

Response Curves for Plane Poiseuille Flow

DANIEL D. JOSEPH[†]

*Department of Aerospace Engineering and Mechanics
University of Minnesota, Minneapolis, Minnesota*

I. Introduction	241
II. The Solution of the Basic Equations for Laminar Poiseuille Flow	245
III. Global Stability of Laminar Poiseuille Flow	246
IV. The Fluctuation Motion and the Mean Motion	249
V. Steady Causes and Stationary Effects	250
VI. Laminar and Turbulent Comparison Theorems	251
VII. Turbulent Plane Poiseuille Flow—An Upper Bound for the Response Curve	253
VIII. The Response Function near the Point of Bifurcation	256
IX. Some Properties of the Bifurcating Solution	261
X. Inferences and Conjectures	263
Appendix: A Formal Bifurcation Theory for Nearly Parallel Flows	268
References	276

I. Introduction

A response function for a fluid motion can be defined as a scalar function that measures the response of the flow to the external forces which induce the motion. For example, in problems of thermal convection, the response function can be taken as the heat transported and the external forces can be regarded as the applied temperature difference. The dimensionless response function relates the Nusselt and Rayleigh numbers. In the flow between rotating cylinders, the response function relates the torque and angular

[†]This work was supported in part by a grant from the National Science Foundation. I had the good fortune to be able to complete the work at the University of Sussex through a grant from the British Science Research Council and the hospitality of the Partial Differential Equations group. I am indebted to Dr. T. S. Chen for many constructive suggestions and for computing the values given in Table I.

velocity. In the example considered below—flow through a plane channel—the external force is the pressure gradient and the response function is the mass flux. The dimensionless response function relates friction factors and Reynolds numbers.

The response function is generally obtained by evaluating a response functional on a suitably defined set of solutions. We study statistically stationary solutions of the Navier Stokes equations for flow through a plane channel. The solutions are defined in Section V; their chief property is that the horizontal average of such solutions is time independent. This is trivially true of laminar Poiseuille flow; we show in Section IX that it is also true of the time-periodic motion which bifurcates from laminar Poiseuille flow, and we shall assume that other solutions observed as turbulence have the property of statistical stationarity. This assumption gives a sense in which fluctuating flow in a steady environment can have steady average properties.

The purpose of this study is best served by drawing a distinction between laminar Poiseuille flow and all the other statistically stationary flows, including the time-periodic bifurcating flow. The subscript l will be used to designate laminar Poiseuille flow. The analysis is conveniently framed in terms of the friction-factor discrepancy $f - f_l$ and the Reynolds number R (or in terms of the friction factor and a Reynolds number or mass-flux discrepancy). Response curves define relations between f and R . In Figs. 1 and 2 we have given various response curves for flow in a channel. The experimental results of Walker *et al.* (1957) are shown as circles in the diagram. The reader should note that for a given channel, the experimental points appear to fall on a single curve. The essential ideas to be explored here are all represented in Figs. 1 and 2. We aim at an understanding of these figures.

To understand the response diagram, it is necessary to develop the concept of stable turbulence. Stable turbulence may not be a stable solution; rather, we envision a stable set of solutions; actually each solution in the stable set need not be very stable but each solution exchanges its stability only with other members of the stable set, and other solutions, outside the stable set, are never realized. The concept of stable turbulence is at the heart of the conjectures of Landau (1944) and Hopf (1948) about the “transition to turbulence through repeated branching of solutions.”

Landau and Hopf regard repeated branching as a process involving continuous bifurcation of manifolds of solutions with N frequencies into manifolds with $N + 1$ frequencies. Here the attractive property of the stable solution is replaced with the attractive property of the manifold. For example, when the data are steady and Reynolds number is small, all solutions are attracted to the steady basic flow. For higher Reynolds numbers, the steady flow is unstable and stability is supposed now to be claimed by an attracting

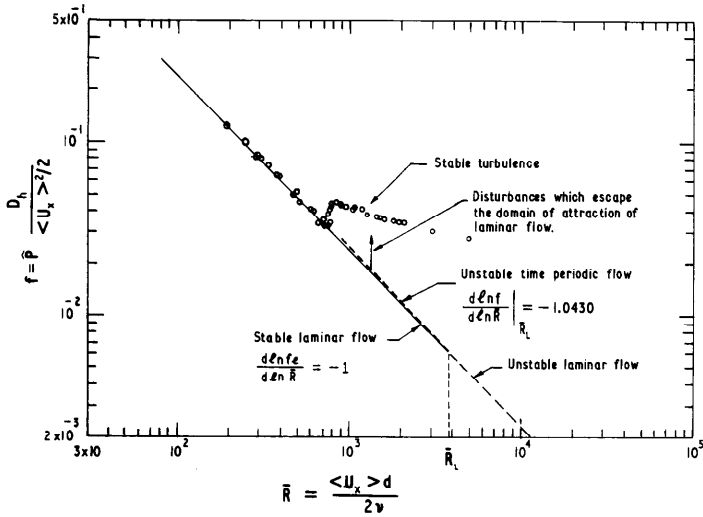


FIG. 1. Response curves for Poiseuille flow. The circles represent the measured response (from Walker *et al.*, 1957). Dashed and solid lines represent unstable and stable solutions, respectively. The reader should verify, using Table 1, that

$$-1.0430 = d \ln f / d \ln \bar{R} |_{\bar{R}_L} < d \ln f / d \ln \bar{R} |_{R(0, \alpha)} \leq -1$$

for $1.021 = \alpha^* < \alpha \leq 1.0964$, where α^* is the minimizing wave number defined by

$$\bar{R}_L = \min_x \bar{R}(0, \alpha) = \bar{R}(0, \alpha^*).$$

It follows that the slopes of successive bifurcating solutions for $\alpha > 1.021$ cross each other. This suggests that the lower branch of the envelope

$$\bar{R}(\epsilon^2) = \min_x \bar{R}(\epsilon^2, \alpha) = \bar{R}(\epsilon^2, \alpha(\epsilon^2))$$

of two-dimensional bifurcating solutions is taken on for wave numbers $\alpha(\epsilon^2) > \alpha^*$.

manifold of time-periodic motions differing from one another in phase alone. Arbitrary solutions of the initial-value problem will be attracted to one or another of the members of the attracting set according to their initial values. At still higher Reynolds numbers, the manifold of periodic solutions loses its stability to a larger manifold of quasiperiodic solutions of independently arbitrary phase. Now, arbitrary solutions of the initial-value problem are attracted to the manifold with two frequencies, and so on.

Important details of the Landau–Hopf conjecture are in need of revision (Joseph, 1973, 1974) but the essential idea—the motion of stable sets of solutions—may yet provide a basis for a correct and fertile mathematical definition of stable turbulence.

In practice, stable turbulence appears to have the property of consistent reproducibility on the average. By this we mean that in a given channel there appears to be a curve, which we have called a response curve, which defines a functional relation between the Reynolds number and the friction factor.

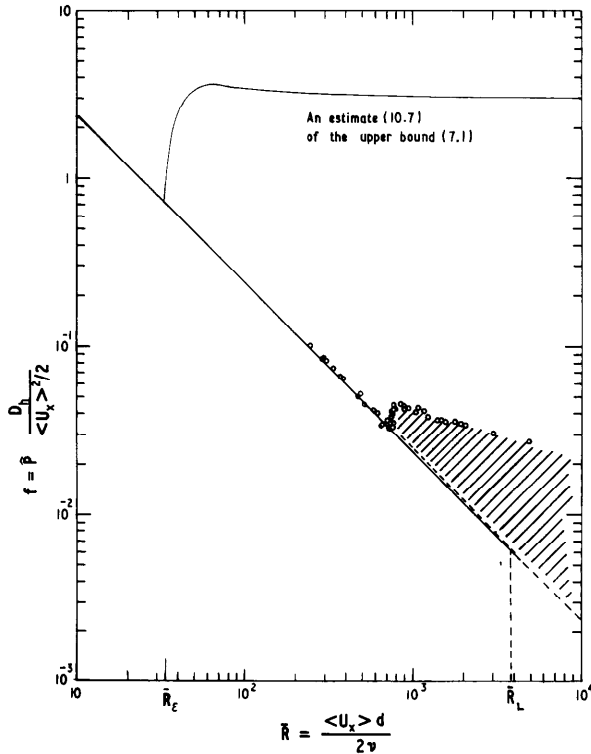


FIG. 2. Response curves for Poiseuille flow. This figure is the same as the previous one except that the estimate (10.7) of the upper bound (7.1) is shown. The shaded region contains many unstable two-dimensional bifurcating solutions.

The existence of such a curve, widely accepted as natural even in elementary books, is actually a remarkable event since the curve is defined over a set of fluctuating turbulent flows each of which differs from its neighbors. In this sense, the response curve may be regarded as giving the steady average response of a fluctuating system subjected to steady external forces.

II. The Solution of the Basic Equations for Laminar Poiseuille Flow

Consider the flow driven through a channel by a constant pressure gradient $\hat{P} > 0$. It is best to visualize this flow as occurring in the annulus between concentric cylinders when the gap is small. The flow proceeds from left to right in the \hat{x} direction increasing where \hat{x} is the axial coordinate. The coordinate \hat{z} is perpendicular to the bounding planes and \hat{y} is the other coordinate. The equations which govern the motion of the fluid in the channel are

$$\frac{\partial \hat{\mathbf{V}}}{\partial \hat{t}} + \hat{\mathbf{V}} \cdot \nabla \hat{\mathbf{V}} = -\nabla \hat{\pi} + \mathbf{e}_{\hat{x}} \hat{P} + \nu \Delta \hat{\mathbf{V}}, \tag{2.1a}$$

$$\text{div } \hat{\mathbf{V}} = 0 \tag{2.1b}$$

in

$$\mathcal{V} = [\hat{x}, \hat{y}, \hat{z} \mid -\infty < \hat{x}, \hat{y} < \infty, -\frac{1}{2}d \leq \hat{z} \leq \frac{1}{2}d]$$

and

$$\hat{\mathbf{V}}(\hat{x}, \hat{y}, \pm \frac{1}{2}d, \hat{t}) = 0. \tag{2.1c}$$

Here $\hat{P} > 0$ is a constant whose value is determined by the applied pressure drop and the total pressure at a point in the fluid is

$$\hat{\pi}(\hat{x}, \hat{y}, \hat{z}, \hat{t}) - \hat{P}\hat{x},$$

where $\hat{\pi}(\hat{x}, \hat{y}, \hat{z}, \hat{t})$ is uniformly bounded in \mathcal{V} .

We shall work with a dimensionless statement of the problem (2.1). The dimensional variables

$$[\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{V}_{\hat{x}}, \hat{V}_{\hat{y}}, \hat{V}_{\hat{z}}, \hat{\pi}, \hat{P}]$$

are obtained from dimensionless variables

$$[x, y, z, t, V_x, V_y, V_z, \pi, P]$$

by multiplying the dimensionless variables by the scale factors

$$\left(\frac{d, d, d, d^2}{\nu}, \frac{\nu}{d}, \frac{\nu}{d}, \frac{\nu}{d}, \frac{\nu^2}{d^2}, \frac{\nu^2}{d^3} \right).$$

The domain occupied by the fluid, described in dimensionless variables, is

$$\mathcal{V} = [x, y, z \mid -\infty < x, y < \infty, -\frac{1}{2} \leq z \leq \frac{1}{2}].$$

We shall need to designate the horizontal average

$$\bar{f}(z, t) = \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L f(x, y, z, t) dx dy$$

and the over-all average

$$\langle f \rangle = \int_{-1/2}^{1/2} \bar{f} dz.$$

It will be convenient to introduce a Reynolds number based on the average of the velocity over a cross section

$$\bar{R}[\bar{V}_x] = \frac{1}{2} \langle \bar{V}_x \rangle. \quad (2.2)$$

The average velocity is proportional to an average mass flux. In a loose terminology we shall call $\langle \bar{V}_x \rangle$ a mass flux.

In dimensionless variables the equations which govern the motion of the fluid are

$$(\partial \mathbf{V} / \partial t) + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla \pi + \mathbf{e}_x P + \Delta \mathbf{V}, \quad (2.3a)$$

$$\operatorname{div} \mathbf{V} = 0, \quad (2.3b)$$

$$V(x, y, \pm \frac{1}{2}, t) = 0. \quad (2.3c)$$

The simplest solution of (2.3) is laminar Poiseuille flow

$$\mathbf{V} = (V_x, V_y, V_z) = [U_x(z), 0, 0],$$

where $U_x(z)$ is the function

$$U_x(z) = \langle U_x \rangle \frac{3}{2} (1 - 4z^2) \equiv \langle U_x \rangle U_\circ(z), \quad (2.4)$$

which solves (2.3) when

$$P = P_l[U_x] = -(d^2 U_x / dz^2) = 12 \langle U_x \rangle. \quad (2.5)$$

III. Global Stability of Laminar Poiseuille Flow

When \bar{R} is small, laminar Poiseuille flow is stable and unique. To prove this, consider the difference between any solution of (2.3) and the laminar solution (2.4). This difference we call a disturbance of laminar Poiseuille flow

$$\mathbf{u} = \mathbf{V} - \mathbf{e}_x U_x(z) = (u, v, w).$$

The disturbance satisfies the evolution equation

$$(\partial \mathbf{u} / \partial t) + U_x \mathbf{e}_x \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla U_x \mathbf{e}_x + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \pi + \Delta \mathbf{u}, \quad (3.1a)$$

and \mathbf{u} and π are elements of the space H of kinematically admissible functions:

$$H = \{ \mathbf{u}, \pi \mid \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}(x, y, \pm \frac{1}{2}) = 0, \quad (\mathbf{u}, \pi) \in AP(x, y) \} \quad (3.1b)$$

where $AP(x, y)$ designates almost periodic functions of x and y .† Initial conditions $\mathbf{u} = \mathbf{u}_0$ (3.1a) are elements of H .

