\rangle+\tfrac{1}{2}]+\langle Y_3\mathscr{L}_1\psi_0\rangle=0 \tag{7.17}$$

are proportional. To prove proportionality, we first show that the determinent

$$\mathscr{D}=\begin{vmatrix}\langle Y_1\psi_0\rangle-\dfrac{1}{2G_0} & \langle Y_2\psi_0\rangle\\ \langle Y_1\mathscr{L}_1\psi_0\rangle & \langle Y_2\mathscr{L}_1\psi_0\rangle+\tfrac{1}{2}\end{vmatrix} \tag{7.18}$$

of the coefficients of C_1 and C_2 in (7.15) and (7.17) vanishes. To show this we note that

$$\begin{aligned}\langle Y_1\mathscr{L}_1\psi_0\rangle&=\langle\psi_0\mathscr{L}_1 Y_1\rangle=\langle\psi_0\mathscr{J}_0^{-1}[L_0-G_0\langle\psi_0^2\rangle+3\mu_0\psi_0^2]\,Y_1\rangle\\&=-\mathscr{J}_0^{-1}\langle\psi_0^2+G_0\langle\psi_0^2\rangle Y_1\psi_0-3\mu_0\psi_0^3 Y_1\rangle\\&=\mathscr{J}_0^{-1}[\langle\psi_0^2\rangle/2-G_0\langle\psi_0^2\rangle\langle Y_1\psi_0\rangle],\end{aligned} \tag{7.19}$$

where the last step follows from the equation

$$0=\langle\psi_0(L_0Y_1+\psi_0)\rangle=\langle\psi_0(\mathscr{M}_0Y_1-2\mu_0\psi_0^2Y_1+\psi_0)\rangle=\langle-2\mu_0\psi_0^3Y_1+\psi_0^2\rangle.$$

In addition, we note that

$$\begin{aligned}\langle Y_2\mathscr{L}_1\psi_0\rangle&=\langle\psi_0\mathscr{L}_1Y_2\rangle=\mathscr{J}_0^{-1}\langle\psi_0[L_0-G_0\langle\psi_0^2\rangle+3\mu_0\psi_0^2]\,Y_2\rangle\\&=\mathscr{J}_0^{-1}\langle-\psi_0\mathscr{L}_1\psi_0-G_0\langle\psi_0^2\rangle\psi_0Y_2+3\mu_0\psi_0^3Y_2\rangle\\&=-\tfrac{1}{2}-\mathscr{J}_0^{-1}G_0\langle\psi_0^2\rangle\langle\psi_0Y_2\rangle,\end{aligned} \tag{7.20}$$

where the last step of (7.20) follows from the equation

$$\begin{aligned}0&=\langle\psi_0(L_0Y_2+\mathscr{L}_1\psi_0)\rangle=\langle\psi_0(\mathscr{M}_0Y_1-2\mu_0\psi_0^2Y_1+\mathscr{L}_1\psi_0)\rangle\\&=\langle-2\mu_0\psi_0^3Y_1+\psi_0\mathscr{L}_1\psi_0\rangle.\end{aligned}$$

A direct computation using (7.18) and (7.20) shows that $\mathscr{D}=0$.

To complete the proof of Lemma 4, we need to show that the ratio of inhomogeneous terms is equal to the ratio of the coefficients of C_1 (or C_2) in (7.15) and (7.16). We shall prove that

$$\frac{\langle Y_3\psi_0\rangle}{\langle Y_3\mathscr{L}_1\psi_0\rangle}=\frac{\langle Y_2\psi_0\rangle}{\langle Y_2\mathscr{L}_1\psi_0\rangle+\tfrac{1}{2}}=\frac{-\mathscr{J}_0}{G_0\langle\psi_0^2\rangle}. \tag{7.21}$$

The second of the equations (7.21) follows directly from (7.20). It is necessary to prove the first equality. We note that

$$\langle Y_3\mathscr{L}_1\psi_0\rangle=\frac{\mu_0}{\mathscr{J}_0}\langle Y_3\psi_0^3\rangle-\frac{G_0\langle\psi_0^2\rangle}{\mathscr{J}_0}\langle Y_3\psi_0\rangle, \tag{7.22}$$

and since $\langle g\psi_0\rangle=0$ and

$$0=\langle\psi_0(L_0Y_3+g)\rangle=\langle\psi_0(\mathscr{M}_0-2\mu_0\psi_0^2)\,Y_3\rangle=-2\mu_0\langle\psi_0^3Y_3\rangle,$$

the first term on the right of (7.22) vanishes and the ratio on the left of (7.21) reduces to the ratio on the right side of (7.21).

