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Friction factors in the theory of bifurcating Poiseuille flow through annular ducts

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The objective of this paper is to show how to formulate a bifurcation theory for pipe flows in terms of the friction factor. We compute the slope of the friction factor *vs.* Reynolds number curve and the frequency change for the time-periodic solution which bifurcates from Poiseuille flow through annular ducts.

1. Introduction

A response function for a fluid motion can be defined as a scalar function which measures the response of the flow to the external forces which induce the motion. For example, in problems of thermal convection the response function can be taken as the heat transported and the external forces can be regarded as the applied temperature difference. The dimensionless response function relates the Nusselt and Rayleigh numbers. In another example, flow down a pipe, the external force is the pressure gradient and the response function can be taken as the mass flux. The dimensionless response function relates friction factors and Reynolds numbers. We are going to study the response function for flow down an annular duct.

The response function is generally obtained by evaluating a response functional on a suitably defined set of solutions. In this paper we shall study statistically stationary solutions of the Navier–Stokes equations for flow through annular ducts. These solutions are defined in §2; their chief property is that the spatial average over cylinders of such solutions is time independent. The bifurcating solutions which we shall construct have a time-dependent spatial average (see remarks closing §9). We shall assume that other solutions which are observed as turbulence have the property of statistical stationarity; this is the basic assumption of the variational theory of turbulence (Howard 1963, 1971; Busse 1969, 1970). The assumption gives a sense in which fluctuating flow in a steady environment can have steady average properties.

The purpose of our analysis is best served by drawing a distinction between laminar Poiseuille flow and all the other statistically stationary flows, including time-periodic bifurcating flow. We shall call all these other flows turbulent. The analysis is conveniently framed in terms of the friction-factor discrepancy, which is the difference $f_T - f_L$, where f_T is the friction factor for turbulent flow and f_L the friction factor for laminar flow.

The main contribution of this work is to show how to introduce the response function, the friction-factor discrepancy (a function of the Reynolds number), as an expansion parameter for constructing the time-periodic solutions which bifurcate from laminar Poiseuille flow. The friction-factor discrepancy will appear as the 'amplitude' in our perturbation study. The introduction of the friction factor allows a direct comparison of stability and perturbation results with experiments. The relation of analysis to experiments may be better understood once one is in possession of a proper understanding of the response function (§§2, 3, 4 and 11), the direction of bifurcation (§§7, 10 and 11), the instability of subcritical bifurcating solutions (§7) and the concept of the snap-through instability (§§7 and 11).

The basic theory for our study, apart from matters relating to the response function which are new, is a specialization of that given first by Joseph & Sattinger (1972)†. The instability of subcritical bifurcating solutions and the concept of a snap-through instability are explained more fully there than here.

Our numerical study of the bifurcation problem is essentially an application of the method used by Chen & Joseph (1973) to study subcritical bifurcation of plane Poiseuille flow to the study of bifurcation of flow in annular ducts. The relation of these methods to those used in previous studies as well as certain other points of interest not treated here are discussed in the Chen & Joseph paper.

2. Decomposition of the whole motion into a mean motion plus a fluctuation with a zero mean

We consider flow between concentric cylinders. The axis of the cylinders coincides with the x axis, where (x, θ, r) are polar cylindrical co-ordinates and

$$r_2 \geqslant r \geqslant r_1$$

is the range of values of the radius between the outer and inner cylinder. The reduced pressure Π (pressure divided by the constant density) is assumed to be of the form

$$\Pi(x, \theta, r, t) = \Pi_1(x, \theta, r, t) - G_T x, \tag{2.1}$$

where Π_1 is bounded as $x \rightarrow \pm\infty$ and $G_T > 0$ is a constant which is determined by the applied pressure drop.

The equations which govern the motion of the fluid in the annulus are

$$\partial \mathbf{V}/\partial t + \mathbf{V}.\nabla\mathbf{V} = -\nabla\Pi_1 + \mathbf{e}_x G_T + \nu\Delta\mathbf{V}, \tag{2.2a}$$

$$\operatorname{div}\mathbf{V} = 0 \tag{2.2b}$$

† This theory is an extension to partial differential equations of the Hopf bifurcation theorem for systems of ordinary differential equations. Hopf (1942) considers bifurcation from a real eigenvalue (steady bifurcation) and bifurcation from a complex eigenvalue (time-periodic bifurcation). Hopf was the first to prove that subcritical bifurcating solutions are unstable and that supercritical bifurcating solutions are stable. Hopf felt that his results might apply to partial differential equations and he makes reference to the Taylor problem and other famous problems of hydrodynamic stability.

and

$$\mathbf{V}|_{r_1} = \mathbf{V}|_{r_2} = 0. \tag{2.2c}$$

Here $\mathbf{e}_x$ is a unit vector parallel to the x axis.

The motion $(\mathbf{V}, \Pi_1)$ may be decomposed in several ways; one convenient way is to introduce a cylinder average

$$\bar{f} = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} dx\left\{\frac{1}{2\pi}\int_0^{2\pi} f\,d\theta\right\} \tag{2.3}$$

and an overall average

$$\langle f\rangle = \frac{2}{r_2^2 - r_1^2}\int_{r_1}^{r_2} \bar{f} r\,dr, \tag{2.4}$$

for any quantity f. Then, we may set

$$(\mathbf{V}, \Pi_1) = (\bar{\mathbf{V}} + \mathbf{V}', \bar{\Pi}_1 + \Pi'), \tag{2.5}$$

where primed quantities are fluctuations and have a zero mean. The average of the velocity V_x will be denoted by $\langle \bar{V}_x\rangle$.

By *statistically stationary* we shall understand that the time derivatives of cylinder averages are zero and that there is no mean circumferential motion ($\bar{V}_\theta = 0$). The vanishing of the mean radial component of velocity ($\bar{V}_r = 0$) may be deduced from the continuity equation.