The proof of global stability of $U_x(y)$ starts from the energy identity

$$\frac{1}{2}(d/dt)\langle |\mathbf{u}|^2 \rangle + \langle \mathbf{u} \cdot \nabla U_x \cdot \mathbf{u} \rangle = -\langle |\nabla \mathbf{u}|^2 \rangle. \tag{3.2}$$

The energy identity follows by integration by parts of the equation $\langle \mathbf{u} \cdot (3.1a) \rangle = 0$, using the properties of H . The first term of (3.2) is proportional to the energy, the second term is an energy-production integral, and the third term is an energy-dissipation integral. The production integral may be written as

$$\langle \mathbf{u} \cdot \nabla U_x \cdot \mathbf{u} \rangle = -12\langle U_x \rangle \langle z w u \rangle = -24\bar{R}[U_x] \langle z w u \rangle. \tag{3.3}$$

We shall define the values

$$1/\tilde{\lambda} = \max_H [\langle z w u \rangle / \langle |\nabla \mathbf{u}|^2 \rangle] \tag{3.4}$$

and

$$\tilde{\Lambda} = \min_H [2\langle |\nabla \mathbf{u}|^2 \rangle / \langle |\mathbf{u}|^2 \rangle]. \tag{3.5}$$

Energy Stability Theorem. Suppose that

$$24\bar{R}[U_x] < \tilde{\lambda} \equiv 24\bar{R}_e \tag{3.6}$$

then

$$\langle |\mathbf{u}|^2 \rangle \leq \langle |\mathbf{u}_0|^2 \rangle \exp\{-\tilde{\Lambda}[1 - (24\bar{R}/\tilde{\lambda})]t\}. \tag{3.7}$$

The field which solves problem (3.4) gives the form of the disturbance whose energy increases initially at the smallest value of

$$\bar{R} > \tilde{\lambda}/24. \tag{3.8}$$

Remark. $\tilde{\lambda} > 96$. Hence, global monotonic stability holds when $\bar{R} < 4$. Numerical computations (Busse, 1969) show that $\tilde{\lambda} = 24\bar{R}_e \simeq 793.6$ corresponding to $\bar{R}_e \simeq 33.3$.

Proof. The inequality (2.7) follows directly from (3.2) written as

$$\begin{aligned} d\langle |\mathbf{u}|^2 \rangle/dt &= -2\langle |\nabla \mathbf{u}|^2 \rangle(1 - 24\bar{R}\langle z w u \rangle / \langle |\nabla \mathbf{u}|^2 \rangle) \\ &\leq -2\langle |\nabla \mathbf{u}|^2 \rangle(1 - 24\bar{R}/\tilde{\lambda}). \end{aligned}$$

† An almost periodic function is a uniformly bounded function whose value at an x, y point (for fixed t and z) is repeated, nearly, at some distant point of the x, y plane. The graph of an almost periodic function may appear chaotically irregular. The overbar average of an AP function always exists and, in fact, forms the natural scalar product for the space of almost periodic functions; an $AP(x, y)$ function $f(x, y)$ vanishes uniformly if and only if $\bar{f}^2 = 0$.

If (3.6) holds, then

$$d\langle |\mathbf{u}|^2 \rangle / dt \leq -\langle |\mathbf{u}|^2 \rangle \tilde{\lambda} (1 - 24R/\tilde{\lambda})$$

and the inequality (3.7) follows by integration. Suppose that initially the disturbance is in the form given by the solution of the maximum problem (3.4). Then if (3.8) holds,

$$d\langle |\mathbf{u}|^2 \rangle / dt = -2\langle |\nabla \mathbf{u}|^2 \rangle (1 - 24\bar{R}/\tilde{\lambda}) \geq 0.$$

This proves the theorem.

The result just given shows that the laminar flow (2.4) is the only steady flow which is possible when $\bar{R} < \tilde{\lambda}/24 \simeq 33.3$.

To prove that $\tilde{\lambda} > 96$ we note that when $-\frac{1}{2} \leq z \leq 0$:

$$u(x, y, z, t) = \int_{-1/2}^z \frac{\partial u}{\partial z'} dz' \leq \left(z + \frac{1}{2}\right)^{1/2} \left[\int_{-1/2}^0 \left(\frac{\partial u'}{\partial z'}\right)^2 dz' \right]^{1/2}$$

with an identical inequality for w . Then

$$\bar{u}^2 \leq \left(z + \frac{1}{2}\right) \int_{-1/2}^0 |\bar{\nabla} \mathbf{u}|^2 dz' \quad (3.9)$$

with an identical inequality for $|\bar{w}|^2$. Then

$$\begin{aligned} z\bar{w}\bar{u} &\leq |z| (\bar{w}^2 \bar{u}^2)^{1/2} \leq \frac{1}{2} |z| (\bar{w}^2 + \bar{u}^2) \\ &\leq \frac{1}{2} |z| \left(z + \frac{1}{2}\right) \int_{-1/2}^0 |\bar{\nabla} \mathbf{u}|^2 dz' \end{aligned}$$

and

$$\begin{aligned} \int_{-1/2}^0 z' \bar{w}\bar{u} dz' &\leq \frac{1}{2} \left[\int_{-1/2}^0 |z'| \left(z' + \frac{1}{2}\right) dz' \right] \left[\int_{-1/2}^0 |\bar{\nabla} \mathbf{u}|^2 dz' \right] \\ &\leq (1/96) \int_{-1/2}^0 |\bar{\nabla} \mathbf{u}|^2 dz'. \end{aligned} \quad (3.10a)$$

A similar argument relative to the wall at $z = \frac{1}{2}$ gives

$$\int_0^{1/2} z' \bar{w}\bar{u} dz' \leq (1/96) \int_0^{1/2} |\bar{\nabla} \mathbf{u}|^2 dz'. \quad (3.10b)$$

Adding (3.10a) and (3.10b) we find that

$$\langle z w u \rangle \leq (1/96) \langle |\nabla \mathbf{u}|^2 \rangle.$$

Better *a priori* estimates than $\tilde{\lambda} > 96$ can be derived using the Fourier series for *AP* functions and the constraint $\text{div } \mathbf{u} = 0$.

In the proof of the energy stability theorem, we compared flows in a family characterized by one and the same constant part of the pressure gradient $P = P_1$. This pressure gradient corresponds to the unbounded (as $x \rightarrow \pm \infty$)

or non-almost-periodic component of the pressure. There is no loss of generality in dividing the different flows in this way; we get all the possible flows by allowing P to range over positive values. We could just as well have considered families of flows having different pressure gradients and the same mass flux $\langle V_x \rangle = \langle U_x \rangle$. Though a term $(P - P_t)\mathbf{e}_x$ would then appear in (3.1a), it would again disappear in the energy identity (3.2) because $\langle \mathbf{u} \rangle = 0$.

The energy criterion guarantees monotonic and global stability (monotonic stability to all disturbances when $\tilde{R} < 33.3$). This criterion gives sufficient conditions for stability but is silent about instability. In fact, experiments (see Fig. 1) suggest that all disturbances eventually decay when $\bar{R} < \bar{R}_G \simeq 650$. There is certainly a best limit \bar{R}_G for global stability. Theoretical methods for obtaining the value \bar{R}_G are at present unknown.

IV. The Fluctuation Motion and the Mean Motion

The motion of the fluid in the channel may be decomposed in several ways. In Section III the motion \mathbf{V} was divided into laminar flow $U_x(y)$ plus a disturbance $\mathbf{u}(x, y, z, t)$. Now we want to resolve this motion into mean and fluctuating parts

$$[\mathbf{V}, \pi] = [\bar{\mathbf{V}} + \mathbf{u}, \bar{\pi} + p], \quad (4.1)$$

where the fluctuations \mathbf{u} and p have a zero mean $\bar{\mathbf{u}} = \bar{p} = 0$.

An elementary consequence of the horizontal average of the continuity equation is that

$$\bar{V}_z = 0. \quad (4.2)$$

To obtain the equations for the mean and fluctuating motion, insert (4.1) into (2.3). Using (4.2) we find that

$$\begin{aligned} \frac{\partial \bar{\mathbf{V}}}{\partial t} + \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot [\bar{\mathbf{V}}\mathbf{u} + \mathbf{u}\bar{\mathbf{V}} + \mathbf{u}\mathbf{u}] \\ = -\nabla p - \frac{\partial \bar{\pi}}{\partial z} \mathbf{e}_z + P\mathbf{e}_x + \frac{\partial^2 \bar{\mathbf{V}}}{\partial z^2} + \Delta \mathbf{u}. \end{aligned} \quad (4.3)$$

The average of (4.3) is

$$(\partial \bar{\mathbf{V}}/\partial t) + \nabla \cdot \bar{\mathbf{u}}\mathbf{u} = -\mathbf{e}_z(\partial \bar{\pi}/\partial z) + \mathbf{e}_x P + (\partial^2 \bar{\mathbf{V}}/\partial z^2), \quad (4.4)$$

and the difference [(4.4) - (4.3)] is

$$(\partial \mathbf{u}/\partial t) + \nabla \cdot (\mathbf{V}\mathbf{u} + \mathbf{u}\mathbf{v} + \mathbf{u}\mathbf{u} - \bar{\mathbf{u}}\mathbf{u}) = -\nabla p + \Delta \mathbf{u}, \quad (4.5)$$

where $\nabla \cdot \mathbf{u} = 0$ in \mathcal{V} and, at $z = \pm \frac{1}{2}$,

$$\mathbf{V} = (\bar{V}_x, \bar{V}_y, 0) = \mathbf{u} = 0 \quad (4.6)$$

It is possible to consider an initial-value problem for the mean and fluctuating motions. Since the initial velocity field (V_x, V_y, V_z) is arbitrary, we may assign $(\bar{V}_x, \bar{V}_y, 0)$ and (u, v, w) arbitrarily, provided only that the fluctuations have a zero horizontal average.

Associated with the decomposition into a fluctuation and mean motion are two energy identities, one for the mean motion and one for the fluctuation motion. These identities follow from $\langle \bar{\mathbf{V}} \cdot (4.4) \rangle = 0$ and $\langle \mathbf{u} \cdot (4.5) \rangle = 0$, respectively:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle |\bar{V}_x|^2 + |\bar{V}_y|^2 \rangle + \left\langle \bar{V}_x \frac{\partial \overline{wu}}{\partial z} + \bar{V}_y \frac{\partial \overline{wv}}{\partial z} \right\rangle \\ = P \langle \bar{V}_x \rangle - \left\langle \left| \frac{\partial \bar{V}_x}{\partial z} \right|^2 + \left| \frac{\partial \bar{V}_y}{\partial z} \right|^2 \right\rangle, \end{aligned} \quad (4.7)$$

$$\frac{1}{2} \frac{d}{dt} \langle |\mathbf{u}|^2 \rangle + \left\langle \overline{wu} \frac{\partial \bar{V}_x}{\partial z} + \overline{wv} \frac{\partial \bar{V}_y}{\partial z} \right\rangle = - \langle |\nabla \mathbf{u}|^2 \rangle. \quad (4.8)$$

The total energy is

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle |\bar{V}_x|^2 + |\bar{V}_y|^2 + |\mathbf{u}|^2 \rangle \\ = P \langle \bar{V}_x \rangle - \left\langle |\nabla \mathbf{u}|^2 + \left| \frac{\partial \bar{V}_x}{\partial z} \right|^2 + \left| \frac{\partial \bar{V}_y}{\partial z} \right|^2 \right\rangle. \end{aligned} \quad (4.9)$$

The energy source for turbulent Poiseuille flow is $P \langle \bar{V}_x \rangle$ work of the pressure gradient on the mean flow.

V. Steady Causes and Stationary Effects

We face the task of describing in some useful sense all the solutions of the Navier–Stokes equations that can arise when the steady external conditions are those giving rise to Poiseuille flow. Some progress with this hard problem can be made if we admit the basic assumption that steady external conditions can have a stationary effect even when the motion is fluctuating. This assumption is supported by the consistent reproducibility of certain average values in turbulent flow.

The stationary effects of steady external conditions need not imply unique steady solutions. Only when $\bar{R} < \bar{R}_G$ do all flows tend to a unique steady

flow. When $\bar{R} > \bar{R}_G$, there are at least two solutions possible: the unstable laminar flow and any one of the motions that replace laminar flow. The limiting flows that actually occur when $\bar{R} > \bar{R}_G$ are those that are in some sense stable. The stable solutions with $\bar{R} > \bar{R}_G$ need not be unique; indeed we envision stable *sets* of solutions. Though such solutions would lack uniqueness in the ordinary sense, it is consistent with observations to postulate the existence of stable sets of solutions sharing common properties in the average.