This completes the proof of Lemma 4.

Lemma 4 implies that with G_n chosen so that $G_n\langle\psi_0^2\rangle^2+\langle A_n\psi_0\rangle=0$, we may always find ψ_n satisfying (6.6) in the form

$$\psi_n=C_{n1}Y_1+C_{n2}Y_2+Y_{n3} \tag{7.23}$$

where C_{n1} and C_{n2} satisfy a linear relation

$$a_nC_{n1}+b_nC_{n2}+d_n=0. \tag{7.24}$$

To solve (6.1) and (6.2) we simultaneously make ψ_n unique and satisfy the solvability condition (7.1b) required by the Fredholm alternative for (6.7). Thus,

$$\begin{aligned}\langle\phi_0B_n\rangle&=-2\langle\phi_0\mathscr{K}(\psi_1)\phi_0\rangle=-2C_{n1}\langle\phi_0\mathscr{K}(Y_1)\phi_0\rangle\\&\quad-2C_{n2}\langle\phi_0\mathscr{K}(Y_2)\phi_0\rangle-2\langle\phi_0\mathscr{K}(Y_{n3})\phi_0\rangle.\end{aligned} \tag{7.25}$$

Equations (7.24) and (7.25) together determine unique values of C_{n1} and C_{n2} and imply the uniqueness of ψ_n.

The solutions of (6.7) are not unique; to any solution we may add a multiple of ϕ_0. To determine a unique solution in the form $\phi_n=\phi_{np}+\phi_0K_n$, we choose K_n so as to satisfy (6.1c).

Theorem. *The problems* (6.1a, b, c) *and* (6.2) *are uniquely solvable. Moreover, the solutions* $\psi_n(z)$ *and* $\phi_n(z)$ *are elements of* $C^\infty[0, \frac{1}{2}]$.

The infinite differentiability of $\psi_0(z)$ follows immediately from its representation as an elliptic function (see BJ). $\phi_0(z)$ satisfies a Sturm-Liouville eigenvalue problem with $C^\infty[0,\frac{1}{2}]$ coefficients and, therefore, $\phi_0\in C^\infty[0,\frac{1}{2}]$. The infinite differentiability of $\psi_n(z)$ and $\phi_n(z)$ then follows from a bootstrap argument using Lemmas 1 and 3.

The problem of bifurcation of the N-α solution into an $N+1$-α solution is similar to the problem just considered. The basic perturbation parameter would again be

$$\varepsilon=\mu-\mu_{0N}$$

where μ_{0N} is the value of μ at the N^{th} point of bifurcation. A normalizing condition like (5.5)

$$\left\langle \sum_{n=1}^{N} \theta_{0n}^2 \right\rangle = \left\langle \sum_{n=1}^{N} \theta_n^2 \right\rangle + \varepsilon b \langle \phi^2 \rangle$$

would again apply with θ_{0n} denoting the functions $\theta_n(\mu_{0N})$ and $\sqrt{\varepsilon b}\,\phi = \theta_{N+1}$. In the general problem we are not yet able to replace the given boundary conditions (3.3b) with derived boundary conditions like (3.6b). In this case it is necessary to retain the orthogonality condition (3.3d) which is an identity in ε and should be differentiated along with the other equations.

In closing, we note that the perturbation method evolved here is valuable for computations only in the immediate neighborhood of a point of bifurcation. It is precisely in such regions that the numerical methods introduced in BJ and GJ are least useful. The numerical methods are, of course, computationally superior for most values of μ. The main value of the perturbation theory is that it gives a rigorous mathematical foundation to the multi-α solutions and reveals important properties of these solutions.