To study statistically stationary turbulent flow through the annulus we form the cylinder averages of the equations of motion. The equations governing the mean motion are

$$\frac{1}{r}\frac{d}{dr}(r\,\overline{V_r'^2}) - \frac{1}{r}\overline{V_\theta'^2} = -\frac{d\bar{\Pi}_1}{dr}, \tag{2.6a}$$

$$d(r^2\overline{V_r' V_\theta'})/dr = 0, \tag{2.6b}$$

$$\frac{d}{dr}\left[r\,\overline{V_r' V_x'} - \nu r\frac{d\bar{V}_x}{dr} - G_T\frac{r^2}{2}\right] = 0. \tag{2.6c}$$

The equations governing the fluctuations are

$$\frac{\partial\mathbf{V}'}{\partial t} + \bar{V}_x\frac{\partial\mathbf{V}'}{\partial x} + V_r'\frac{d}{dr}\mathbf{e}_x\bar{V}_x + \mathbf{V}'.\nabla\mathbf{V}' - \overline{\mathbf{V}'.\nabla\mathbf{V}'} = -\nabla\Pi' + \nu\Delta\mathbf{V}'. \tag{2.7}$$

All velocity components vanish on the solid cylinders at $r = r_1$ and $r = r_2$ and $\operatorname{div}\mathbf{V} = \operatorname{div}\mathbf{V}' = 0$. Equation (2.6$b$) shows that $\overline{V_r' V_\theta'}$ is a constant whose value is zero at the boundary and elsewhere. We find the energy identity

$$\langle V_r' V_x' d\bar{V}_x/dr\rangle = -\nu\langle|\nabla\mathbf{V}'|^2\rangle \tag{2.8}$$

from the overall average of (2.7).

A basic and important consequence of the assumption of statistical stationarity is that (2.6c) has a first integral; after introducing the equation (2.10) for laminar flow with a constant pressure gradient G_L this integral may be written as

$$r\overline{V_r' V_x'} + \frac{r_2^2 - r_1^2}{2\ln\eta}\left\langle\frac{\overline{V_r' V_x'}}{r}\right\rangle - \frac{G_T - G_L}{2}\left[r^2 + \frac{r_2^2 - r_1^2}{2\ln\eta}\right] = \nu r\frac{d}{dr}(\bar{V}_x - U_x), \tag{2.9}$$

where $\eta = r_1/r_2$ and

$$\frac{G_L}{2}\left[r^2+\frac{r_2^2-r_1^2}{2\ln\eta}\right] = -\nu r\frac{dU_x}{dr}. \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) we find that

$$\frac{G_T}{2}\langle h(r)\,V_r'\,V_x'\rangle = \nu^2\langle|\nabla \mathbf{V}'|^2\rangle+\left\langle\left[\overline{V_r'V_x'}+\frac{r_2^2-r_1^2}{2r\ln\eta}\left\langle\frac{V_r'V_x'}{r}\right\rangle\right]^2\right\rangle, \tag{2.11}$$

where

$$h(r) = r+(r_2^2-r_1^2)/(2r\ln\eta).$$

Equation (2.11) shows that

$$\langle h(r)\,V_r'\,V_x'\rangle \geqslant 0 \tag{2.12}$$

with equality only if $\mathbf{V}' = 0$.

Relations (2.9) and (2.12) are the basis for the following laminar–turbulent comparison theorems: (i) *statistically stationary turbulent Poiseuille flow has a smaller mass flux* ($\langle\overline{V}_x\rangle > \langle U_x\rangle$) *than the laminar Poiseuille flow with the same applied pressure gradient* ($G_T = G_L$); (ii) *statistically stationary turbulent Poiseuille flow has a larger applied pressure gradient* ($G_T > G_L$) *than the laminar flow with the same mass flux* ($\langle\overline{V}_x\rangle = \langle U_x\rangle$). The first of these theorems was proved by Thomas (1942); the proof given below follows Busse (1969, 1970). One notes that the integral of r times (2.9) may be written as

$$\langle h(r)\,V_r'\,V_x'\rangle-\tfrac{1}{4}(G_T-G_L)\,[r_2^2+r_1^2+(r_2^2-r_1^2)/\ln\eta] = -2\nu\,\langle\overline{V}_x-U_x\rangle. \tag{2.13}$$

Equation (2.13) together with (2.12) proves the comparison theorem.

3. Decomposition of the whole motion into laminar Poiseuille flow plus a disturbance

We want to compare two different resolutions of the same motion $\mathbf{V}$:

$$\mathbf{V} = \overline{V}_x(r)\,\mathbf{e}_x+\mathbf{V}'(x,\theta,r,t) = U_x(r)\,\mathbf{e}_x+\mathbf{U}'(x,\theta,r,t), \tag{3.1a, b}$$

where (3.1*b*) gives the decomposition into laminar flow plus a disturbance and

$$U_x(r) = 2\langle U_x\rangle\frac{r_2^2-r^2-2r_{\max}^2\ln(r_2/r)}{r_2^2+r_1^2-2r_{\max}^2}, \quad r_{\max}^2 = \frac{r_1^2-r_2^2}{2\ln\eta}. \tag{3.2}$$

Equations (3.1*a*) and (3.1*b*) imply that

$$V_r' = U_r', \quad V_\theta' = U_\theta', \quad \overline{V}_x+V_x' = U_x+U_x'. \tag{3.3}$$

The relation

$$\overline{V}_x(r) = U_x(r)+\overline{U}_x' \tag{3.4}$$

follows directly from the cylinder average of (3.1*a, b*). We note that (3.4) implies that $\overline{U}_x'$ is statistically stationary (cf. remarks closing §9).

We shall make use of the following relation:

$$\overline{V_x'V_r'} = \overline{U_x'U_r'}. \tag{3.5}$$

To prove (3.5) we use (3.3) to write

$$\overline{V_x'V_r'} = \overline{V_x'U_r'} = \overline{(U_x-\overline{V}_x+U_x')\,U_r'} = (U_x-\overline{V}_x)\,\overline{U_r'}+\overline{U_x'U_r'} = \overline{U_x'U_r'}.$$

Using (3.5) we may rewrite (2.13) as

$$\langle h(r)\,U_x'\,U_r'\rangle = \tfrac{1}{4}(G_T-G_L)\,[r_2^2+r_1^2+(r_2^2-r_1^2)/\ln\eta]-2\nu\langle\overline{V}_x-U_x\rangle. \tag{3.6}$$

This basic relation will be used in the next section to define the pressure-gradient discrepancy functional for flows with a constant mass flux.

4. The response function near the point of bifurcation

The linear stability problem associated with Poiseuille flow through annular ducts has been considered by Mott & Joseph (1968); they restrict their study to axisymmtric disturbances. Here we shall construct the axisymmetric flows which bifurcate from laminar flow. To construct the bifurcating solution we shall use the basic perturbation method of Joseph & Sattinger (1972) in its stream-function formulation given by Chen & Joseph (1973) in their study of plane Poiseuille flow.