The basic property we shall assume here is (1) that all horizontal averages are time independent. This assumption says that a consequence of steady exterior conditions (boundary conditions and pressure drop) is that horizontal averages are steady. The fluctuation fields themselves can be very unsteady. We also shall assume (2) that velocity components have a zero mean value unless a nonzero mean value is forced externally. Property (2) implies that $\bar{V}_y = 0$. Following Howard (1963), we call fields which share properties (1) and (2) statistically stationary.†

VI. Laminar and Turbulent Comparison Theorems

Assuming properties (1) and (2) of statistical stationarity, equations (4.4) may be written as

$$d/dz[\overline{wu} - (d\bar{V}_x/dz) - Pz] = 0 \quad (6.1)$$

$$d/dz[\overline{wv}] = 0, \quad (6.2)$$

$$d/dz[\overline{w^2} + \bar{\pi}] = 0, \quad (6.3)$$

and Eq. (4.8) becomes

$$\langle \overline{wu}(d\bar{V}_x/dz) \rangle = -\langle |\nabla \mathbf{u}|^2 \rangle. \quad (6.4)$$

Equation (6.2) shows that \overline{wv} is a constant whose value is zero at the boundary and elsewhere.

A basic and important consequence of statistical stationarity is that (6.1) has a first integral

$$\overline{wu} - \langle \overline{wu} \rangle = Pz + (d\bar{V}_x/dz). \quad (6.5)$$

† The observation that steady exterior conditions should be expected to lead to stationary turbulence needs qualification. Stationary turbulence evidently cannot exist in Hagen-Poiseuille flow and plane Couette flow when the fluctuations are infinitesimal (the linearized stability theory shows that *all* infinitesimal disturbances decay). The analysis given here applies to stationary turbulence when it exists. It should also be noted that motions which are here called statistically stationary need not be turbulent. Steady laminar motions fit our definitions and are to be included in the class of statistically stationary turbulence.

Using (2.5) we may write this integral as

$$\overline{wu} - \langle wu \rangle = (P[\bar{V}_x] - P_l[U_x])z + (d/dz)(\bar{V}_x - U_x). \quad (6.6)$$

Combining (6.4) and (6.5) we find that†

$$P\langle zwu \rangle = \langle |\nabla \mathbf{u}|^2 \rangle + \langle [\overline{wu} - \langle wu \rangle]^2 \rangle. \quad (6.7)$$

Forming $\langle z(6.6) \rangle = 0$, we find, using (6.7) that

$$\langle zwu \rangle = (P[\bar{V}_x] - P_l[U_x])/12 + \langle U_x - \bar{V}_x \rangle > 0. \quad (6.8)$$

Equation (6.8) relates the pressure-gradient discrepancy $P[\bar{V}_x] - P_l[U_x]$ and the mass-flux discrepancy $\langle U_x - \bar{V}_x \rangle$ and forms the basis for the following laminar-turbulent comparison theorems.

Mass-Flux-Discrepancy Theorem. Statistically stationary turbulent Poiseuille flow has a smaller mass flux ($\langle U_x \rangle > \langle \bar{V}_x \rangle$) than the laminar Poiseuille flow with the same constant component of the pressure gradient ($P[\bar{V}_x] = P_l[U_x]$). This theorem was first proved by Thomas (1942); the proof given here is essentially due to Busse (1969, 1970).

Pressure-Gradient-Discrepancy Theorem. Statistically stationary turbulent Poiseuille flow has a larger pressure gradient $P[\bar{V}_x] > P_l[\bar{V}_x]$ than the laminar flow with same mass flux ($\langle U_x \rangle = \langle \bar{V}_x \rangle$).

For the case of equal pressure gradients one may prove the following theorem.

Equal-Shear-Stress Theorem. Suppose that $P[\bar{V}_x] = P_l[U_x]$ and $\bar{V}_x(z)$ is an even function of z , then

$$d\bar{V}_x/dz = (dU_x/dz)|_{z=\pm 1/2}.$$

In Fig. 3 we have sketched the comparison for the equal-pressure-gradient case.

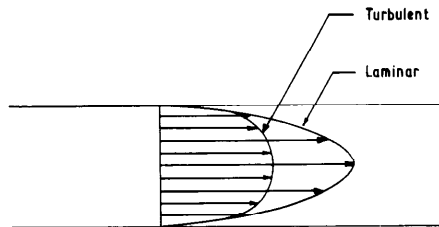


FIG. 3. Mass-flux discrepancy theorem. If the pressure gradient is fixed, the laminar flow U_x and turbulent mean flow \bar{V}_x have the same slope at the wall. The mass efflux of the laminar flow exceeds the turbulent flow by an amount $\langle U_x - \bar{V}_x \rangle = \langle zwu \rangle > 0$.

† Reynolds (1895) used Eq. (6.7) to find critical values of P . He notes the presence of the quartic integrals makes (6.7) "... very complex and difficult of interpretation except in so far as showing that the resistance varies as a power of velocity higher than the first."

VII. Turbulent Plane Poiseuille Flow—An Upper Bound for the Response Curve

Busse (1969, 1970), following earlier work of Howard (1963) on turbulent convection, and Howard (1972) have considered the variational problem for the response curve which is implied by (6.7). In a convenient formulation one introduces the mass-flux discrepancy

$$\mu = \langle U_x - \bar{V}_x \rangle = \langle zwu \rangle > 0$$

as a parameter and seeks the minimum value

$$\tilde{P}(\mu) = \min_H \mathcal{P}[\mathbf{u}; \mu] \tag{7.1}$$

of the functional

$$\begin{aligned} \mathcal{P}[\mathbf{u}; \mu] - 12\mu &= \frac{\langle |\nabla \mathbf{u}|^2 \rangle}{\langle zwu \rangle} + \mu \left(\frac{\langle [\bar{w}u - wu]^2 \rangle}{\langle zwu \rangle^2} - 12 \right) \\ &= \frac{\langle |\nabla \mathbf{u}|^2 \rangle}{\langle zwu \rangle} + \mu \frac{\langle [\bar{w}u - \langle wu \rangle - 12z\langle zwu \rangle]^2 \rangle}{\langle zwu \rangle} \end{aligned} \tag{7.2}$$

over the space \bar{H} of kinematically admissible fluctuation fields; \bar{H} is the subspace of H with zero mean values $\bar{\mathbf{u}} = 0$.

The values $\tilde{P}(\mu)$ give a lower bound for the pressure gradients possible in statistically stationary turbulent Poiseuille flow. The reader is referred to Busse (1970) and to Howard (1972) for further details. We note when $\mu = 0$, the variational problem for $\tilde{P}^{-1}(0)$ coincides with the problem (3.4) except that the competitors for the maximum of (3.4) do not need to have $\bar{\mathbf{u}} = 0$. The zero mean condition is satisfied by the winner of the competition (3.4) and therefore

$$\tilde{P}(0) = \tilde{\lambda} \simeq 793.6. \tag{7.3}$$

Using a method given by Howard (1963) one can show that

$$d\tilde{P}/d\mu = \langle [\bar{w}u - \langle wu \rangle]^2 \rangle / \langle zwu \rangle^2$$

and $d\tilde{P}/d\mu$ is a decreasing function of μ .

A complete mathematical solution of (7.1) is not yet known [see Howard (1972) for the most recent discussion]. The solution seems to generate repeated bifurcations as μ is increased in a manner reminiscent of the Landau–Hopf conjectures discussed in the Introduction (see Joseph, 1974).

Fortunately it is possible to construct *a priori* estimates of the solutions of

(7.1) which give explicit upper bounds for the response function. One such estimate is constructed below.†

The response function $P(\mu)$, $\mu = \langle U_x - \bar{V}_x \rangle$, satisfies the inequalities

$$P(\mu) \geq \tilde{P}(\mu) \geq \tilde{\tilde{P}}(\mu) \\ = 12\mu + \begin{cases} \tilde{P}(0) + 576\mu/[\tilde{P}(0) + 48] & \text{for } \mu \leq \mu^* \\ 48\sqrt{\mu} - 48 & \text{for } \mu \geq \mu^* \end{cases} \quad (7.4)$$

where

$$\mu^* = [\tilde{P}(0) + 48]^2/576.$$

The bound (7.4) is reinterpreted in terms of the friction factor and Reynolds number in Section X and is shown graphically in Fig. 2.

To prove (7.4) we must first establish the estimate

$$\frac{\langle [\bar{w}u - \langle wu \rangle - 12z\langle zwu \rangle]^2 \rangle}{\langle zwu \rangle^2} \geq \frac{576}{D + 48}, \quad (7.5)$$

where

$$D = \langle |\nabla \mathbf{u}|^2 \rangle / \langle zwu \rangle.$$

Assuming (7.5), for the moment, we note that from (7.1) and (7.2)

$$\tilde{P}(\mu) - 12\mu \geq D + 576\mu/(D + 48) \geq \min_D [D + 576\mu/(D + 48)]. \quad (7.6)$$

This minimum is attained when $D + 48 = (576\mu)^{1/2} = 24\mu^{1/2}$ and is equal to $2(576\mu)^{1/2} - 48 = 48\sqrt{\mu} - 48$. However, by (7.1), $D \geq \tilde{P}(0)$ and therefore $D + 48$ cannot equal $24\mu^{1/2}$ if $24\mu^{1/2} < \tilde{P}(0) + 48$; that is, if $\mu < [\tilde{P}(0) + 48]^2/576 = \mu^*$. Therefore when $\mu < \mu^*$, $D + 576\mu/(D + 48)$ is an increasing function D which must be minimum when D has its smallest positive value $D = \tilde{P}(0)$.

Hence we may continue (7.6) as

$$= \begin{cases} \tilde{P}(0) + 576\mu/[\tilde{P}(0) + 48] \\ 48\sqrt{\mu} - 48 \end{cases}$$

completing the proof of (7.4).

It remains to prove the estimate (7.5). We first note that the relation

$$\langle \bar{w}u - \langle wu \rangle - 12z\langle zwu \rangle \rangle = 0 \quad (7.7)$$

implies that there is a value $z = \hat{z}$, $-\frac{1}{2} \leq \hat{z} \leq \frac{1}{2}$, such that

$$\bar{w}u - \langle wu \rangle - 12\hat{z}\langle zwu \rangle = 0.$$

It is convenient to use the coordinate $\zeta = z + \frac{1}{2}$ and to define

$$f(\zeta) = \bar{w}u - \langle wu \rangle - 12(\zeta - \frac{1}{2})\langle zwu \rangle.$$

† The estimate (7.4) is due to V. Gupta and D. Joseph.

At $\zeta = \hat{\zeta} + \frac{1}{2}$ we have $f(\hat{\zeta}) = 0$.

Define

$$D_{0\hat{\zeta}} = \int_0^{\hat{\zeta}} |\overline{\nabla \mathbf{u}}|_{\text{av}}^2 d\zeta;$$

clearly

$$D_{0\hat{\zeta}} = (D_{0\hat{\zeta}}/D_{01})D_{01} = \alpha D_{01}$$

and

$$D_{\hat{\zeta}1} = D_{01}[1 - (D_{0\hat{\zeta}}/D_{01})] = (1 - \alpha)D_{01},$$

where $\alpha = D_{0\hat{\zeta}}/D_{01}$.

Following the derivation of (3.5) we may find

$$|\overline{wu}| \leq |\overline{w^2 u^2}|^{1/2} \leq \zeta \left[\int_0^{\hat{\zeta}} \left(\frac{\widehat{cw}}{\widehat{c\zeta}} \right)^2 d\zeta \right]^{1/2} \left[\int_0^{\hat{\zeta}} \left(\frac{\widehat{cu}}{\widehat{c\zeta}} \right)^2 d\zeta \right]^{1/2} \leq \frac{\zeta}{2} D_{0\hat{\zeta}}.$$

When $\hat{\zeta} > \zeta$,

$$|\overline{wu}| \leq \frac{\zeta}{2} \frac{D_{0\hat{\zeta}}}{D_{0\hat{\zeta}}} \frac{D_{0\hat{\zeta}}}{D_{01}} D_{01} \leq \frac{\alpha \zeta}{2} D_{01}, \tag{7.8}$$

and with $\beta = \langle wu \rangle / \langle zwu \rangle$ and $D = D_{01} / \langle zwu \rangle$,

$$\begin{aligned} \frac{|f(\zeta)|}{\langle zwu \rangle} &= \left| \frac{\overline{wu}}{\langle zwu \rangle} - \beta - 12 \left(\zeta - \frac{1}{2} \right) \right| \geq |6 - \beta| - \left| 12\zeta - \frac{\overline{wu}}{\langle zwu \rangle} \right| \\ &\geq |6 - \beta| - \zeta \left(12 + \frac{\alpha}{2} D \right) \equiv \frac{g(\zeta)}{\langle zwu \rangle}. \end{aligned} \tag{7.9}$$

At $\zeta = 0$ we have $|f(0)| = g(0) = |6 - \beta| \langle zwu \rangle$. At $\zeta = \tilde{\zeta} = |6 - \beta| / (12 + \frac{1}{2}\alpha D)$ we have $g(\tilde{\zeta}) = 0$. Since $|f(\zeta)| \geq g(\zeta)$ when $0 < \zeta < \hat{\zeta}$ [$f(\hat{\zeta}) = 0$] we must have

$$\hat{\zeta} > \tilde{\zeta}. \tag{7.10}$$

When $\zeta < \tilde{\zeta}$, (7.8) holds and $|f(\zeta)| \geq |g(\zeta)|$.