8. Convergence

In this section we shall use the implicit operator theorem of HILDEBRANDT and GRAVES (see VAINBERG & TRENOGIN [1962] or SATTINGER [1973]) to prove the analyticity in ε of solutions of (5.7) and (5.8).

The proof to be given is actually a slight generalization of the solvability proof for the first-order perturbation problem (6.3) and (6.4). To facilitate this proof, we note that Lemma 4 may be recast so as to apply directly to (7.8) which generalizes (6.3); the solution of this problem is in the form

$$F = d_1 X_1 + d_2 X_2 + X_3 \tag{8.1}$$

where

$$L_0 X_1 + 2\mu_0 \phi_0^2 \psi_0 / \langle \phi_0^2 \rangle - 2g(\phi_0) \mathscr{L}_1 \psi_0 / \langle \phi_0^2 \rangle = 0, \tag{8.2a}$$

$$L_0 X_2 + \mathscr{L}_1 \psi_0 = 0, \tag{8.2b}$$

$$L_0 X_3 + J = 0, \tag{8.2c}$$

$$X_i(0) = X_i'(\tfrac{1}{2}) = 0, \tag{8.2d}$$

and

$$F_3 = A_1 d_1 + A_2 d_2 + \langle X_3 \psi_0 \rangle = 0 \tag{8.3}$$

where

$$A_1 = \langle X_1 \psi_0 \rangle - \frac{1}{2G_0} \cong 2.5409626 \times 10^{-4},$$

$$A_2 = \langle X_2 \psi_0 \rangle \cong -1.3442708 \times 10^{-2}$$

and

$$\langle X_3 \psi_0 \rangle \cong 3.93089 \times 10^{-5}.$$

With this preliminary aside, we turn to the main proof. Introducing

$$\begin{aligned} \psi &= \psi_0 + \varepsilon \hat{\psi}, \\ \phi &= \phi_0 + \varepsilon \hat{\phi}, \\ \tilde{g} &= (G - G_0)/\varepsilon \end{aligned} \tag{8.4}$$

and

$$\tilde{\mathscr{A}} = \tilde{g} \langle \psi_0^2 \rangle - \psi_0^2$$

into (6.3) and (6.4) we find that

$$L_0 \hat{\psi} + 2I(\hat{\psi}) + \tilde{\mathscr{A}} \psi_0 + \varepsilon R_1 [\hat{\psi}, \hat{\phi}] = 0 \tag{8.5}$$

where $R_1 [\hat{\psi}, \hat{\phi}]$ is a quadratic form in $\hat{\psi}$ and $\hat{\phi}$ and

$$\mathscr{N}_0 \hat{\phi} + 2\mathscr{K}(\hat{\psi}) \phi_0 + \tilde{\mathscr{A}} \phi_0 + \varepsilon R_2 [\hat{\psi}, \hat{\phi}, \tilde{g}] = 0. \tag{8.6}$$

Here R_2 is a quadratic form in $\hat{\phi}$ and $\hat{\psi}$, linear in $\tilde{g}$. Equations (8.5) and (8.6) may be regarded as a nonlinear mapping from $C^2[0, \frac{1}{2}]$ to $C^0[0, \frac{1}{2}]$. We are looking for solutions with two derivatives.

According to Lemmas 1 and 4, (8.5) and (8.6) are solvable when R_1 and R_2 are given functions if and only if

$$\langle \psi_0 (\tilde{\mathscr{A}} \psi_0 + \varepsilon R_1 [\hat{\psi}, \hat{\phi}]) \rangle = 0 \tag{8.7}$$

and

$$2 \langle \phi_0 \mathscr{K}(\hat{\psi}) \phi_0 \rangle + \langle \phi_0 (\tilde{\mathscr{A}} \phi_0 + \varepsilon R_2) \rangle = 0. \tag{8.8}$$

Then, by Lemma 4 in the form 8.1, we have

$$F_4 = -\hat{\psi} + d_1 X_1 + d_2 X_2 + X_3 = 0. \tag{8.9}$$

Given (8.8) the problem (8.6) is uniquely invertible on the complement to the null space of $\mathscr{N}_0$. Thus,

$$\hat{\phi} + 2\mathscr{N}_0^{-1} [2\mathscr{K}(\hat{\psi}) \phi_0 + \tilde{\mathscr{A}} \phi_0 + \varepsilon R_2] = 0 \tag{8.10}$$

where $\mathscr{N}_0^{-1}$ is a Green function operator which maps $C^n[0, \frac{1}{2}]$ into $C^{n+2}[0, \frac{1}{2}]$.