One aim of the present work is to show how to enrich the physical content of the perturbation theory by a proper choice of the amplitude parameter.

For any axisymmetric motion we may introduce a stream function $\tilde{\tilde{\Psi}}$ such that

$$V_x = \frac{1}{r}\frac{\partial\tilde{\tilde{\Psi}}}{\partial r}, \quad V_r = -\frac{1}{r}\frac{\partial\tilde{\tilde{\Psi}}}{\partial x}. \tag{4.1}$$

The resolution of the motion into Poiseuille flow plus a disturbance

$$\tilde{\tilde{\Psi}} = \tilde{\Psi}+\Psi', \quad \partial\tilde{\Psi}/\partial r = rU_x, \tag{4.2}$$

leads to the following nonlinear boundary-value problem for Ψ':

$$\frac{\partial}{\partial t}D^2\Psi'+U_x\frac{\partial}{\partial x}D^2\Psi'-r\frac{d}{dr}\left[\frac{1}{r}\frac{dU_x}{dr}\right]\frac{\partial}{\partial x}\Psi'+J(\Psi', D^2\Psi') = \nu D^4\Psi' \tag{4.3a}$$

and

$$\Psi'|_{r_1} = \Psi'|_{r_2} = \left.\frac{\partial\Psi'}{\partial r}\right|_{r_1} = \left.\frac{\partial\Psi'}{\partial r}\right|_{r_2} = 0, \tag{4.3b}$$

where

$$D^2 = \frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2}$$

and

$$J(\Psi', D^2\Psi') = \frac{1}{r}\left(\frac{\partial\Psi'}{\partial r}\frac{\partial}{\partial x}D^2\Psi'-\frac{\partial\Psi'}{\partial x}\frac{\partial}{\partial r}D^2\Psi'\right)+\frac{2}{r^2}\frac{\partial\Psi'}{\partial x}D^2\Psi'.$$

We note that *every solution of* (4.3*a, b*) *has the same mass flux as the laminar flow* (3.2):†

$$\langle\overline{V}_x-U_x\rangle = \langle\overline{U}_x'\rangle = \frac{2}{r_2^2-r_1^2}\int_{r_1}^{r_2}\frac{\partial\overline{\Psi'}}{\partial r}\,dr = \frac{2}{r_2^2-r_1^2}\,\overline{\Psi'}\Big|_{r_1}^{r_2} = 0. \tag{4.4}$$

† There is no loss of generality in restricting the bifurcation analysis to flows which satisfy (4.3*b*). All the other possibilities may be reduced to this one (see Joseph 1974); for example, the bifurcating flow with the same pressure gradient G_L as the laminar flow $U_x(r)$ may be obtained by adding a suitably determined laminar flow to the solution of (4.3*a, b*).

It follows from (4.4) and (3.6) that the pressure-gradient discrepancy is related to solutions of (4.3a, b) by

$$\tfrac{1}{4}(G_T - G_L)\left[r_2^2 + r_1^2 + \frac{r_2^2 - r_1^2}{\ln \eta}\right] = -\left\langle \frac{h(r)}{r^2}\frac{\partial \Psi'}{\partial r}\frac{\partial \Psi''}{\partial x}\right\rangle. \tag{4.5}$$

It is, of course, well known that the friction factor f and pressure gradient are related by the formula

$$G = \tfrac{1}{2}f\langle U_x\rangle^2/D_h, \tag{4.6}$$

where $D_h = 4\delta = 2(r_2 - r_1)$ is the hydraulic diameter (the ratio of four times the area to the wetted perimeter). We introduce dimensionless variables

$$[t, x, \theta, r, U(r), \epsilon u, \epsilon v, \epsilon\hat{\Psi}] = \left[\frac{t\langle U_x\rangle}{\delta}, \frac{x}{\delta}, \theta, \frac{r}{\delta}, \frac{U_x}{\langle U_x\rangle}, \frac{V_x'}{\langle U_x\rangle}, \frac{V_r'}{\langle U_x\rangle}, \frac{\Psi''}{\delta^2\langle U_x\rangle}\right]. \tag{4.7}$$

Here,

$$\epsilon^2 = f_T - f_L > 0 \tag{4.8}$$

is the friction-factor discrepancy (the friction factor for turbulent flow minus the friction factor for laminar flow with the same mass flux). The symbols t, r and x are being used for both dimensional and dimensionless variables. From this point on only the dimensionless variables are used and

$$-\infty < x < \infty, \quad a \equiv \frac{2\eta}{1-\eta} \leqslant r \leqslant \frac{2}{1-\eta} \equiv b. \tag{4.9}$$

The Reynolds number

$$R = \langle U_x\rangle \delta/\nu \tag{4.10}$$

as well as the dimensionless function of the radius ratio

$$b(\eta) = \tfrac{1}{8}\left[\frac{1+\eta^2}{(1-\eta)^2} + \frac{1+\eta}{(1-\eta)\ln\eta}\right], \quad b(0) = \tfrac{1}{8} \geqslant b(\eta) \geqslant b(1) = \tfrac{1}{12}, \tag{4.11}$$

appear in the dimensionless statement of the boundary-value problem governing the bifurcating solution. This follows from (4.3) as

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]D^2\hat{\Psi} - r\frac{d}{dr}\left[\frac{1}{r}\frac{dU}{dr}\right]\frac{\partial}{\partial x}\hat{\Psi} + \epsilon J(\hat{\Psi}, D^2\hat{\Psi}) - \frac{1}{R}D^4\hat{\Psi} = 0, \tag{4.12a}$$

$$\hat{\Psi} = \partial\hat{\Psi}/\partial r = 0 \quad \text{at} \quad r = a, b \tag{4.12b}$$

and, using (4.5)–(4.8),

$$b(\eta) = \langle h(r)\, vu\rangle = -\left\langle \frac{h(r)}{r^2}\frac{\partial\hat{\Psi}}{\partial x}\frac{\partial\hat{\Psi}}{\partial r}\right\rangle. \tag{4.12c}$$

We shall seek a spatially periodic solution of (4.12) with a period $2\pi/\alpha$ in x. The bifurcating solution is necessarily time periodic when the solution of the spectral problem (Orr–Sommerfeld problem, see §5) is both unique and time periodic.

There is a unique curve $R(\epsilon^2)$ and associated frequencies $\omega(\epsilon^2)$ for which time-periodic solutions of (4.12) exist. $R(\epsilon^2) = R(f_T - f_L)$ gives the response curve for Poiseuille flow near the point of bifurcation; this is the relation between the Reynolds number and the friction factor. There is abundant literature reporting experimental measurements of the response curve.