Using (7.9) and (7.10) we find that

$$\begin{aligned} \int_0^{\hat{\zeta}} f^2 d\zeta &\geq \int_0^{\hat{\zeta}} |f|^2 d\zeta \geq \int_0^{\hat{\zeta}} |g|^2 d\zeta \\ &= \langle zwu \rangle^2 \int_0^{\hat{\zeta}} \{ |6 - \beta| - \zeta(12 + \frac{1}{2}\alpha D) \}^2 d\zeta \\ &= \langle zwu \rangle^2 |6 - \beta|^2 \int_0^{\hat{\zeta}} (1 - \zeta/\tilde{\zeta})^2 d\zeta = \langle zwu \rangle^2 |6 - \beta|^2 \frac{4}{3} \tilde{\zeta} \\ &= \langle zwu \rangle^2 |6 - \beta|^3 / 3(12 + \frac{1}{2}\alpha D). \end{aligned} \tag{7.11}$$

Analysis relative to the wall at $z = \frac{1}{2}$ (introduce $\zeta = z - \frac{1}{2}$) leads to an inequality of the form (7.11) with $|6 - \beta|$ replaced with $|6 + \beta|$ and α replaced by $1 - \alpha$.

We have

$$\begin{aligned} & \frac{\langle [\overline{wu} - \langle wu \rangle - 12z\langle zwu \rangle]^2 \rangle}{\langle zwu \rangle^2} \\ &= \frac{1}{\langle zwu \rangle^2} \int_{-1/2}^{1/2} f^2(z) dz = \frac{1}{\langle zwu \rangle^2} \int_0^1 f^2(\zeta) d\zeta \\ &\geq \frac{1}{3} \left(\frac{|6 + \beta|^3}{12 + \frac{1}{2}(1 - \alpha)D} + \frac{|6 - \beta|}{12 + \frac{1}{2}\alpha D} \right) \\ &\geq \frac{1}{3} \min_{\alpha, \beta} \left(\frac{|6 + \beta|^3}{12 + \frac{1}{2}(1 - \alpha)D} + \frac{|6 - \beta|}{12 + \frac{1}{2}\alpha D} \right), \end{aligned} \quad (7.12)$$

where $0 \leq \alpha \leq 1$, $-\infty < \beta < \infty$. The minimum of the right-hand side of (7.12) is found at $\beta = 0$ and $\alpha = \frac{1}{2}$. Hence

$$\frac{\langle [\overline{wu} - \langle wu \rangle - 12z\langle zwu \rangle]^2 \rangle}{\langle zwu \rangle^2} \geq \frac{2}{3} \frac{6^3}{12 + \frac{1}{4}D} = \frac{576}{D + 48}.$$

This completes the proof of (7.5).

A slightly better estimate than (7.4) which is valid for large values of the mass-flux discrepancy μ (but not small values) has been given by Busse (1969). Busse considers the variational problem (7.1) in a weakened class of fluctuation fields \mathbf{u} which need not be solenoidal. He solves the Euler equations for this problem in the limit $\mu \rightarrow \infty$ and finds that

$$P(\mu) > 12\mu + 96\sqrt{\mu}/\sqrt{3} + O(\mu^{-1/2})$$

instead of (7.4). The present estimate (7.4) holds for all $\mu \geq 0$. Yet better estimates can be obtained (Busse, 1970) when additional assumptions are made about the form of the minimizing solution of the variational problem (7.1).

VIII. The Response Function near the Point of Bifurcation

Now we turn away from the energy estimates. We shall consider the linear theory of instability and the time-periodic solution which arises from this instability in the neighborhood of the point of bifurcation. We want first to show how to enrich the physical content of the perturbation theory by a proper choice of the amplitude parameter. This is accomplished by defining

the amplitude as a friction-factor discrepancy. To obtain an expression through which the friction-factor discrepancy may be related to the bifurcating solution, it is necessary to compare two different resolutions of the same motion V :

$$\mathbf{V} = \bar{V}_x(z)\mathbf{e}_x + \mathbf{u}(x, y, z, t) = U_x(z)\mathbf{e}_x + \mathbf{u}'(x, y, z, t), \quad (8.1)$$

where $\bar{V}_x(z)$ is the mean motion, \mathbf{u} is the fluctuation velocity, $U_x(z)$ is laminar flow with mass flux $\langle U_x \rangle$, and \mathbf{u}' is the disturbance of $U_x(z)$ (formerly called \mathbf{u}). Equations (8.1) imply that

$$w = w', \quad v = v', \quad \text{and} \quad \bar{V}_x + u = U_x + u'. \quad (8.2)$$

The relation

$$\bar{V}_x(z) = U_x(z) + \bar{u}'(z) \quad (8.3)$$

follows directly from the horizontal average of (8.1). We note that the equality (8.1) leading to (8.3) is possible only if the bifurcating flow is statistically stationary (the stationary property is verified in the remarks closing Section IX).

We shall make use of the following relation:

$$\overline{uw} = \overline{u'w'}. \quad (8.4)$$

To prove (8.4), we use (8.3) and (8.2) to write

$$\overline{uw} = \overline{w\bar{u}'} = \overline{(U_x - \bar{V}_x + u')w'} = (U_x - \bar{V}_x)\bar{w}' + \overline{u'w'} = \overline{u'w'}.$$

Using (8.4) we may rewrite (6.8) as

$$\langle zw'u' \rangle = \frac{1}{12} [P[\bar{V}_x] - P_l[U_x]] + \langle U_x - \bar{V}_x \rangle. \quad (8.5)$$

This basic relation will be used to relate the time periodic bifurcating solutions to the response diagrams measured in the experiments.

In Section IX we shall construct statistically stationary periodic flows which bifurcate from laminar Poiseuille flow at the critical point of the linear theory of stability. We will restrict our analysis to two-dimensional motions. This restriction is actually forced by Squire's theorem; this theorem shows that the solution which bifurcates at the smallest value of R is actually two dimensional. The bifurcating solution is unstable and would not be easily observed in experiments. The relation of the two-dimensional solutions to experiments will be discussed in Section X.

For any two-dimensional motion we may introduce a stream function $\tilde{\Psi}$. The resolution of the motion into Poiseuille flow with mass flux $\langle U_x \rangle$ plus a disturbance

$$\tilde{\Psi} = \tilde{\Psi} + \Psi', \quad U_x = \partial\tilde{\Psi}/\partial z, \quad (u', -w') = [(\partial\Psi'/\partial z), (\partial\Psi'/\partial x)] \quad (8.6)$$

leads to the following problem for Ψ' :

$$\frac{\hat{c}}{\hat{c}t} \Delta \Psi' + \langle U_x \rangle \left(U_{\diamond}(z) \frac{\hat{c}}{\hat{c}x} \Delta \Psi' - \frac{d^2 U_{\diamond}}{dz^2} \frac{\partial \Psi'}{\partial x} \right) + J(\Psi', \Delta \Psi') - \Delta^2 \Psi' = 0, \quad (8.7a)$$

$$\Psi' = \partial \Psi' / \partial z = 0 \Big|_{z=-1/2}, \quad \partial \Psi' / \partial z = 0 \Big|_{z=1/2}, \quad \Psi \Big|_{z=1/2} = \langle \bar{u}' \rangle, \quad (8.7b,c,d)$$

where

$$J(\Psi', \Delta \Psi') = \left(\frac{\partial \Psi'}{\partial z} \frac{\hat{c}}{\hat{c}x} \Delta \Psi' - \frac{\partial \Psi'}{\partial x} \frac{\partial}{\partial z} \Delta \Psi' \right).$$

The total mass flux $\langle \bar{V}_x \rangle$ of any two-dimensional statistically stationary solution of (8.7) may be written as

$$\langle \bar{V}_x \rangle = \langle U_x \rangle + \Psi' \Big|_{z=1/2}, \quad (8.8)$$

where $\langle U_x \rangle$ is the mass flux for a suitably chosen laminar flow with pressure gradient $P_{II}[U_x]$. The bifurcating flow and laminar flow have the same mass flux $\langle U_x \rangle = \langle \bar{V}_x \rangle$ if and only if $\Psi' \Big|_{z=1/2} = 0$.

It is convenient, and completely general, to restrict one's attention to the special case $\Psi' \Big|_{1/2} = 0$ in the construction of the bifurcating solution. To completely specify the bifurcation problem, it will be necessary to fix the spatial periodicity. Then the time-periodic solution which bifurcates from laminar flow is determined uniquely to within an arbitrary phase. This unique solution may be computed relative to a laminar flow for which $\langle U_x \rangle = \langle \bar{V}_x \rangle$. To show this we will now reduce problem (8.7) to the study of bifurcation of laminar flow with $\langle U_x \rangle = \langle \bar{V}_x \rangle$.

The stream function for the bifurcating solution may always be written as

$$\tilde{\Psi} = \tilde{\Psi}(z) + \Psi', \quad \Psi' = \Phi(z) + \Psi'', \quad (8.9)$$

where $\Phi(z)$ is a function of z alone which can be chosen so that

$$\Phi(-\tfrac{1}{2}) = d\Phi(-\tfrac{1}{2})/dz = d\Phi(\tfrac{1}{2})/dz = 0, \quad \Phi(\tfrac{1}{2}) = \langle \bar{u}' \rangle, \quad (8.10)$$

and $\Psi''(x, z, t)$ is a flow satisfying (8.7a)–(8.7c) with zero mass flux

$$\langle \hat{c} \Psi'' / \hat{c} z \rangle = \Psi'' \Big|_{z=1/2} = 0. \quad (8.11)$$

$\Phi(z)$ is a Poiseuille flow. The mass flux for the Poiseuille flow $\tilde{\Psi}(z) + \Phi(z)$ is $\langle \bar{V}_x \rangle$. Moreover,

$$\frac{1}{12}P_i[U_x] + \langle \bar{V}_x - U_x \rangle = \frac{1}{12}P_i[\bar{V}_x] \tag{8.12}$$

and

$$\langle zw''u'' \rangle = \frac{1}{12}(P[\bar{V}_x] - P_i[\bar{V}_x]). \tag{8.13}$$

Proof. Substitute (8.9) into (8.7) to find that $d^4\Phi/dz^4 = 0$ and, using (8.10),

$$\Phi = \frac{3}{2}\langle \bar{u}' \rangle(z - \frac{4}{3}z^3) + \frac{1}{2}\langle \bar{u}' \rangle. \tag{8.14}$$

It follows that

$$\begin{aligned} (\partial/\partial z)(\tilde{\Psi} + \Phi) &= U_x(z) + d\Phi/dz \\ &= \frac{3}{2}\langle U_x + \bar{u}' \rangle(1 - 4z^2) \\ &= \frac{3}{2}\langle \bar{V}_x \rangle(1 - 4z^2) = \langle \bar{V}_x \rangle U_\diamond(z). \end{aligned}$$

The pressure gradient for this laminar flow is

$$P_i[U_x + \bar{u}'] = -(d^3\tilde{\Psi} + \Phi)/dz^3 = 12\langle \bar{V}_x \rangle = P_i[\bar{V}_x].$$

This proves (8.12). Noting next that $\langle w''f(z) \rangle = 0$ for any function of z alone, and using (8.9) and (8.12), we may rewrite (8.5) as

$$\begin{aligned} \frac{1}{12}(P[\bar{V}_x] - P_i[U_x]) + \langle U_x - \bar{V}_x \rangle \\ = \frac{1}{12}(P[\bar{V}_x] - P_i[\bar{V}_x]) = \langle zw'u' \rangle \\ = \langle zw''[u'' + (d\Phi/dz)] \rangle = \langle zw''u'' \rangle, \end{aligned}$$

proving (8.13).

The bifurcation problem for the double-primed variables may be obtained from (8.7) using (8.9) and a suitably scaled stream function Ψ :

$$\Psi'' = 2\langle \bar{V}_x \rangle \Psi = 2\varepsilon \bar{R} \Psi, \tag{8.15}$$

where $\bar{R}[\bar{V}_x] = \bar{R}[U_x]$ and

$$\varepsilon^2 \langle \bar{V}_x \rangle^2 \langle zwu \rangle = \frac{1}{12}(P[\bar{V}_x] - P_i[\bar{V}_x]). \tag{8.16}$$

Now, by choosing a normalizing condition in the form

$$\langle zwu \rangle = 1/48$$

we find that

$$\varepsilon^2 = f - f_l \quad (8.17)$$

is the friction-factor discrepancy where, in dimensional variables,

$$f \equiv \hat{P}[\hat{V}_x] 2D_h \left/ \frac{1}{l} \int_{-l/2}^{l/2} \hat{V}_x dz \right. = 4P[\bar{V}_x] / \langle \bar{V}_x \rangle^2 \quad (8.18)$$

and $D_h = 2l$ is the hydraulic diameter (the ratio of four times the area of the cross section of the annulus to the wetted perimeter and $P = \hat{P}^{1/3}/\nu$).