We may combine (8.8) and (8.9) to get

$$B_1 d_1 + B_2 d_2 + 2 \langle \phi_0 \mathscr{K}(X_3) \phi_0 \rangle + \langle \phi_0 (\tilde{\mathscr{A}} \phi_0 + \varepsilon R_2) \rangle = 0, \tag{8.11}$$

where

$$B_1 = 2 \langle \phi_0 \mathscr{K}(X_1) \phi_0 \rangle \cong 627.92802$$

and

$$B_2 = 2 \langle \phi_0 \mathscr{K}(X_2) \phi_0 \rangle \cong -6.3199661;$$

combining (8.3) and (8.11) we find that

$$\begin{aligned} F_2 = (B_2 A_1 - A_2 B_1) d_1 + B_2 \langle X_3 \psi_0 \rangle - 2A_2 \langle \phi_0 \mathscr{K}(X_3) \phi_0 \rangle \\ - A_2 \langle \phi_0 (\tilde{\mathscr{A}} \phi_0 + \varepsilon R_2) \rangle = 0. \end{aligned} \tag{8.12}$$

Finally, using Lemma 3, we note that we may invert (8.2c) and eliminate

$$X_3 = \mathscr{G}(J) = \mathscr{G}(\tilde{\mathscr{R}}\psi_0 + \varepsilon R_1) \tag{8.13}$$

from (8.3), (8.9) and (8.12).

Collecting equations, we have from (8.4) and (8.7) that

$$F_1 = \tilde{g}\langle\psi_0^2\rangle^2 - \langle\psi_0^4\rangle + \varepsilon\langle\psi_0 R_1\rangle = 0, \tag{8.14}$$

and from (8.9) and (8.10) that

$$\begin{aligned} F_5 = \hat{\phi} + 2\mathscr{N}_0^{-1}[&2d_1\mathscr{K}(X_1)\phi_0 + 2d_2\mathscr{K}(X_2)\phi_0 \\ &+ 2\mathscr{K}(X_3)\phi_0 + \tilde{\mathscr{R}}\phi_0 + \varepsilon R_2] = 0. \end{aligned} \tag{8.15}$$

To prove that the original problem (5.7) and (5.8) can be solved, it will suffice to show that the equations (8.14, 12, 3, 9, 15) with X_3 expressed as in (8.13) can be solved for $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \equiv (\tilde{g}, d_1, d_2, \hat{\psi}, \hat{\phi})$. These equations are in the form

$$F_i(\gamma_j; \varepsilon) = 0 \quad (i, j = 1\text{–}5) \tag{8.16}$$

and can be regarded as a mapping of Banach spaces

$$\begin{aligned} &R \times R \times R \times C^2[0, \tfrac{1}{2}] \times C^2[0, \tfrac{1}{2}] \times R \\ &\quad \to R \times R \times R \times C^2[0, \tfrac{1}{2}] \times C^2[0, \tfrac{1}{2}]. \end{aligned} \tag{8.17}$$

The mapping (8.17) is continuous because the inverting operators $\mathscr{G}$ and $\mathscr{N}_0^{-1}$ are continuous. The implicit operator theorem therefore applies to (8.16). We interpret

$$\left[\frac{\partial F_i}{\partial \gamma_j}\right] \tag{8.18}$$

as a matrix Fréchet derivative evaluated at $\varepsilon = 0$, $\tilde{g} = G_1$, $\hat{\phi} = \phi_1$, $\hat{\psi} = \psi_1$ with d_1 and d_2 determined. We find that (8.18) is a triangular matrix with a non-zero leading diagonal