5. Instability of the basic flow

The spectral problem for the instability of the annular duct flow described by (3.2) is obtained from (4.12a, b) by setting $\epsilon = 0$ in (4.12a). Equations (4.12a, b) can then be written as

$$\left(\frac{\partial}{\partial t}D^2 + \mathscr{L}\right)\hat{\Psi} = 0, \quad \hat{\Psi} = \frac{\partial\hat{\Psi}}{\partial r} = 0 \quad \text{at} \quad r = a, b, \quad \text{periodicity in } x, \tag{5.1a,b,c}$$

where

$$\mathscr{L}[U, \lambda] = U\frac{\partial}{\partial x}D^2 - r\frac{d}{dr}\left(\frac{1}{r}\frac{dU}{dr}\right)\frac{\partial}{\partial x} - \lambda D^4 \tag{5.2}$$

and $\lambda = 1/R$.

To obtain the spectral problem, solutions of (5.1) are sought in the form

$$\hat{\Psi}(x, r, t, \lambda) = e^{-\gamma(\lambda)t}\psi(x, r, \lambda). \tag{5.3}$$

This leads to

$$-\gamma D^2\psi + \mathscr{L}\psi = 0, \quad \psi = \partial\psi/\partial r = 0 \quad \text{at} \quad r = a, b, \quad \text{periodicity in } x. \tag{5.4a,b,c}$$

The values $\gamma(\lambda) = \xi(\lambda) + i\hat{\eta}(\lambda)$ are eigenvalues of the spectral problem (5.4). If $\xi(\lambda) < 0$ then the flow U is unstable. For large values of λ (i.e. small R), $\xi(\lambda) > 0$ for all eigenvalues $\gamma(\lambda)$. The border between stability and instability is defined by the critical value $\lambda = \lambda_0 = 1/R_c$ where $\xi(\lambda_0) = 0$. At criticality $\gamma(\lambda_0) = i\hat{\eta}(\lambda_0) = i\omega_0$ and $\mathscr{L}(U, \lambda_0) = \mathscr{L}_0$. If $\gamma(\lambda)$ is a simple eigenvalue of (5.4), then $\bar{\gamma}(\lambda) = \xi(\lambda) - i\hat{\eta}(\lambda)$, the complex conjugate of $\gamma(\lambda)$, is also an eigenvalue. The functions $\psi(x, r, \lambda)$ and $\bar{\psi}(x, r, \lambda)$ are eigenfunctions of (5.4) belonging, respectively, to the eigenvalues of γ and $\bar{\gamma}$ of (5.4).

For the solutions $\psi(x, r, \lambda)$ and $\bar{\psi}(x, r, \lambda)$ of (5.4) to be periodic in x with period $2\pi/\alpha$, where α is the wavenumber, it is necessary that these solutions are expressible in the form

$$\psi(x, r, \lambda(\alpha)) = e^{i\alpha x}\phi(r, \lambda(\alpha)). \tag{5.5}$$

Substitution of (5.5) into (5.4) leads to the Orr–Sommerfeld problem for the annular duct flow.

6. The adjoint problem and a perturbation formula for γ'

For any functions $\hat{a}$ and $\hat{b}$ which are $2\pi/\alpha$ periodic in x and such that

$$\hat{a} = \hat{b} = \partial\hat{a}/\partial r = \partial\hat{b}/\partial r = 0$$

at $r = a$ and b, the scalar product $(\hat{a}, \hat{b})$ is defined as

$$(\hat{a}, \hat{b}) = \frac{\alpha\pi^{-1}}{b^2 - a^2}\int_0^{2\pi/\alpha}\int_a^b \hat{a}\,\bar{\hat{b}}\,r\,dr\,dx. \tag{6.1}$$

The operator $-\bar{\gamma}D^{*2} + \mathscr{L}^*$, the adjoint of $-\gamma D^2 + \mathscr{L}$, is defined by the requirement that

$$(\hat{a}, [-\gamma D^2 + \mathscr{L}]\hat{b}) = ([-\bar{\gamma}D^{*2} + \mathscr{L}^*]\hat{a}, \hat{b}). \tag{6.2}$$

With the aid of (5.2) and (6.2), one finds

$$\mathscr{L}^*\hat{a} = -D^{*2}\left(U\frac{\partial \hat{a}}{\partial x}\right) + r\frac{d}{dr}\left(\frac{1}{r}\frac{dU}{dr}\right)\frac{\partial \hat{a}}{\partial x} - \lambda D^{*4}\hat{a}, \tag{6.3}$$

where

$$D^{*2} \equiv \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}. \tag{6.4}$$

The adjoint eigenvalue problem is thus described by the system

$$-\bar{\gamma}D^{*2}\psi^* + \mathscr{L}^*\psi^* = 0, \quad \psi^* = \partial\psi^*/\partial r = 0 \quad \text{at} \quad r = a, b, \quad \text{periodicity in } x. \tag{6.5a, b, c}$$

It is well established from the linear theory that the flow loses stability as R is increased past R_c at a fixed value of α along the lower branch of the neutral-stability curve up to the point of maximum α; on the other hand, the flow gains stability as R is increased past R_c at a fixed α on the upper branch of the neutral curve beyond the point of maximum α. We shall next derive the perturbation formula which gives the value of $\gamma' = d\gamma(\lambda)/d\lambda$ at a fixed value of α.

To proceed, one differentiates (5.4) with respect to λ and finds

$$(-\gamma D^2 + \mathscr{L})\,d\psi/d\lambda - \gamma' D^2\psi - D^4\psi = 0, \tag{6.6a}$$

$$\frac{d\psi}{d\lambda} = \frac{\partial}{\partial r}\left(\frac{d\psi}{d\lambda}\right) = 0 \quad \text{at} \quad r = a, b, \quad \text{periodicity in } x. \tag{6.6b, c}$$