With these definitions established we may now state the bifurcation problem in terms of the friction-factor discrepancy:

$$\left(\omega \frac{\partial}{\partial s} + U_{\diamond}(z) \frac{\partial}{\partial x} \right) \Delta \Psi - \frac{d^2 U_{\diamond}}{dz^2} \frac{\partial \Psi}{\partial x} + \varepsilon J(\Psi, \Delta \Psi) - \frac{1}{2\bar{R}} \Delta^2 \Psi = 0, \quad (8.19a)$$

$$\Psi = \partial \Psi / \partial z = 0 \Big|_{z=\pm 1/2}, \quad (8.19b)$$

$$\Psi \text{ is } 2\pi \text{ periodic in } \alpha x \text{ and } s = \omega t, \quad (8.19c)$$

$$\langle zwu \rangle = 1/48. \quad (8.19d)$$

Equations (8.19) are a complete statement of the mathematical problem for the bifurcating solution. With α given we seek solutions

$$[\Psi(x, z; \varepsilon, \alpha), \bar{R}(\varepsilon^2, \alpha), \omega(\varepsilon^2, \alpha)] \quad (8.20)$$

of (8.19). Joseph and Sattinger (1972) have shown that the bifurcating solution is necessarily time periodic when the solution of the spectral problem associated with (8.19) is unique and time periodic. The solution (8.20) of (8.19a,b,c) may be developed in a series of powers of ε . The normalization (8.19d) leading to the definition (8.17) was given by Joseph and Chen (1973).

The spectral problem for (8.19) may be reduced to the familiar Orr-Sommerfeld problem; uniqueness of solutions to the Orr-Sommerfeld problem means uniqueness to within a multiplicative constant fixed by normalization; time periodicity refers to the fact that at criticality

$\sigma = i\omega_0 = i\omega(0, \alpha) \neq 0$. Yih (1973) has recently shown ω_0/α lies between the maximum and minimum values of $U(z)$.†

IX. Some Properties of the Bifurcating Solution

It follows from the theory of Joseph and Sattinger that the solution (8.20) of (8.19) may be developed as a power series in ε :

$$\begin{bmatrix} \Psi(x, z, s; \varepsilon, \alpha) \\ \omega(\varepsilon; \alpha) \\ \lambda(\varepsilon; \alpha) \end{bmatrix} = \sum_{\mu=0}^{\infty} \varepsilon^{\mu} \begin{bmatrix} \Psi_{\mu}(x, z, s; \alpha) \\ \omega_{\mu}(\alpha) \\ \lambda_{\mu}(\alpha) \end{bmatrix}, \quad (9.1)$$

where $\lambda = 1/\bar{R}$. This solution is unique to within an arbitrary change in the

† The earliest studies of the spectral problem of linearized theory led to contradictory results with some investigators claiming stability and others instability. Lin (1945), following the asymptotic theory developed by Heisenberg (1924) and Tollmein (1929) calculated a critical Reynolds number. Lin's results were checked and confirmed by the numerical computation of L. H. Thomas (1952). Nowadays critical values and eigenfunctions for the spectral problem are computed by digital computation.

The earliest nonlinear studies of the stability go back to Meksyn and Stuart (1951). They used an energy approximation in which the shape of the bifurcating solution is taken as given by the critical eigenfunction of the spectral problem; the unknown amplitude of the disturbance is then determined by the over-all energy balance. Stuart (1960) and Watson (1960) gave a formal algorithm for determining finite amplitude solutions for disturbances of Poiseuille flow and other parallel flows. Their construction presumes an amplitude equation which at lowest significant order reduces to Landau's (1944) equation. Landau had conjectured the form of this amplitude equation in the context of his more general conjectures about transition to turbulence through repeated supercritical branching. The perturbation method of Stuart and Watson was extended and clarified by Reynolds and Potter (1967); they computed the first bifurcation results for plane Poiseuille flow; more computational results were then given by McIntire and Lin (1972).

Closely related to Stuart's (1960) formal method of amplitude expansions is the formal theory of Eckhaus; this method was applied to the problem of bifurcation of plane Poiseuille flow. A relatively thorough review of these formal theories and more references can be found in the review paper of Stuart (1971).

Rigorous theories of bifurcation of time periodic flow from steady flow, assuming simplicity of the principal eigenvalue of the spectral problem at criticality, have been by Yudovich (1971), Iooss (1972), and Joseph and Sattinger (1972). The work of Joseph and Sattinger extends the Hopf (1942) bifurcation theorem for systems of ordinary differential equations to systems of partial differential equations. Hopf's results are complete and rigorous. Hopf felt that his results might apply to partial differential equations, and he refers to the Taylor problem and other famous problems of hydrodynamic stability. The work of Yudovich extends the theory of Lyapounov-Schmidt to the case of bifurcation from a simple, complex-valued eigenvalue. His theory has been applied to the problem of bifurcating Poiseuille flow by Andreichikov and Yudovich (1972).

origin of the time. The odd-order derivatives all vanish:

$$\omega_{2\mu+1} = \lambda_{2\mu+1} = 0.$$

The first nonlinear corrections are given by the values ω_2 and λ_2 .

TABLE 1
NONLINEAR STABILITY CHARACTERISTICS OF PLANE POISEUILLE FLOW

α	$\bar{R}(0, \alpha)^a$	ω/α	$\hat{\xi}_1$	λ_2	ω_2
0.650	14950	0.2483	192.37	-0.3798	311.89
0.700	10903	0.2734	158.45	-0.3711	264.38
0.750	8307	0.2975	133.03	-0.3426	241.39
0.800	6588	0.3204	113.35	-0.2860	237.45
0.850	5427	0.3417	97.59	-0.1883	251.56
0.900	4643	0.3612	84.44	-0.0280	287.17
0.925	4360	0.3701	78.50	0.0870	316.11
0.950	4138	0.3782	72.80	0.2356	355.89
1.000	3877	0.3918	61.37	0.7014	492.57
1.021*	3848	0.3961	56.12	1.0220	595.05
1.050	3927	0.3996	47.43	1.7650	852.01
1.075	4209	0.3988	36.32	3.3149	1442.00
1.090	4683	0.3935	23.67	6.9112	2921.63
1.095	5075	0.3886	14.60	13.3214	5660.52
1.0964	5298	0.3858	9.70	21.7853	9322.37
1.0964	6238	0.3745	-10.01	-27.6987	-12344.70
1.095	6597	0.3705	-17.33	-17.3983	-7885.29
1.090	7478	0.3615	-35.00	-10.2626	-4840.71
1.075	9538	0.3438	-75.58	-6.4708	-3305.22
1.050	12907	0.3221	-140.66	-4.9225	-2755.16
1.020	17573	0.3008	-228.75	-4.2075	-2536.25
1.000	21264	0.2881	-296.80	-3.9373	-2450.08

^a $\bar{R} \equiv \langle V_x \rangle d/2r$ (based on average velocity and half-height)

* Minimum critical point.

The values of λ_2 and ω_2 for Plane Poiseuille flow were computed by Chen and Joseph (1973); they use a different definition of ε . The values of λ_2 and ω_2 using the friction-factor discrepancy to define ε [Eq. (8.17)] were computed by Joseph and Chen (1973) for flow through annular ducts. In Table 1 we have listed values of the parameters for plane Poiseuille flow when $\varepsilon^2 = f - f_1$. The working equations for these computations will not be given here. The solutions to the perturbation problem at zeroth order ($\varepsilon = 0$) depends on x and t only in the combination $\alpha x + \omega t = \theta$. This property is retained in the higher-order solutions. It follows from this that the bifurcating solutions are statistically stationary.

A most important result of bifurcation studies is the determination of the

direction of bifurcation. Bifurcating solutions which exist when the point (λ, α) is in a region where laminar Poiseuille flow is stable according to the linearized theory of stability are called subcritical. Supercritical solutions exist for points (λ, α) for which laminar flow is stable.

The Floquet stability analysis (Joseph and Sattinger, 1972) of the time-periodic flow which bifurcates from laminar Poiseuille flow shows that subcritical bifurcating solutions are unstable when $[\lambda(\varepsilon^2, \alpha), \varepsilon]$ is near the point of bifurcation $(\lambda_0, 0)$. Supercritical solutions are stable to small two-dimensional disturbances of the same basic periodicity as the bifurcating solution.† This sense of stability is too weak to be physically meaningful.

Referring to Table 1 we note that solutions for which $\lambda_2 \tilde{\xi}_1 > 0$ are subcritical.‡ In particular the solution which bifurcates first (the one marked with an asterisk) is subcritical and unstable. This unstable subcritical bifurcating solution is shown in Fig. 1 as a dotted line on the left emanating from the point of bifurcation. By choosing other values of α , and in other ways, we can show that an unstable solution bifurcates subcritically from every point on the laminar flow line (the 45° in Fig. 1) when $\bar{R} > \bar{R}_L \equiv \bar{R}(0, \alpha_*)$.

X. Inferences and Conjectures

Figure 1 is a bifurcation diagram in the plane (\bar{R}, f) of the response curve. The experimental points, shown as circles, are the values observed by Walker *et al.* (1957) in experiments on turbulent flow in annular ducts. The ratio of the inner to outer radius is 1/1.01, and we shall accept that this configuration is sufficiently representative of flow in a plane channel. In this regard we note that values computed by Chen (Table 1) for plane Poiseuille flow are only very slightly different than the values for the annular duct of Fig. 1 (see Joseph and Chen, 1973). The coordinates Re_2 and f_2 used by Walker *et al.* are related to \bar{R} and f by

$$\bar{R} = Re_2/4, \quad f = 4f_2.$$

The response curve for laminar flow $f_l = 24/\bar{R}$ appears as a straight line in a log-log plot:

$$d \ln f_l / d \ln \bar{R} = -1.$$

The 45° laminar line is a lower bound for the friction factors in all possible motions with steady average values.

† Hopf (1942) was the first to prove that subcritical bifurcating solutions are unstable and that supercritical bifurcating solutions are stable. His construction for the Floquet exponents is not clear; a clear construction has been given by Joseph and Sattinger (1972).

‡ $\tilde{\xi}_1$ is the real part of $\sigma(\lambda)$ for solutions of the linearized equations proportional to $e^{-\sigma(\lambda)t}$. $\tilde{\xi}_1 = d\tilde{\xi}(\lambda)/d\lambda$ when $\lambda = \lambda_0 = \lambda(0, \alpha) = R^{-1}(0, \alpha)$.

In Fig. 2 we have also graphed an upper bound for friction factors in statistically stationary turbulent flow. To convert the bound (7.4) into (R, f) coordinates we note that (7.4) holds for flows with equal pressure gradients:

$$P[\bar{V}_x] = P_i[U_x] = 12\langle U_x \rangle.$$

Then, using (8.18) and $\langle \bar{V}_x \rangle = 2\bar{R}$ we note that

$$f = P/\bar{R}^2 = 12\langle U_x \rangle/\bar{R}^2$$

and

$$\mu = \langle U_x \rangle - \langle \bar{V}_x \rangle = f\bar{R}^2/12 - 2\bar{R} \geq 0.$$

These two relations are used to change variables $(\mu, P) \rightarrow (\bar{R}, f)$ in (7.4); we find that

$$f \leq \tilde{f} \equiv \begin{cases} (24/\bar{R}) + (12/\bar{R}^2)(\bar{R}_e + 2)(\bar{R} - \bar{R}_e) & \text{for } \bar{R}_e \leq \bar{R} \leq \gamma \\ 3 + (36/\bar{R}) + (12/\bar{R}^2) & \text{for } \bar{R} \geq \gamma, \end{cases} \quad (10.1)$$

where $\gamma = 2(\bar{R}_e + 1)$, $\bar{R}_e \simeq 33.3$.

Summarizing; statistically stationary flow through a channel can exist only if

$$f_l = 24/\bar{R} \leq f(\bar{R}) \leq \tilde{f}(\bar{R}). \quad (10.2)$$

The slope of the response function for the time-periodic bifurcating solution at the minimum critical Reynolds number is (see Table 1)

$$d \ln f / d \ln \bar{R} = -1.0430. \quad (10.3)$$

This slope appears as the dashed line in Fig. 1 (the bifurcation is subcritical and is therefore unstable).

The slope (10.3) is computed as follows: One notes that

$$\varepsilon^2 = f - f_l = (\lambda - \lambda_0)/\lambda_2 + O(\varepsilon^4).$$

Then, using (10.2) we find that

$$\ln f = \ln f_l + \ln\{1 + 1/24\lambda_2(\alpha)[1 - \bar{R}(\varepsilon^2, \alpha)/\bar{R}(0, \alpha)]\} + O(\varepsilon^4).$$

Differentiation with respect to $\ln \bar{R}(\varepsilon^2, \alpha)$, when α and $\bar{R}(0, \alpha)$ are fixed gives

$$d \ln f / d \ln \bar{R} = -1 - [24\lambda_2(\alpha)]^{-1}. \quad (10.4)$$

Bifurcation always takes place above the 45° laminar line. When the slope is negative, the solution bifurcates to the left; when the slope is positive, the solution bifurcates to the right.