$$\begin{bmatrix} \langle\psi_0^2\rangle^2 & & & & 0 \\ \partial F_2/\partial\tilde{g} & B_2A_1 - A_2B_1 & & & \\ \partial F_3/\partial\tilde{g} & \partial F_3/\partial d_1 & A_2 & & \\ \partial F_4/\partial\tilde{g} & \partial F_4/\partial d_1 & \partial F_4/\partial d_2 & -1 & \\ \partial F_5/\partial\tilde{g} & \partial F_5/\partial d_1 & \partial F_5/\partial d_2 & \partial F_5/\partial\hat{\psi} & 1 \end{bmatrix} \tag{8.19}$$

where $B_2A_1 - A_2B_1 = -8.4426589$.

The matrix operator (8.19) is therefore invertible and the implicit operator theorem guarantees that it can be solved for $(\tilde{g}(\varepsilon), d_1(\varepsilon), d_2(\varepsilon), \hat{\psi}(\varepsilon), \hat{\phi}(\varepsilon))$ and that these quantities are analytic functions of ε for sufficiently small ε. Moreover, the implicit operator theorem guarantees that $\hat{\psi}(\varepsilon)$, $\hat{\phi}(\varepsilon)$ and $\tilde{g}(\varepsilon)$ are unique solutions to the original problems (5.7) and (5.8).

Theorem 2. *The series solutions* (5.11) *constructed in sections* 6 *and* 7 *converge when* ε *is sufficiently small. The functions* $\psi(z; \varepsilon)$ *and* $\phi(z; \varepsilon)$ *together with* $G(\varepsilon)$ *solve* (5.7) *and* (5.8) *uniquely.*

Remark. To prove that $(\hat{\psi}, \hat{\phi}) \in C^\infty[0, \frac{1}{2}]$ we first express X_3 as in (8.13); we then apply a bootstrap argument to $\hat{\psi}$ and $\hat{\phi}$ as given by (8.9) and (8.15) using the fact that $\mathscr{G}$ and $\mathscr{N}_0^{-1}$ gain two derivatives.

9. Perturbation of the Wave Numbers

In the derivation of the perturbation theory we fixed the wave numbers α and β at optimizing values α_0 and β_0 for the point of bifurcation ($\varepsilon = 0$). The optimizing wave numbers when $\varepsilon > 0$ do not stay constant but vary in accord with (3.3c). Thus,

$$\alpha^2(\varepsilon) = \langle\psi'^2\rangle / \langle\psi^2\rangle \tag{9.1}$$

and

$$\beta^2(\varepsilon) = \langle\phi'^2\rangle / \langle\phi^2\rangle. \tag{9.2}$$

We shall not develop a general perturbation theory when the wave numbers vary with ε. The essential parts of this theory can be understood from the computation of the derivatives of $\alpha^2(\varepsilon)$ and $\beta^2(\varepsilon)$ at the point of bifurcation. We first note that when the wave numbers are chosen optimally, as in (9.1) and (9.2), it is necessary to account for the variation of the solutions

$$\begin{aligned} &\psi(z; \alpha(\varepsilon), \beta(\varepsilon), \varepsilon), \\ &\phi(z; \alpha(\varepsilon), \beta(\varepsilon), \varepsilon), \end{aligned} \tag{9.3}$$

and

$$G(\alpha(\varepsilon), \beta(\varepsilon), \varepsilon)$$

with the wave numbers. Then, differentiating (9.1) and (9.2) with respect to ε, we find that

$$\alpha_{,\varepsilon} = \frac{-\langle\psi_{,\varepsilon}(\psi_0'' + \alpha_0^2\psi_0)\rangle}{[\alpha_0\langle\psi_0^2\rangle + \langle\psi_{,\alpha}(\psi_0'' + \alpha_0^2\psi_0)\rangle]} \cong -7.205267 \times 10^{-4} \tag{9.4}$$

and

$$\beta_{,\varepsilon} = \frac{-\langle\phi_{,\varepsilon}(\phi_0'' + \beta_0^2\phi_0)\rangle - \alpha_{,\varepsilon}\langle\phi_{,\alpha}(\phi_0'' + \beta_0^2\phi_0)\rangle}{[\beta_0\langle\phi_0^2\rangle + \langle\phi_{,\beta}(\phi_0'' + \beta_0^2\phi_0)\rangle]} \cong 9.069256 \times 10^{-2}. \tag{9.5}$$

Here the comma denotes differentiation with respect to the variable named by the subscript following the comma evaluated at the point of bifurcation.* To obtain the derivatives of ψ and ϕ with respect to α and β at the point of bifurcation, we may confine our attention to the problems (5.5), (5.7) and (5.8) when $\varepsilon = 0$ but when α and β are allowed to vary independently.