Since

$$-\gamma\left(D^2\frac{d\psi}{d\lambda}, \psi^*\right) + \left(\mathscr{L}\frac{d\psi}{d\lambda}, \psi^*\right) = -\gamma\left(\frac{d\psi}{d\lambda}, D^{*2}\psi^*\right) + \left(\frac{d\psi}{d\lambda}, \mathscr{L}^*\psi^*\right) = 0 \tag{6.7}$$

it can be seen that

$$\gamma' = -(D^4\psi, \psi^*)/(D^2\psi, \psi^*). \tag{6.8}$$

7. The perturbation series; subcritical bifurcating solutions and their stability

We now turn our attention to the basic problem (4.12), and shall obtain the nonlinear solutions which are periodic in time with period $2\pi/\omega(\epsilon)$. By introducing $\omega(\epsilon)t = s$ the problem (4.12) may be reformulated as

$$\mathscr{J}\hat{\Psi} + \epsilon J(\hat{\Psi}, D^2\hat{\Psi}) = 0, \quad \hat{\Psi} = \partial\hat{\Psi}/\partial r = 0 \quad \text{at} \quad r = a, b, \tag{7.1a, b}$$

$\hat{\Psi}(x, r, s)$ is periodic with period $2\pi/\alpha$ in x and 2π in s, (7.1c)

$$b(\eta) = -\left\langle \frac{h(r)}{r^2}\frac{\partial\hat{\Psi}}{\partial r}\frac{\partial\hat{\Psi}}{\partial x}\right\rangle, \tag{7.1d}$$

where

$$\mathscr{J}[\omega, \lambda] = \omega\,\partial D^2/\partial s + \mathscr{L}[U, \lambda]. \tag{7.2}$$

The condition (7.1d) is a normalizing condition which will relate ϵ^2 directly to the friction-factor discrepancy [see (4.8)].

The solutions of (7.1) can be constructed as a Taylor series

$$\{\hat{\Psi}(x, r, s, \epsilon), \omega(\epsilon), \lambda(\epsilon)\} = \sum_{l=0} \epsilon^l\{\Psi_l(x, r, s), \omega_l, \lambda_l\}, \tag{7.3}$$

where $\lambda_l = (1/l!)\,\partial^l\lambda/\partial\epsilon^l$, etc. It is not hard to establish [see (8.15)] that $\lambda(\epsilon)$ and $\omega(\epsilon)$ are even functions. Then when ϵ is small $\lambda - \lambda_0 > 0$ if $\lambda_2 > 0$ (see tables 1, 2 and 3). If $\xi'(\lambda_0) > 0$ then Poiseuille flow loses stability as λ is decreased or as R is increased. The condition $\xi'(\lambda_0) > 0$ is of greatest interest; in this case the bifurcating solution is subcritical if $\lambda_2 > 0$ and supercritical if $\lambda_2 < 0$. In the subcritical case a time-periodic solution will exist when $R < R_c$. It has been demonstrated by Joseph & Sattinger (1972), using Floquet theory, that subcritical bifurcating solutions are unstable. It follows from this that in the case of greatest interest ($\lambda_2 > 0$, $\xi'(\lambda_0) > 0$) any disturbance of laminar Poiseuille flow which does not decay will snap through the periodic bifurcating solution with the small value of the friction-factor discrepancy $f_T - f_L = \epsilon^2$ and be attracted to larger norm solutions which we have called stable turbulence (see figure 1a).

Substitution of (7.3) into (7.1) leads to the following sequence of problems.

At zeroth order,

$$\mathscr{J}_0\Psi_0 = 0, \Psi_0 = \partial\Psi_0/\partial r = 0 \quad \text{at} \quad r = a, b, \tag{7.4a, b}$$

$$b(\eta) = -\left\langle \frac{h(r)}{r^2}\frac{\partial\Psi_0}{\partial r}\frac{\partial\Psi_0}{\partial x}\right\rangle. \tag{7.4c}$$

At first order,

$$\left.\begin{aligned} &\mathscr{J}_0\Psi_1 + \mathscr{J}_1\Psi_0 + J_0 = 0, \ \Psi_1 = \partial\Psi_1/\partial r = 0 \quad \text{at} \quad r = a, b, \\ &0 = \left\langle \frac{h(r)}{r^2}\left(\frac{\partial\Psi_0}{\partial r}\frac{\partial\Psi_1}{\partial x} + \frac{\partial\Psi_1}{\partial r}\frac{\partial\Psi_0}{\partial x}\right)\right\rangle. \end{aligned}\right\} \tag{7.5}$$

At second order,

$$\left.\begin{aligned} &\mathscr{J}_0\Psi_2 + \mathscr{J}_1\Psi_1 + \mathscr{J}_2\Psi_0 + J_1 = 0, \quad \Psi_2 = \partial\Psi_2/\partial r = 0 \quad \text{at} \quad r = a, b, \\ &0 = \left\langle \frac{h(r)}{r^2}\left(\frac{\partial\Psi_0}{\partial r}\frac{\partial\Psi_2}{\partial x} + \frac{\partial\Psi_1}{\partial r}\frac{\partial\Psi_1}{\partial x} + \frac{\partial\Psi_2}{\partial r}\frac{\partial\Psi_0}{\partial x}\right)\right\rangle. \end{aligned}\right\} \tag{7.6}$$

Here,

$$\mathscr{J}_0 = \mathscr{J}[\omega_0, \lambda_0], \quad \mathscr{J}_l = \omega_l\,\partial D^2/\partial s - \lambda_l D^4, \tag{7.7), (7.8}$$

$$J_0 = J(\Psi_0, D^2\Psi_0), \quad J_1 = J(\Psi_1, D^2\Psi_0) + J(\Psi_0, D^2\Psi_1). \tag{7.9), (7.10}$$

The only two possible solutions of (7.4) when $\gamma(\lambda_0)$ is a simple eigenvalue of $\mathscr{L}_0$ are

$$z_1 = e^{-is}\psi_0(x, r, \lambda_0(\alpha)), \quad z_2 = \bar{z}_1. \tag{7.11}$$

The unique real-valued solution of (7.4), therefore, takes the form

$$\Psi_0 = 2\,\mathrm{Re}\,(z_1) = e^{-is}\psi_0 + e^{is}\bar{\psi}_0. \tag{7.12}$$

The amplitude of this solution is fixed by the normalizing condition (7.4c); the solution is unique to within an arbitrary translation of the origin of s.

8. The solvability conditions

To determine the solvability condition for systems (7.5) and (7.6), one needs to define another scalar product

$$[\hat{a}, \hat{b}] = \frac{1}{2\pi}\int_0^{2\pi}(\hat{a}, \hat{b})\,ds \tag{8.1}$$

for 2π-periodic functions of s. For notational convenience, let

$$[\hat{a}] \equiv [\hat{a}, z_1^*], \tag{8.2}$$

where

$$z_1^* = e^{-is}\psi_0^*, \quad z_2^* = \bar{z}_1^* \tag{8.3}$$

are the solutions of (6.5) at criticality, when $\bar{\gamma} = -i\omega_0$.