It is clear from (10.4) and Table 1 that even when three-dimensional bifurcating solutions are disallowed, there are a continua of solutions with $\bar{R}(\alpha, \eta) \geq \bar{R}_L$ which bifurcate in every direction above the 45° laminar line.

Many of these solutions may be closer to the laminar line when $\varepsilon \neq 0$ than the solution which bifurcates first at $\alpha = \alpha_*$. Further study of bifurcating solutions in two dimensions requires the computation of the envelope $\lambda(\varepsilon^2, \alpha(\varepsilon^2))$ of bifurcation curves $\lambda(\varepsilon^2, \alpha)$ depending on the parameter α . The envelope condition

$$\partial\lambda/\partial\alpha = 0 \tag{10.5}$$

is to be an identity in ε^2 . Points on the envelope give the smallest values $\bar{R}(\varepsilon^2, \alpha(\varepsilon^2)) = \lambda^{-1}(\varepsilon^2, \alpha(\varepsilon^2))$ for which two-dimensional bifurcating solutions can be found. At second order in powers of ε^2 we have

$$\lambda(\varepsilon^2, \alpha(\varepsilon^2)) = \lambda(0, \alpha_*) + \lambda_2(\alpha_*)\varepsilon^2 + [\lambda_4(\alpha_*) + \lambda_{2,\alpha}(\alpha_*)\alpha_2]\varepsilon^4 + O(\varepsilon^6), \tag{10.6}$$

where the derivatives

$$\lambda_2 = \partial\lambda/\partial\varepsilon^2, \quad \lambda_4 = \frac{1}{2} \partial^2\lambda/\partial(\varepsilon^2)^2, \quad \lambda_{2,\alpha} = \frac{1}{2} \partial^2\lambda/\partial\varepsilon^2\partial\alpha, \quad \alpha_2 = d\alpha/d\varepsilon^2$$

are all evaluated at $\varepsilon = 0, \alpha = \alpha_*$. These derivatives can all be computed by analytic perturbation theory; in particular, the value of α_2 arises from differentiation of the identity (10.5) with respect to ε^2 at $\varepsilon = 0$:

$$\lambda_{,\alpha\alpha}\alpha_2 + \lambda_{2,\alpha} = 0,$$

where

$$\lambda_{,\alpha\alpha} = \partial^2\lambda/\partial\alpha^2.$$

We shall now summarize what can be inferred and conjectured from analysis and experiments about the response curve for flow through channels. The major points are most graphically explained as an interpretation of results shown in Figs. 1 and 2.

First, when $R < \bar{R}_e \simeq 33.3$ (or $f > f_e \simeq 24/\bar{R}_e$) all disturbances of plane Poiseuille flow decay monotonically from the initial instant; if $\bar{R}_e < \bar{R} < \bar{R}_G$, then all disturbances decay eventually but the decay need not be monotonic. Plane Poiseuille flow can be unstable when $\bar{R} > \bar{R}_G$. Experiments suggest that $\bar{R}_G \simeq 650$. For $R_G < R < R_L$ laminar plane Poiseuille flow is stable to small disturbances; the experiments indicate that the stable disturbances must be very small. If natural disturbances are suppressed, however, one can achieve laminar flow with $\bar{R}_G < \bar{R} < \bar{R}_L$. For $\bar{R} > \bar{R}_L$, laminar flow is unstable. At $\bar{R} = \bar{R}_L$, a time-periodic solution bifurcates from laminar flow. The bifurcating solution is subcritical and unstable. Unstable bifurcating solutions cannot attract disturbances and solutions which escape the domain of attraction of laminar Poiseuille flow snap through the

bifurcating solutions and are attracted to a stable set of solutions with much larger values of the friction-factor discrepancy.[†]

The experiments suggest that when $\bar{R} > R_G$, there are a stable set of solutions, called stable turbulence, which appear to share a common response curve. The surprising and noteworthy observation is that at a given $\bar{R} > \bar{R}_G$ it is possible to reproduce the same value, as far as "sameness" can be ascertained from experiments, of the friction-factor discrepancy. The surprise stems from the fact that the solutions which are observed are turbulent and all different; despite this, each of these infinitely many turbulent solutions leads to an apparently common value of the friction factor.

Assuming that the stable turbulence is statistically stationary, the response function is bounded from above by the estimate (10.5) which arises from estimates of functionals defined in the variational theory of turbulence. The response of statistically stationary solutions is bounded from below by the response of laminar flow (the 45° line in Figs. 1 and 2). There are surely very many statistically stationary solutions in the region between the 45° line and the upper bound. The bifurcating solution shown in Fig. 1 is but one example; there are also at least a continua of solutions depending on α with $\bar{R}(0, \alpha) > \bar{R}_L = \bar{R}(0, \alpha^*)$ which bifurcate subcritically. Many of these solutions are demonstrably unstable and all of them may be unstable.

The significance of the solution bifurcating from Poiseuille flow at the lowest critical value \bar{R}_L is that it is the first solution to bifurcate. The heavy dashed line shown in Fig. 1 gives the slope of the response curve of the two-dimensional bifurcating solution at the point of bifurcation. The computation of the actual curve requires, at least, the computation of more terms in the power series for $\lambda(\varepsilon^2, \alpha)$.

The physical significance of the bifurcation analysis is limited in the case of subcritical bifurcations by the fact that subcritical bifurcating solutions are unstable when ε^2 is small and therefore cannot be achieved in permanent form. In addition, there are many who hold that the bifurcation analysis is of little physical significance since the theory leads to the conclusion that two-dimensional disturbances bifurcate first, and two-dimensional disturbances are not observed in real turbulent flows. This criticism ignores completely the fact that this same criticism could be applied to all of the unstable three-dimensional solutions which exist in the shaded region of Fig. 2. In

[†] It may be possible to observe the time-periodic bifurcating solution as a transient of the snap-through instability. Small disturbances of laminar flow with a fixed value of $\bar{R}(\bar{R}_G < \bar{R} < \bar{R}_L)$ which are marginally attracted from Poiseuille flow may take on the properties of the two-dimensional bifurcating solution: given \bar{R} , the solution could be expected to oscillate with a frequency $\omega(\varepsilon^2(\bar{R}))$. The measured friction factor for this flow would be given by $f(\bar{R}) = f_L(\bar{R}) + \varepsilon^2(\bar{R}) = 24\sqrt{\bar{R}} + \varepsilon^2(\bar{R})$. This transient periodic solution might exist for a time before being destroyed by instabilities.

fact, most kinds of solutions are not observed in turbulent flows; the description of what is observed fills tens of thousands of printed pages. The really striking feature of regularity in the observations is the response curve for stable turbulence. Solutions which may exist in the shaded region of Fig. 2 are not observed in experiments. They are unstable as in the case of time-periodic bifurcating flow or they are weakly stable with only a small domain of attraction as in laminar Poiseuille flow. At a given \bar{R} only those solutions which have the value (or small range of values) of the friction factor on the response curve for stable turbulence seem to persist. These are the circles in Figs. 1 and 2.

I want to raise the possibility that the response curve for stable turbulence (the circles) can be achieved on subsets of a stable set of solutions of permanent form. I do not know what limits can be placed on the definition of solutions of "permanent form." Examples of the kind of permanence I have in mind are periodic or almost periodic functions of time. The usual objection that physically realized turbulence does not have this or that analytical property specified by permanence is beside the point being made here. It is almost certain that observed turbulence is never realized on solutions of permanent form. The domains of attraction of the stable permanent solutions are probably too small to capture all the disturbances which occur in real flows. Given this, we should expect that at each instant the observed solutions are transients which tend now toward one and then another solution in the stable set of permanent form. Though the permanent solutions are not fully realizable, all the realized transients are attracted by permanent solutions which lie in the stable set.

The exciting possibility is that the statistics of observed turbulence could be computed on elements of the stable set of solutions of permanent form. If true, this would enormously simplify aspects of the analytical problem associated with turbulent flow. For example, we could investigate the possibility that the response curve for stable turbulence can be computed on the envelope of two-dimensional, time-periodic bifurcating solutions. Numerical studies (for example, see Zahn *et al.* 1973) suggest that the envelope $f(\bar{R})$ is a double-valued function of \bar{R} . The lower branch lies close to the dashed lines representing the bifurcating solution in Figs. 1 and 2; the lower branch is almost certainly unstable. The upper branch is more problematic; experience with other stability problems suggests that the upper branch of the envelope is stable, at least to more disturbances than the lower branch. The interesting possibility is that the upper branch of the envelope is stable to small disturbances and coincides with the response curve for stable turbulence (the circles). The computation of higher terms in the perturbation series, and of (10.6) in particular, is one way, though not a decisive one, to further study the possibility.

Appendix: A Formal Bifurcation Theory for Nearly Parallel Flows

In this appendix I am going to present a formal theory of bifurcation for nearly parallel flows. This theory is based on a triple-perturbation series; it uses an extension of the Poincaré–Lindstedt method (described in Section IX) to treat the nonlinear effects and the method of multiple scales to treat the effects of slow spatial variation of the main stream.

Multiple scale theories for treating the linearized stability theory of nearly parallel flows have been given independently by Bouthier (1972) and by Ling and Reynolds (1973). These theories make the quasiparallel approximation which leads to the conventional Orr–Sommerfeld theory at the zeroth order.

The earliest mathematical study of the effects of nonparallelism (Lanchon and Eckhaus, 1964) already indicated that though the quasiparallel approximation is valid in the case of the Blasius boundary layer, this same approximation could not be expected to correctly give the linear stability limit for flows like those in jets.† This observation appears to be sound and its implications for perturbations are great. If a flow is not well represented at the zeroth order, it cannot be approximated by perturbations. It is for this reason that the perturbation method of Ling and Reynolds fails in the case of the jet when the wave number of the disturbance is small.

The problem of the correct zeroth order is basic in developing a perturbation theory which will apply equally to flows in boundary layers and jets. I believe that the solution of this problem lies along the lines laid out by the work of Haaland (1972). Haaland has noted that the difference between flows of the boundary-layer type and flows of the jet type can be characterized by the behavior of the velocity component \tilde{V} normal to the main stream at distances $\tilde{Y} \rightarrow \infty$ far away from the axis $\tilde{Y} = 0$ of the main flow. The boundary layer grows by the diffusion of vorticity and does not require inflow from infinity. On the other hand, the conservation of the axial momentum of the jet

$$M = 2\rho \int_0^{\infty} \tilde{U}^2 d\tilde{Y}$$

together with the slowing of the jet with distance \tilde{X} downstream requires the entrainment of new fluid. The spreading of the jet implies a nonzero inflow ($\tilde{V} \neq 0$) at infinity.

In his study of the linear theory of stability of nonparallel flows, Haaland modifies the Orr–Sommerfeld theory to include some of the effects of inflow. The effect of the retention of these terms is to confine the vorticity of distur-

† Tatsumi and Kakutani (1958) note that the parallel flow approximation may not apply to jets.

bances to the regions of the main flow where viscosity is important and to prevent the spillover of vorticity into regions where the flow is essentially irrotational. These inflow terms make a big difference in the critical Reynolds numbers especially when the wave numbers are small (see Fig. 4).

The formal perturbation theory developed below allows for a certain flexibility in the choice of a zeroth order. For definiteness, however, we shall

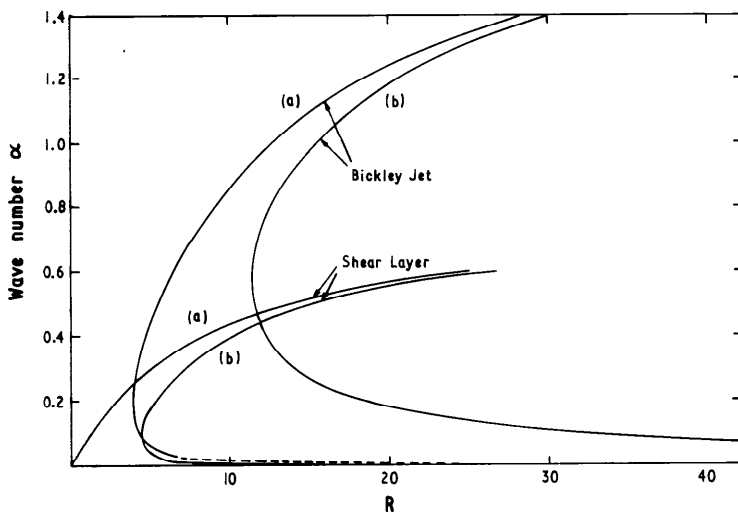


FIG. 4. Neutral curves for the Shear layer and the Bickley jet (after Haaland, 1972). (a) Conventional Orr-Sommerfeld equation; (b) modified Orr-Sommerfeld equation.

consider the stability of Bickley’s jet and use Haaland’s linear theory as the zeroth order. The perturbation scheme then corrects this zeroth order for effects of nonlinear terms and of linear terms which are neglected at the zeroth order.