* The values (9.4) and (9.5) are in good agreement with the values which one obtains by backward extrapolation of the results of the Galerkin approximation method used by BJ. In the notation of BJ, $\alpha = \alpha_1^{(2)}$, $\beta = \alpha_2^{(2)}$ and

$$\frac{d\alpha}{d(R/\pi^2)} = \pi^2\alpha_{,\varepsilon}/(G_{,\varepsilon} - 1) = -0.04865476158$$

and

$$\frac{d\beta}{d(R/\pi^2)} = \pi^2\beta_{,\varepsilon}/(G_{,\varepsilon} - 1) = 0.612416105.$$

The basic problems to be solved when $\varepsilon=0$ are

$$\mathcal{J}(\psi)\mathcal{L}_1\psi+G\langle\psi^2\rangle\psi-\mu_0\psi^3=0,\qquad \psi(0)=\psi'(\tfrac{1}{2})=0 \tag{9.6}$$

and

$$\mathcal{J}(\psi)\mathcal{L}_2\phi+H\langle\psi^2\rangle\phi-\mu_0\psi^2\phi=0,\qquad \phi(0)=\phi(\tfrac{1}{2})=0 \tag{9.7}$$

with

$$\langle\phi^2\rangle=\langle\psi^2\rangle=\langle\psi_0^2\rangle. \tag{9.8}$$

Equation (9.6) is the Euler equation for the minimum value of the functional

$$G[\psi;\alpha]=(\mathcal{J}^2(\psi)+\mu_0\langle\psi^4\rangle)/\langle\psi^2\rangle^2.$$

Equation (9.7) is the Euler equation for the minimum value of the functional

$$H[\phi;\psi;\alpha,\beta]=(\mathcal{J}(\psi)\,g(\phi)+\mu\langle\psi^2\phi^2\rangle)/\langle\psi^2\phi^2\rangle$$

when ψ is regarded as a given function. We note that (9.6) does not depend on β; the minimizing values $G(\alpha)$ and functions $\psi(z;\alpha)$ are not coupled to (9.7).

When $\alpha=\alpha_0$, $\beta=\beta_0$, $\psi=\psi_0$, $\phi=\phi_0$, we find that

$$H(\alpha_0,\beta_0)=G(\alpha_0).$$

We want (9.6), (9.7) to hold with $G=H$. It is not possible to have $G=H$ for arbitrary values of α and β; however, we can find a function $\alpha(\beta)$ such that

$$H(\alpha(\beta),\beta)=G(\alpha(\beta)) \tag{9.9}$$

with $\alpha_0=\alpha(\beta_0)$ when β is near to β_0.

To construct the function $\alpha(\beta)$ in a series of powers of $(\beta-\beta_0)$, we first note that the derivatives

$$\left.\frac{\partial^{n+m}H(\alpha,\beta)}{\partial\alpha^n\,\partial\beta^m}\right|_{\alpha_0,\beta_0},\qquad \left.\frac{\partial^n G(\alpha)}{\partial\alpha^n}\right|_{\alpha_0}$$

may be computed by the perturbation method. Then we may sequentially solve the equations which arise from repeated differentiation with respect to β of (9.10) at the point $\beta=\beta_0$ for the various derivatives of the function $\alpha(\beta)$. For example, we shall show that $\partial G/\partial\alpha=\partial H/\partial\beta=0$ and $\partial H/\partial\alpha\neq0$ when $\beta=\beta_0$. Then

$$\frac{d\alpha}{d\beta}\frac{\partial H}{\partial\alpha}=-\frac{\partial H}{\partial\beta}+\frac{dG}{d\alpha}\frac{d\alpha}{d\beta}=0.$$

Similarly, given $\partial^2H/\partial\beta^2$ we may compute

$$\frac{d^2\alpha}{d\beta^2}=-\frac{\partial^2H}{\partial\beta^2}\Big/\frac{\partial H}{\partial\alpha}$$

when $\beta=\beta_0$, and so on.