The adjoint problem to (7.4a, b) is described by the system

$$\mathscr{J}_0^* \Psi_0^* = 0, \quad \Psi_0^* = \partial\Psi_0^*/\partial r = 0 \quad \text{at} \quad r = a, b, \tag{8.4}$$

where the adjoint operator

$$\mathscr{J}_0^* = -\omega_0 \partial D^{*2}/\partial s + \mathscr{L}_0^* \tag{8.5}$$

is defined relative to (8.1).

Suppose that $\pm i\omega_0$ are the simple eigenvalues of $\mathscr{L}_0$ and consider the equation

$$\mathscr{J}_0 \Psi_n = f_n, \quad n = 1, 2, 3, \ldots, \tag{8.6}$$

where f_n is a periodic function of x and s with periods $2\pi/\alpha$ and 2π, respectively, Ψ_n has the same periodicity as f_n and $\Psi_n = \partial\Psi_n/\partial r = 0$ at $r = a, b$. Then it can be readily shown using (8.4) that a necessary condition for the solvability of (8.6) is

$$[f_n, z_1^*] = [f_n, z_2^*] = 0. \tag{8.7}$$

Equation (8.7) is also a sufficient condition for solvability (Joseph & Sattinger 1972). The periodic solutions of (8.6) are not unique. Any solution of the homogeneous system (7.4) may be added to a solution of (8.6). The normalizing condition

$$b(\eta)\,\delta_{n0} = -\left\langle \frac{h(r)}{r^2}\left(\frac{\partial\Psi}{\partial r}\frac{\partial\Psi}{\partial x}\right)_n \right\rangle \tag{8.8}$$

is sufficient to ensure uniqueness.

By applying the condition (8.7) to (7.5) and (7.6), one finds

$$[\mathscr{J}_1\Psi_0] + [\mathscr{J}_0] = 0 \tag{8.9}$$

and

$$[\mathscr{J}_1\Psi_1] + [\mathscr{J}_2\Psi_0] + [J_1] = 0. \tag{8.10}$$

It is evident that $[J_0] = 0$ by virtue of the integration over the variable s since J_0 contains no terms proportional to e^{-is}. Thus, at first order,

$$\left.\begin{aligned} 0 = [\mathscr{J}_1\Psi_0] &= \omega_1[\partial(D^2\Psi_0)/\partial s] - \lambda_1[D^4\Psi_0] \\ &= (-i\omega_1 + \gamma'\lambda_1)(D^2\psi_0, \psi_0^*), \end{aligned}\right\} \tag{8.11}$$

where the last expression follows from using (6.8) evaluated at criticality with

$$\psi(x, r, \lambda_0(\alpha)) = \psi_0. \tag{8.12}$$

It then follows that

$$\omega_1 = \lambda_1 = \mathscr{J}_1 = 0 \tag{8.13}$$

when $\gamma' \neq 0$.

With the use of (8.13), equation (8.10) becomes

$$-i\omega_2 + \lambda_2\gamma' + [J_1]/(D^2\psi_0, \psi_0^*) = 0. \tag{8.14}$$

To find λ_2 and ω_2, one must solve problems (7.4) and (7.5). It is shown by Joseph & Sattinger (1972) that

$$\lambda_{2l+1} = \omega_{2l+1} = 0, \quad l = 0, 1, 2, \ldots. \tag{8.15}$$

Thus, the first non-zero corrections

$$\lambda = \lambda_0 + \epsilon^2\lambda_2 + O(\epsilon^4) \tag{8.16}$$

and

$$\omega = \omega_0 + \epsilon^2\omega_2 + O(\epsilon^4) \tag{8.17}$$

can be determined from (8.14) once $[J_1]$ is known. To compute J_1, one needs to solve problem (7.5). The ϵ^2 in (8.16) and (8.17) is equal to $f_T - f_L$ as in (4.8).

9. The solution of the first-order equations and expressions for λ_2 and ω_2

Before proceeding to the solution of the first-order equations, the zeroth-order system (7.4) and its adjoint problem (8.4) will be written in terms of their respective amplitude functions $\phi_0(r)$ and $\phi_0^*(r)$ defined by

$$\psi_0(x, r) = \phi_0(r)\,e^{i\alpha x}, \quad \psi_0^*(x, r) = \phi_0^*(r)\,e^{i\alpha x}. \tag{9.1}$$

The system (7.4) reduces to the Orr–Sommerfeld system

$$\left(U - \frac{\omega_0}{\alpha}\right)\left(\phi_0'' - \frac{1}{r}\phi_0' - \alpha^2\phi_0\right) - \left(U'' - \frac{1}{r}U'\right)\phi_0 = \frac{\lambda_0}{i\alpha}\left[\phi_0^{\text{iv}} - \frac{2}{r}\phi_0''' + \left(\frac{3}{r^2} - 2\alpha^2\right)\left(\phi_0'' - \frac{1}{r}\phi_0'\right) + \alpha^4\phi_0\right], \tag{9.2a}$$

$$\phi_0 = \phi_0' = 0 \quad \text{at} \quad r = a, b. \tag{9.2b}$$

The function ϕ_0 is to be normalized by the condition (7.4c) such that

$$\left[\frac{4\alpha}{b^2 - a^2}\int_a^b \text{Im}\,(\phi_0\bar{\phi}_0')\,dr + \frac{2\alpha}{\ln\eta}\int_a^b \text{Im}\,(\phi_0\bar{\phi}_0')\frac{dr}{r^2}\right] = b(\eta), \tag{9.3}$$

where $b(\eta)$ is given by (4.11). The primes denote differentiation with respect to r.