We begin with a mathematical description of the Bickley jet. Throughout the Appendix we denote differentiation by the “comma followed by subscript” convention; e.g.,

$$A_{,\lambda\mu} \equiv \partial^2 A / \partial \lambda \partial \mu.$$

The Bickley jet satisfies the boundary-layer equations

$$\tilde{U}\tilde{U}_{,\tilde{X}} + \tilde{V}\tilde{U}_{,\tilde{Y}} = \nu U_{,\tilde{Y}\tilde{Y}}, \quad U_{,\tilde{X}} + V_{,\tilde{Y}} = 0; \tag{A.1}$$

the axial momentum M of the jet is conserved. In the local theory we fix our attention on a point \tilde{X}_0 which is an arbitrary distance downstream of the origin of the jet. We introduce a scale length \tilde{L}_0 , a scale velocity \tilde{U}_0 , and a

Reynolds number R :

$$\begin{aligned}\tilde{L}_0 &= (48\rho\tilde{X}_0^2 v^2/M)^{1/3}, \\ \tilde{U}_0 &= (3M^2/32\rho^2\tilde{X}_0 v)^{1/3},\end{aligned}$$

and

$$R = \tilde{U}_0 \tilde{L}_0 / \nu.$$

Then we may write the similarity solution of (A.1) in dimensionless variables:

$$\begin{aligned}U(y, x) &= f^{1/2} \operatorname{sech} fy, \\ V = \lambda W(y, x) &= 2\lambda f (2fy \operatorname{sech}^2 fy - \tanh fy),\end{aligned}\quad (\text{A.2})$$

where

$$\begin{aligned}f &= (1 + 6\chi)^{-2/3}, & \chi &= \lambda x, & \lambda &= 1/R, \\ x &= (\tilde{X} - \tilde{X}_0)/\tilde{L}_0, & y &= \tilde{Y}/\tilde{L}_0.\end{aligned}$$

A two-dimensional disturbance

$$u = \hat{\Psi}_{,y}, \quad v = -\hat{\Psi}_{,x}$$

of the similarity solution (A.2) satisfies

$$\begin{aligned}\hat{\zeta}_{,t} + U\hat{\zeta}_{,x} - \hat{\Psi}_{,x}\Delta U + \lambda(W\hat{\zeta}_{,y} - \hat{\Psi}_{,y}\Delta W) \\ + \hat{\Psi}_{,y}\hat{\zeta}_{,x} + \hat{\Psi}_{,x}\hat{\zeta}_{,y} - \lambda\Delta\hat{\zeta} = 0,\end{aligned}\quad (\text{A.3})$$

where

$$\hat{\zeta} = \Delta\hat{\Psi} = \hat{\Psi}_{,xx} + \hat{\Psi}_{,yy}$$

is the disturbance vorticity and

$$\hat{\Psi} \rightarrow 0 \Big|_{y \rightarrow \pm \infty}.\quad (\text{A.4})$$

Equation (A.4) implies that the nonlinear terms in (A.3) become increasingly less important at large distances from the jet.

The conventional quasiparallel assumption requires that ΔU be replaced by $U_{,yy}$ and $\lambda W = 0$. This assumption is usually justified by noting that on the neutral stability curve λ is generally small; this is not the case near the nose of the neutral curves shown in Fig. 4. The quasiparallel assumption is not uniformly valid for jets; all of the coefficients which depend on the main flow in (A.3) do not vanish at large values of $|y|$;

$$\lim_{y \rightarrow \pm \infty} [U, \Delta U, \lambda W, \lambda \Delta W] \rightarrow [0, 0, -2\lambda f, 0].\quad (\text{A.5})$$

Haaland has shown that if the W terms are set to zero at the outset, as in the

conventional quasiparallel theory, the vorticity of disturbances with small wave numbers will decay much less rapidly than the vorticity of the main flow. On the other hand, retention of the inflow terms results in confining the disturbance vorticity to the jet. A rough and not fully correct argument demonstrating this point follows from comparing the asymptotic solutions of

$$\zeta_{,x} - 2\lambda f \zeta_{,y} - \lambda \Delta \zeta = 0 \tag{A.6}$$

and

$$\zeta_{,x} - \lambda \Delta \zeta = 0. \tag{A.7}$$

Equation (A.6) arises from (A.3) by applying (A.5). Equation (A.7) also arises from (A.3) when the additional assumption $W = 0$ is made. Continuing the rough argument with yet another approximation $f = 1$ we find that the decaying solutions of (A.6) are in the form

$$\zeta = c_1 e^{i(\alpha x + \omega t)} e^{-\gamma y},$$

where c_1 is a constant and

$$\gamma = 1 + (1 + \alpha^2 + i\omega/\lambda)^{1/2},$$

whereas for (A.7)

$$\gamma = \alpha + i\omega/\lambda.$$

For small values of α , solutions of (A.7) decay much less rapidly than the vorticity of the main stream. In fact $\omega/\lambda \rightarrow 0$ as $\alpha \rightarrow 0$ on neutral solutions of the Orr–Sommerfeld equation. In the limit $\alpha \rightarrow 0$ the disturbance vorticity associated with (A.7), but not (A.6), exists deep into the region of irrotationality outside the jet. In contrast, the inflow W transports disturbance vorticity to the jet interior and the disturbance vorticity does not escape into irrotational regions of the main flow as in (A.6). A more refined argument, given by Haaland (1972), leads to the same result; neglecting the convection of disturbance vorticity leads to sharply different results in the neutral curves for jets and shear layers when the wave numbers are small (see Fig. 4).

In preparing for the perturbation theory we call attention to the two spatial scales x and $\chi = \lambda x$, where χ is slowly varying. We next introduce a frequency ω and amplitude ε :

$$\tau = \omega t, \quad \hat{\Psi} = \varepsilon \Psi(\tau, x, y; \chi, \lambda) \tag{A.8}$$

and note that

$$\begin{aligned} \Psi_{,x} &= \varepsilon(\Psi_{,x} + \lambda \Psi_{,\chi}), \\ \Delta \Psi &= \varepsilon(\nabla^2 \Psi + \lambda \mathcal{D}^2 \Psi), \end{aligned} \tag{A.9}$$

where

$$\nabla^2\Psi = \Psi_{,xx} + \Psi_{,yy}$$

and

$$\mathcal{D}^\lambda\Psi = 2\Psi_{,x\chi} + \lambda\Psi_{,xx}.$$

Using (A.8) and (A.9) we may rewrite (A.3) as

$$(A + \lambda B + \lambda C^2)\Psi + \varepsilon N(\Psi, \Psi) + \varepsilon\lambda M^\lambda(\Psi, \Psi) = 0, \tag{A.10}$$

where

$$\begin{aligned} A\Psi &= \omega\nabla^2\Psi_{,\tau} + U\nabla^2\Psi_{,x} - U_{,yy}\Psi_{,x}; \\ B\Psi &= -W_{,yy}\Psi_{,y} + W\nabla^2\Psi_{,y} - \nabla^4\Psi; \\ C^\lambda\Psi &= \omega\mathcal{D}^\lambda\Psi_{,\tau} + U\mathcal{D}^\lambda\Psi_{,x} + U\nabla^2\Psi_{,x} - \Psi_{,x}U_{,yy} \\ &\quad + \lambda U\mathcal{D}^\lambda\Psi_{,x} - \lambda U_{,xx}\Psi_{,x} - \lambda^2\Psi_{,x}U_{,xx} \\ &\quad - \lambda^2\Psi_{,y}W_{,xx} + \lambda W\mathcal{D}^\lambda\Psi_{,y} - 2\lambda\nabla^2\mathcal{D}^\lambda\Psi - \lambda^2\nabla^4\Psi; \\ N(\Psi, \Psi) &= \Psi_{,y}\nabla^2\Psi_{,x} - \Psi_{,x}\nabla^2\Psi_{,y}; \\ M^\lambda(\Psi, \Psi) &= \Psi_{,y}\mathcal{D}^\lambda\Psi_{,x} + \Psi_{,y}\nabla^2\Psi_{,x} + \lambda\Psi_{,y}\mathcal{D}^\lambda\Psi_{,x} \\ &\quad - \Psi_{,x}\mathcal{D}^\lambda\Psi_{,y} - \Psi_{,x}\nabla^2\Psi_{,y} - \lambda\Psi_{,x}\mathcal{D}^\lambda\Psi_{,y}. \end{aligned}$$

The next step in our theory is to introduce the notion of a *local solution of permanent form*. This is a solution of (A.10) valid near $\chi = 0$ (local) which vanishes at infinity and is 2π periodic in αx and τ (permanent). Local solutions of permanent form are different from the localized transient solutions which were studied by Stewartson and Stuart (1971) in their effort to explain turbulent bursts. We seek mathematical expressions which will describe the Tollmien-Schlichting waves which are frequently observed as a permanent feature in certain boundary-layer flows. Such waves first appear with zero amplitude at a critical distance down from the leading edge; their amplitude, wave length, and period may all change with distance downstream. At each station there is a characteristic amplitude, spatial period, and frequency which may be permanently maintained. In our construction we fix the wave number and allow the frequency to vary. Then at each station downstream, there is a family of nonlinear solutions depending on the amplitude and wave number. The values of α which minimize $R = 1/\lambda$ on permanent solutions of fixed amplitude, of course, depend on the position downstream so that our method of computation does not preclude (nor assume) spatially varying wave numbers or constant frequencies.

The construction of a local solution of permanent form proceeds by a *method of false problems*. This method is introduced so that we may pivot the perturbation series around a problem which can be computed by separating

variables but does not neglect inflow (the modified Orr–Sommerfeld problem studied by Haaland, 1972). This requirement then leaves *extra terms* which are proportional to λ . λ is small (at worst, in the shear layer $\lambda \cong \frac{1}{4}$) but decidedly not zero. We just replace λ with μ in the extra terms; then we perturb with μ . The solution of the false problem coincides (formally) with the true solution when $\lambda = \mu$.

We shall start the construction of local solutions of permanent form with a complete statement, followed by explanation, of the false problem:

$$\begin{aligned} (A + \lambda B + \mu C^\mu)\Psi + \varepsilon N(\Psi, \Psi) + \varepsilon \mu M^\mu(\Psi, \Psi) &= 0, \\ \Psi \rightarrow 0 \quad \text{as} \quad y \rightarrow \pm \infty, \\ \Psi \text{ is } 2\pi \text{ periodic in } \alpha x \text{ and } \tau, \\ 1 &= -[\nabla^2 \Psi, Z_1^*]. \end{aligned} \tag{A.11}$$

The operators C^μ and M^μ are obtained from C^λ and M^λ by replacing λ with μ and \mathcal{C}^λ with \mathcal{C}^μ . The terms involving μ are the extra terms. The square bracket is used to designate the scalar product

$$[a, b] = \frac{1}{2\pi} \int_0^{2\pi} (a, b) \, d\tau,$$

where

$$(a, b) = \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} dx \int_{-\infty}^{\infty} a \bar{b} \, dy$$

and the overbar designates complex conjugate. The last condition of (A.11) is a normalizing condition which defines ε ; explanation of this condition requires a little preparation.

The pivot problem for the perturbation may be obtained from (A.11) by putting

$$[\varepsilon, \mu, \chi, \lambda, \omega, U, W] = [0, 0, 0, \lambda_{000}, \omega_{000}, U_0, W_0], \tag{A.12}$$

where

$$U_0 = U(y, 0), \quad W_0 = W(y, 0),$$

and the first equation of (A.11) becomes

$$L_{000} \Psi_{000} = (A_{000} + \lambda_{000} B_{000}) \Psi_{000} = 0. \tag{A.13}$$

The pivot problem (A.13) is autonomous in the periodic variables αx and τ and accommodates separable solutions of the Orr–Sommerfeld type

$$\Psi_{000} = Z_1 + \bar{Z}_1, \quad Z_1 = e^{i(\alpha x + \tau)} \Phi(y). \tag{A.14}$$

We call (A.13) the Orr–Sommerfeld problem with inflow. Solutions (A.14) of (A.13) and the side conditions of (A.11) exist when the parameters $\omega_{000}(\alpha)$ and $\lambda_{000}(\alpha) = 1/R$ have certain values which were computed by Haaland (see Fig. 4).