The boundary-value problem for the first α derivatives of ψ, ϕ, G and H are formed from (9.6), (9.7) and (9.8) by direct differentiation. Thus,

$$L_0\psi_{,\alpha}-2\langle\psi_{,\alpha}\mathcal{L}_1\psi_0\rangle\mathcal{L}_1\psi_0+2G_0\langle\psi_{,\alpha}\psi_0\rangle\psi_0 -\mathcal{J}_0\left(\frac{\psi_0''}{\alpha^2}+\psi_0\right)+\frac{\partial G}{\partial\alpha}\langle\psi_0^2\rangle\psi_0=0, \tag{9.10a}$$

$$\psi_{,\alpha}(0)=\psi'_{,\alpha}(\tfrac{1}{2})=0 \tag{9.10b}$$

and

$$\mathcal{N}_0\phi_{,\alpha}-2\langle\psi_{,\alpha}\mathcal{L}_1\psi_0\rangle\mathcal{L}_2\phi_0-2\mu_0\psi_0\psi_{,\alpha}\phi_0+\frac{\partial H}{\partial\alpha}\langle\psi_0^2\rangle\phi_0=0, \tag{9.11a}$$

$$\phi_{,\alpha}(0)=\phi_{,\alpha}(\tfrac{1}{2})=0, \tag{9.11b}$$

$$\langle\phi_0\phi_{,\alpha}\rangle=\langle\psi_0\psi_{,\alpha}\rangle=0. \tag{9.12}$$

It is easy to deduce from the integration of $\langle\psi_0(9.10\text{a})\rangle$ that

$$\frac{\partial G}{\partial\alpha}=0.$$

The problem (9.10) is a differential-integral problem of the type to which Lemma 4 applies; using this lemma in the form (8.1), we have

$$\psi_{,\alpha}=C_{\alpha1}X_1+C_{\alpha2}X_2+X_{\alpha3}$$

where $C_{\alpha1}$ and $C_{\alpha2}$ are completely determined by (8.3) and (9.12).

To solve (9.1) we apply the solvability condition of Lemma 1 to (9.11a) and find that

$$\frac{\partial H}{\partial\alpha}=\frac{2\langle\psi_{,\alpha}\mathcal{L}_1\psi_0\rangle\langle\phi_0\mathcal{L}_2\phi_0\rangle+2\mu_0\langle\psi_0\psi_{,\alpha}\phi_0^2\rangle}{\langle\psi_0^2\rangle\langle\phi_0^2\rangle}\cong17.735895.$$

To find $\phi_{,\beta}$ and $H_{,\beta}$, we must solve

$$\mathcal{N}_0\phi_{,\beta}+H_{,\beta}\langle\psi_0^2\rangle\phi_0-\mathcal{J}_0(\phi_0''/\beta^2+\phi_0)=0, \tag{9.15a}$$

$$\phi_{,\beta}(0)=\phi_{,\beta}(\tfrac{1}{2})=0 \tag{9.15b}$$

and

$$\langle\phi_0\phi_{,\beta}\rangle=0. \tag{9.16}$$

Application of the solvability condition $\langle\phi_0\mathcal{N}_0\phi_{,\beta}\rangle=0$ leads to $H_{,\beta}=0$. Equation (9.16) is satisfied by properly choosing the constant multiple c of the homogenous solution $c\phi_0$ which may be added to any particular solution of (9.15a, b).

The graphs of the functions $\psi_{,\alpha}$, $\phi_{,\alpha}$ and $\phi_{,\beta}$ are shown in Figs. 8, 9 and 10. The values (9.4) and (9.5) may be computed from these and earlier perturbation results.