The adjoint problem (8.4) is expressible as

$$\left(U - \frac{\omega_0}{\alpha}\right)\left(\phi_0^{*\prime\prime} + \frac{3}{r}\phi_0^{*\prime} - \alpha^2\phi_0^*\right) + 2U'\left(\phi_0^{*\prime} + \frac{2}{r}\phi_0^*\right) = -\frac{\lambda_0}{i\alpha}\left[\phi_0^{*\text{iv}} + \frac{6}{r}\phi_0^{*\prime\prime\prime} + \left(\frac{3}{r^2} - 2\alpha^2\right)\phi_0^{*\prime\prime} - \frac{1}{r}\left(\frac{3}{r^2} - 6\alpha^2\right)\phi_0^{*\prime} + \alpha^4\phi_0^*\right], \tag{9.4a}$$

$$\phi_0^* = \phi_0^{*\prime} = 0 \quad \text{at} \quad r = a, b. \tag{9.4b}$$

Next, attention is directed to the first-order system. With $\mathscr{J}_1 = 0$, one can write (7.5) as

$$\mathscr{J}_0\Psi_1 + J_0 = 0, \quad \Psi_1 = \partial\Psi_1/\partial r = 0 \quad \text{at} \quad r = a, b, \tag{9.5a, b}$$

$$0 = \left\langle \frac{h(r)}{r^2}\left(\frac{\partial\Psi_0}{\partial r}\frac{\partial\Psi_1}{\partial x} + \frac{\partial\Psi_1}{\partial r}\frac{\partial\Psi_0}{\partial x}\right)\right\rangle. \tag{9.5c}$$

Now, from (7.9) one finds with the use of (7.12)

$$J_0 = J(\Psi_0, D^2\Psi_0) = A\,e^{2i(\alpha x - s)} + \bar{A}\,e^{-2i(\alpha x - s)} + B + \bar{B}, \tag{9.6}$$

where

$$A = \frac{i\alpha}{r}\left[\left(\phi_0' + \frac{2}{r}\phi_0\right)\left(\phi_0'' - \frac{1}{r}\phi_0' - \alpha^2\phi_0\right) - \phi_0\left(\phi_0''' - \frac{1}{r}\phi_0'' + \frac{1}{r^2}\phi_0' - \alpha^2\phi_0'\right)\right] \tag{9.7}$$

and

$$B = \frac{i\alpha}{r}\left[\left(\bar\phi_0' - \frac{2}{r}\bar\phi_0\right)\left(\phi_0'' - \frac{1}{r}\phi_0' - \alpha^2\phi_0\right) + \bar\phi_0\left(\phi_0''' - \frac{1}{r}\phi_0'' + \frac{1}{r^2}\phi_0' - \alpha^2\phi_0'\right)\right]. \tag{9.8}$$

By linearity, one may write

$$\Psi_1 = \phi_{11}(r)\, e^{2i(\alpha x - s)} + \bar\phi_{11}(r)\, e^{-2i(\alpha x - s)} + \phi_{12} + \bar\phi_{12} + a_0\Psi_0, \tag{9.9}$$

where $a_0\Psi_0$ is a multiple of the solution of the system (7.4). Application of (9.5c) leads to $a_0 = 0$. This results in

$$\Psi_1 = \phi_{11}(r)\, e^{2i(\alpha x - s)} + \bar\phi_{11}(r)\, e^{-2i(\alpha x - s)} + \phi_{12} + \bar\phi_{12}. \tag{9.10}$$

Substitution of (9.10) and (9.6) into (9.5a, b) leads to

$$\mathscr{J}_0(\phi_{11}e^{2i(\alpha x - s)}) + A\, e^{2i(\alpha x - s)} = 0, \quad \phi_{11} = \phi_{11}' = 0 \quad \text{at} \quad r = a, b \tag{9.11a, b}$$

and

$$\mathscr{J}_0\phi_{12} + B = 0, \quad \phi_{12} = \phi_{12}' = 0 \quad \text{at} \quad r = a, b. \tag{9.12a, b}$$

Working out (9.11a) and (9.12a), one obtains

$$\left(U - \frac{\omega_0}{\alpha}\right)\left(\phi_{11}'' - \frac{1}{r}\phi_{11}' - 4\alpha^2\phi_{11}\right) - \left(U'' - \frac{1}{r}U'\right)\phi_{11}$$
$$- \frac{\lambda_0}{2i\alpha}\left[\phi_{11}^{\text{iv}} - \frac{2}{r}\phi_{11}'' + \left(\frac{3}{r^2} - 8\alpha^2\right)\left(\phi_{11}'' - \frac{1}{r}\phi_{11}'\right) + 16\alpha^4\phi_{11}\right] = -\frac{A}{2i\alpha} \tag{9.13}$$

and

$$\lambda_0\left(\phi_{12}^{\text{iv}} - \frac{2}{r}\phi_{12}''' + \frac{3}{r^2}\phi_{12}'' - \frac{3}{r^3}\phi_{12}'\right) = B. \tag{9.14}$$

To calculate λ_2 and ω_2 from (8.14), one begins with the evaluation of $[J_1]$. By employing (7.12), (9.1) and (9.10) one finds

$$[J_1] = \frac{2}{b^2 - a^2}\int_a^b H\bar\phi_0^*\, r\, dr, \tag{9.15}$$

where

$$H = \frac{i\alpha}{r}\Big\{(\phi_{12}' + \bar\phi_{12}')\left(\phi_0'' - \frac{1}{r}\phi_0' - \alpha^2\phi_0\right) - \phi_{11}'\left(\bar\phi_0'' - \frac{1}{r}\bar\phi_0' - \alpha^2\bar\phi_0\right)$$
$$- 2\phi_{11}\left[\bar\phi_0''' - \frac{1}{r}\bar\phi_0'' + \left(\frac{1}{r^2} - \alpha^2\right)\bar\phi_0'\right] + \frac{4}{r}\phi_{11}\left(\bar\phi_0'' - \frac{1}{r}\bar\phi_0' - \alpha^2\bar\phi_0\right)$$
$$+ 2\bar\phi_0'\left(\phi_{11}'' - \frac{1}{r}\phi_{11}' - 4\alpha^2\phi_{11}\right) - \phi_0\left[\phi_{12}''' + \bar\phi_{12}''' - \frac{1}{r}(\phi_{12}'' + \bar\phi_{12}'')\right.$$
$$\left. + \frac{1}{r^2}(\phi_{12}' + \bar\phi_{12}')\right] + \bar\phi_0\left[\phi_{11}''' - \frac{1}{r}\phi_{11}'' + \left(\frac{1}{r^2} - 4\alpha^2\right)\phi_{11}'\right]$$
$$+ \frac{2}{r}\phi_0\left[\phi_{12}'' + \bar\phi_{12}'' - \frac{1}{r}(\phi_{12}' + \bar\phi_{12}')\right] - \frac{2}{r}\bar\phi_0\left(\phi_{11}'' - \frac{1}{r}\phi_{11}' - 4\alpha^2\phi_{11}\right)\Big\}. \tag{9.16}$$