The adjoint L_{000}^* to L_{000} is defined by the requirement that

$$[a, L_{000} b] = [L_{000}^* a, b]$$

for all functions a and b which vanish at infinity and are 2π periodic in αx and τ . We find that

$$L_{000}^* a = -\omega_{000} \nabla^2 a_{,\tau} - \nabla^2 (U a_{,x}) + U_{0,yy} a_{,x} + \lambda_{000} \{-\nabla^4 a - \nabla^2 (a W_0)_{,y} + (a W_{0,yy})_{,y}\}.$$

The adjoint pivot problem is then defined as

$$L_{000}^* \Psi_{000}^* = 0,$$

where Ψ_{000}^* satisfies the same decay and periodicity conditions as Ψ_{000} . One may verify that

$$\Psi_{000}^* = Z_1^* + \bar{Z}_1^*, \quad Z_1^* = e^{i(\alpha x + \tau)} \Phi(y).$$

We may now interpret the last of the conditions (A.11) as equivalent to requiring that ε be the projection of the vorticity of bifurcating solution into the eigen subspace of the pivot problem; that is,

$$\varepsilon = -[\Delta \hat{\Psi}, Z_1^*] = -\varepsilon[\Delta \Psi, Z_1^*].$$

We should like to have the solution of the false problem in a series

$$\begin{bmatrix} \Psi(\tau, x, y; \varepsilon, \mu, \chi) \\ \lambda(\varepsilon, \mu, \chi) \\ \omega(\varepsilon, \mu, \chi) \end{bmatrix} = \sum_{\substack{n=0, \\ l=0}} \varepsilon^n \mu^l \begin{bmatrix} \Psi_{nl}(\tau, x, y) \\ \lambda_{nl} \\ \omega_{nl} \end{bmatrix}. \tag{A.15}$$

This series would correct the solution of the Orr–Sommerfeld problem with inflow for the extra terms and for the effects of the nonlinear terms. Given the series for $\lambda(\varepsilon, \mu, 0)$ we should take the solution of the false problem with

$$\mu = \lambda(\varepsilon, \mu, \chi)$$

as best approximation of the true solution.

It is not possible however to carry out the solution given by (A.15) because the perturbation computation for the extra terms requires differentiation with respect to the parameter χ [see (c) below]. To circumvent this difficulty we develop the basic flow (U, W), as well as the solution, in powers

of χ :

$$\begin{bmatrix} U(y, \chi) \\ W(y, \chi) \end{bmatrix} = \sum_{p=0} \chi^p \begin{bmatrix} U_p(y) \\ W_p(y) \end{bmatrix}, \tag{A.16}$$

$$\begin{bmatrix} \Psi(\tau, x, y; \varepsilon, \mu, \chi) \\ \lambda(\varepsilon, \mu, \chi) \\ \omega(\varepsilon, \mu, \chi) \end{bmatrix} = \sum_{n, l, p=0} \varepsilon^n \mu^l \chi^p \begin{bmatrix} \Psi_{nlp}(\tau, x, y) \\ \lambda_{nlp} \\ \omega_{nlp} \end{bmatrix}. \tag{A.17}$$

Substitution of (A.16) and (A.17) into the false problem (A.11) leads to a sequence of perturbation problems in the form

$$L_{000} \Psi_{nlp} = G_{nlp}, \tag{A.18}$$

where G_{nlp} is of lower order and Ψ_{nlp} vanishes as $y \rightarrow \pm \infty$ and is 2π periodic in αx and τ with

$$[\nabla^2 \Psi_{nlp}, Z_1^*] = 0$$

if $l + n + p > 0$. If there is a solution of (A.18) it must necessarily satisfy the condition that

$$\begin{aligned} [G_{nlp}] &\equiv [G_{nlp}, Z_1^*] = [L_{000} \Psi_{nlp}, Z_1^*] \\ &= [\Psi_{nlp}, L_{000}^* Z_1^*] = 0. \end{aligned} \tag{A.19}$$

Equation (A.19) is usually complex valued and is equivalent to two real-valued conditions. These may be satisfied by an appropriate choice of the constants ω_{nlp} and λ_{nlp} which appear in the functions G_{nlp} .

Let us consider some typical problems which arise in the perturbation. (a) Problems with $(n, l, p) = (n, 0, 0)$ give nonlinear corrections of the Orr-Sommerfeld problem with inflow without accounting for the extra terms. These problems are analogous to those considered in Section IX. The first of these problems is stated below:

$$L_{000} \Psi_{100} + \lambda_{100} B_{000} \Psi_{000} + \omega_{100} \nabla^2 \Psi_{000,\tau} + N(\Psi_{000}, \Psi_{000}) = 0.$$

As in Poiseuille flow, $[N(\Psi_{000}, \Psi_{000})] = 0$ and (A.19) gives

$$\lambda_{100} [B_{000} \Psi_{000}] + \omega_{100} [\nabla^2 \Psi_{000,\tau}] = 0.$$

Hence, $\lambda_{100} = \omega_{100} = 0$. (b) Problems with $(n, l, p) = (0, 0, p)$ correct the Orr-Sommerfeld problem with inflow for variations with χ . This perturbation can be avoided by redefining variables; χ may be swept into similarity variables so that one computation of the Orr-Sommerfeld problem with inflow for all $\chi > 0$ requires one computation and no perturbation. However the expansion in χ is required when $n + l \neq 0$. (c) Problems with $(n, l, p) =$

$(0, l, 0)$ correct the Orr–Sommerfeld problem with inflow for the extra terms. It is not possible to compute corrections for the extra terms without prior computation of terms which involve χ derivatives. To illustrate this we shall show that the first correction for the extra terms requires prior computation of Ψ_{001} . When $(n, l, p) = (0, 1, 0)$ we have

$$L_{000}\Psi_{010} + \lambda_{010}B_{000}\Psi_{000} + \omega_{010}\nabla^2\Psi_{000,\tau} + C_{000}^0\Psi_{000} = 0, \quad (\text{A.20})$$

where

$$C_{000}^0 = \omega_{000}\mathcal{L}^0\Psi_{000,\tau} + U^0\mathcal{L}^0\Psi_{000,x} + U_0\nabla^2\Psi_{000,x} - \Psi_{000,\chi}U_{0,yy}$$

and

$$\mathcal{L}^0\Psi_{000} = 2\Psi_{000,x\chi} = 2\Psi_{001,x}.$$

Hence to form the Eq. (A.20) for the first correction for the extra terms, we need first to find the functions $\Psi_{000,\chi} = \Psi_{001}$. A similar prior computation of the derivatives with respect to the slow spatial parameter χ is required at higher orders. The perturbation problems may be solved sequentially provided that an appropriate order in the computation is observed.

The final result of computations carried out for solutions which are 2π periodic in αx and τ is the series (A.17). A χ -dependent wave number $\alpha(\varepsilon, \mu, \chi)$ is automatically generated by maximizing λ over $\alpha > 0$:

$$\lambda(\varepsilon, \mu, \chi, \alpha(\varepsilon, \mu, \chi)) = \max_{\alpha > 0} \lambda(\varepsilon, \mu, \chi, \alpha).$$

In the end we compute the values of the series for the first positive root of the equation

$$\mu = \lambda(\varepsilon, \mu, \chi, \alpha(\varepsilon, \mu, \chi)).$$

REFERENCES

- ANDREICHIKOV, I. P. and YUDOVICH, V. I. (1972). Self-oscillating regimes branching from Poiseuille flow in a two-dimensional channel. *Sov. Phys.-Dokl.* **17**.
- BOUTHIER, M. (1972). Stabilité linéaire des écoulements presque parallèles. *J. Mech.* **11**, 599.
- BUSSE, F. (1969). Bounds on the transport of mass and momentum by turbulent flow between parallel plates. *Z. Angew. Math. Phys.* **20**, 1.
- BUSSE, F. (1970). Bounds for turbulent shear flow. *J. Fluid Mech.* **41**, 219.
- CHEN, T. S. and JOSEPH, D. D. (1973). Subcritical bifurcation of plane Poiseuille flow. *J. Fluid Mech.* **58**, 337.
- ECKHAUS, W. (1965). *Studies in Non-linear Stability Theory*. Springer-Verlag, Berlin and New York.
- HAALAND, S. (1972). Contributions to linear stability theory of nearly parallel flows. Ph.D. Thesis. University of Minnesota, Minneapolis, Minnesota.

- HEISENBERG, W. (1924). Über Stabilität und Turbulenz von Flüssigkeitsströmen. *Ann Phys.* **74**, 577.
- HOPF, E. (1942). Abzweigung einer periodischen Lösung von einer Stationären Lösung eines Differentialsystems. *Ber. Math. Phys. K. Säch. Akad. Wiss. Leipzig.* **94**, 1–22.
- HOPF, E. (1948). A mathematical example displaying the features of turbulence. *Commun. Proc. Appl. Math.* **1**, 303.
- HOWARD, L. N. (1963). Heat transport by turbulent convection. *J. Fluid Mech.* **17**, 405.
- HOWARD, L. N. (1972). Bounds on flow quantities. In "Annual Review of Fluid Mechanics," p. 473. Annual Reviews, Palo Alto, California.
- IOOSS, G. (1972). Existence et Stabilité de la Solution Périodique Secondaire Intervenant dans les problèmes d'Évolution du Type Navier-Stokes. *Arch. Ration. Mech. Anal.* **47**, 301.
- JOSEPH, D. D. (1973). Remarks about bifurcation and stability of quasi-periodic solutions which bifurcate from periodic solutions of the Navier–Stokes equations. In "Nonlinear Problems in Physical Science and Biology," Springer Lecture Notes in Mathematics (I. Stakgold, D. Joseph, and D. Sattinger, eds.) No. 322. Springer-Verlag, Berlin and New York.
- JOSEPH, D. D. (1974). Repeated supercritical branching of solutions arising in the variational theory of turbulence. *Arch. Ration. Mech. Anal.* **53**, 101.
- JOSEPH, D. D. and CHEN, T. S. (1973). Friction factors in the theory of bifurcating Poiseuille flow through annular ducts. *J. Fluid Mech.* To be published.
- JOSEPH, D. D. and SATTINGER, D. (1972). Bifurcating time periodic solutions and their stability. *Arch. Ration. Mech. Anal.* **45**, 79–109.
- LANCHON, H. and ECKHAUS, W. (1964). Sur l'analyse de la stabilité des écoulements faiblement divergents. *J. Mech.* **3**, 445.
- LANDAU, L. (1944). On the problem of turbulence. *C. R. Acad. Sci.* **44**, 311; see also L. Landau and E. Lifschitz. (1959). "Fluid Mechanics," Pergamon, Oxford.
- LIN, C. C. (1945). On the stability of two-dimensional parallel flows. *Quart. Appl. Math.* **3**, 117.
- LING, C. H. and REYNOLDS, W. C. (1973). Non-parallel flow corrections for the stability of shear flows. *J. Fluid Mech.* **59**, 571.
- MCINTIRE, L. V. and LIN, C. C. (1972). Finite amplitude instability of second-order fluids in plane Poiseuille flow. *J. Fluid Mech.* **52**, 273.
- MEKSYN, D. and STUART, J. T. (1951). Stability of viscous motion between parallel planes for finite disturbances. *Proc. Roy. Soc. Ser. A*, **208**, 517.
- PEKERIS, C. L. and SHKOLLER, B. (1967). Stability of plane Poiseuille flow to periodic disturbances of finite amplitude in the vicinity of the neutral curve. *J. Fluid Mech.* **29**, 31.
- PEKERIS, C. L. and SHKOLLER, B. (1969). The neutral curves for periodic perturbations of finite amplitude of plane Poiseuille flow. *J. Fluid Mech.* **39**, 629.
- REYNOLDS, W. C. and POTTER, M. C. (1967). Finite-amplitude instability of parallel shear flows. *J. Fluid Mech.* **27**, 465.
- REYNOLDS, O. (1895). On the dynamical theory of incompressible viscous fluids and the determination of the criterion. *Phil. Trans. Roy. Soc. London* **186**, 123.
- STEWARTSON, K. and STUART, J. T. (1971). A nonlinear instability theory for a wave system in plane Poiseuille flow. *J. Fluid Mech.* **48**, 529.
- STUART, J. T. (1960). On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part 1. The basic behaviour in plane Poiseuille flow. *J. Fluid Mech.* **9**, 353.
- STUART, J. T. (1971). Nonlinear stability theory. In "Annual Review of Fluid Mechanics," Vol. 3. Annual Reviews, Palo Alto, California.
- TATSUMI, T. and KAKUTANI, T. (1958). The stability of a two-dimensional laminar jet. *J. Fluid Mech.* **4**, 261.
- THOMAS, L. H. (1953). The stability of plane Poiseuille flow. *Phys. Rev.* **91**, 780.
- THOMAS, T. Y. (1942). Qualitative analysis of the flow of fluids in pipes. *Amer. J. Math.* **64**, 754.

- TOLLMEN, W. (1929). Über die Entstehung der Turbulenz. Nachr. Ges. Wiss. Gottingen. Math-phys. Klasse. pp. 21–44.
- WALKER, J. E., WHAN, A., and ROTHFUS, R. R. (1957). Fluid friction in non-circular ducts. *A.I.Ch.E. J.* **3**, 184.
- WATSON, J. (1960). On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part 2. The development of a solution for plane Poiseuille flow and for plane Couette flow. *J. Fluid Mech.* **9**, 371.
- YIH, C. S. (1973). Wave velocity in parallel flows of a viscous fluid. *J. Fluid Mech.* **58**, 703.
- YUDOVICH, V. I. (1971). The origin of auto-oscillations in fluids. *Prikl. Mat. Mekh.* **35**, 638.
- ZAHN, J., TOOMRE, J., SPIEGEL, E., and GOUGH, D. (1973). Nonlinear cellular motions in Poiseuille channel flow. *J. Fluid Mech.* To be published.