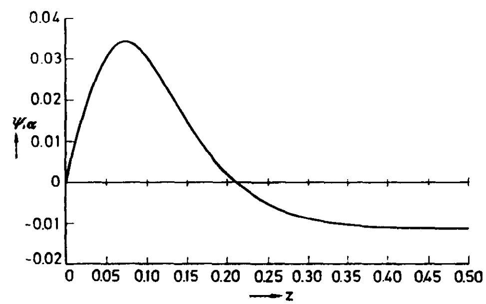

Fig. 8. Graph of the function $\psi_{,\alpha}$ satisfying (9.10)

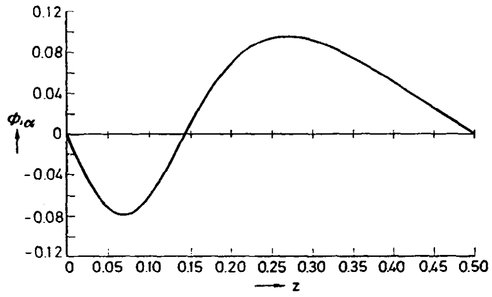

Fig. 9. Graph of the function $\phi_{,\alpha}$ satisfying (9.11)

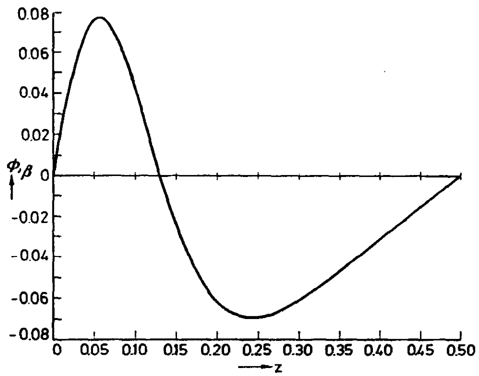

Fig. 10. Graph of the function $\phi_{,\beta}$ satisfying (9.15)

The last matter to be considered is the jump in the second derivative of the bounding heat transport curve at the point of bifurcation. It has already been established that $F_1'(\mu_0)=F_2'(\mu_0)$ (see equation (2.11)) and we seek the value of $F_2''(\mu_0)-F_1''(\mu_0)$. Returning to the notation of (3.1) and (3.3) we note that

$$\frac{dG_2}{d\mu}=\frac{dF_2}{d\mu}+1=\frac{\langle(\theta_1^2+\theta_2^2)^2\rangle}{\langle\theta_1^2+\theta_2^2\rangle^2}=\frac{\langle(\psi^2+\varepsilon b\,\phi^2)^2\rangle}{\langle\psi_0^2\rangle}.$$

At the first point of bifurcation (3.5), we have $\varepsilon=0$, $\mu=\mu_0$, $dG_1/d\mu=dG_2/d\mu$ and

$$\frac{d^2F_2}{d\mu^2}=\frac{d^2G_2}{d\mu^2}=\frac{d^2G_2}{d\varepsilon^2}=4\left\langle\psi_0^2\left(\psi_0\psi_{,\varepsilon}+\alpha_{,\varepsilon}\psi_0\psi_{,\varepsilon}+\frac{b_0}{2}\phi_0^2\right)\right\rangle\Big/\langle\psi_0^2\rangle^2$$

where

$$b_0=-2\langle\psi_0\psi_{,\varepsilon}\rangle/\langle\phi_0^2\rangle.$$

The single-α solution is given with an accuracy of 1% by the asymptotic solution

$$\frac{dF_1}{d\mu}\cong\frac{16}{3}\mu^{\frac{1}{2}}$$

which was given in BJ. The computations of W. J. Sun give

$$F_2''(\mu_0)=\left.\frac{d^2F_2}{d\mu^2}\right|_{\mu_0}=\frac{d^2G_2}{d\varepsilon^2}=-0.001894920 \qquad (9.17\text{a})$$

and

$$F_1''(\mu_0)=\left.\frac{d^2F_1}{d\mu^2}\right|_{\mu_0}\cong-0.00023456. \qquad (9.17\text{b})$$

It is clear that $F_2(\mu)$ falls below $F_1(\mu)$ when $\mu>\mu_0$.

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