Equation (9.15) can be further simplified by integration by parts to yield

$$[J_1] = \frac{2i\alpha}{b^2 - a^2}\Big\{\int_a^b \bar\phi_0^{*\prime}\,[-\bar\phi_0\phi_{11}'' - \bar\phi_0'\phi_{11}' + 2\bar\phi_0''\phi_{11} + \phi_0(\phi_{12}'' + \bar\phi_{12}'') - \phi_0'(\phi_{12}' + \bar\phi_{12}')]\, dr$$
$$+ \int_a^b \bar\phi_0^*\left[-(\phi_{12}' + \bar\phi_{12}')\left(\frac{1}{r}\phi_0' + \frac{3}{r^2}\phi_0 - \alpha^2\phi_0\right) + \frac{3}{r}\phi_0(\phi_{12}'' + \bar\phi_{12}'')\right.$$
$$+ \phi_{11}\left(\frac{6}{r}\bar\phi_0'' - \frac{6}{r^2}\bar\phi_0' - 6\alpha^2\bar\phi_0' + \frac{4}{r}\alpha^2\bar\phi_0\right)$$
$$\left. - \phi_{11}\left(\frac{1}{r}\bar\phi_0' - \frac{3}{r^2}\bar\phi_0 + 3\alpha^2\bar\phi_0\right) - \frac{3}{r}\bar\phi_0\phi_{11}''\right] dr\Big\}. \tag{9.17}$$

With (9.17), the values of λ_2 and ω_2 can be evaluated from (8.14). By separating (8.14) into real and imaginary parts, one finds

$$\lambda_2 = -\operatorname{Re}([J_1]/(D^2\psi_0, \psi_0^*))/\operatorname{Re}\gamma', \tag{9.18}$$

$$\omega_2 = \lambda_2 \operatorname{Im}\gamma' + \operatorname{Im}([J_1]/(D^2\psi_0, \psi_0^*)), \tag{9.19}$$

where $\gamma' = -(D^4\psi_0, \psi_0^*)/(D^2\psi_0, \psi_0^*)$ at criticality and, when use is made of (6.1) and (9.1),

$$(D^2\psi_0, \psi_0^*) = -\frac{2}{b^2 - a^2}\int_a^b \left(\phi_0'\bar\phi_0^{*\prime} + \frac{2}{r}\phi_0'\bar\phi_0^* + \alpha^2\phi_0\bar\phi_0^*\right) r\, dr \tag{9.20}$$

and

$$(D^4\psi_0, \psi_0^*) = \frac{2}{b^2 - a^2}\int_a^b \left[\phi_0''\bar\phi_0^{*\prime\prime} + \frac{4}{r}\phi_0''\bar\phi_0^{*\prime} - \left(\frac{3}{r^2} - 2\alpha^2\right)\phi_0'\bar\phi_0^{*\prime} + \frac{4}{r}\alpha^2\phi_0'\bar\phi_0^* + \alpha^4\phi_0\bar\phi_0^*\right] r\, dr. \tag{9.21}$$

The reader may readily formulate the problems which govern the derivatives of higher order in ϵ of the time-periodic bifurcating solution. It is clear from the constructions given in this section that the stream-function series (7.3) involve x and s only in the combination $\alpha x - s = \theta$. It follows from this that

$$\hat\Psi(x, r, s; \epsilon) = \hat{\hat\Psi}(r, \theta; \epsilon). \tag{9.22}$$

Functions of x and s which are of the form given by (9.22) are statistically stationary (see §2).

10. Numerical solutions and results

The systems (9.2), (9.4), (9.13) and (9.14) were solved numerically by a Runge–Kutta integration scheme, using the Gram–Schmidt orthonormalization procedure to remove the 'parasitic errors' during the integration. This technique was first devised by Wazzan, Okamura & Smith (1967) for the solution of the Orr–Sommerfeld equation for plane flow. In this investigation, this procedure was extended to solve the non-homogeneous equations (9.13) and (9.14).

To begin with, the eigenvalue problem of the zeroth-order equations (9.2) was solved to obtain $c_r = \omega_0/\alpha$ for a given point $(\alpha, R_c) = (\alpha, 1/\lambda_0)$ on the neutral-stability curve. The corresponding amplitude function ϕ_0 and its derivatives were also computed. This was done for a range of parameters (α, R_c) for three radius ratios $\eta = r_1/r_2 = 1/1{\cdot}01$, $\frac{1}{2}$ and $\frac{1}{3}$. The results of the first two cases agreed

α	R_c	c_r	$\hat{\xi}'$	λ_2	ω_2	$\mathrm{Re}\left\{\frac{[J_1]}{(D^2\psi_0,\,\psi_0^*)}\right\}$
0·650	14923	0·2483	200·63	−0·3821	339·89	76·66
0·700	10889	0·2733	162·89	−0·3726	286·26	60·70
0·750	8299	0·2974	135·55	−0·3448	257·55	46·73
0·800	6583	0·3203	114·91	−0·2886	248·50	33·16
0·850	5424	0·3417	98·72	−0·1911	257·50	18·86
0·900	4641	0·3612	85·46	−0·0314	287·17	2·68
0·950	4136	0·3782	73·94	0·2290	347·27	−16·93
1·000	3875	0·3918	62·92	0·6815	467·35	−42·88
1·020†	3846†	0·3959†	58·22†	0·9696†	551·11†	−56·44†
1·025	3848	0·3967	56·97	1·0580	577·77	−60·28
1·050	3924	0·3996	49·96	1·6718	773·16	−83·53
1·075	4206	0·3987	40·03	3·0007	1238·91	−120·13
1·090	4676	0·3935	29·08	5·6075	2234·54	−163·08
1·095	5064	0·3886	21·46	9·0251	3602·77	−193·69
1·09707	6000	0·3771	4·88	53·4886	22073·0	−261·13
1·09094	7333	0·3626	−16·78	−20·8865	−9156·86	−350·42
1·07654	9333	0·3450	−47·18	−10·1464	−4835·26	−478·69
1·05163	12667	0·3229	−94·31	−7·2881	−3877·26	−687·36
1·02528	16667	0·3036	−147·24	−6·3231	−3696·57	−931·01

† Parameters evaluated at this minimum critical point are designated with a tilde.

TABLE 1. Parameter values at criticality for $\eta = 1/1{\cdot}01$

well with those reported by Mott & Joseph (1968). As a check, the eigenvalue c_r was also computed from the adjoint problem (9.4). It yielded eigenvalues which agreed with those obtained from the system (9.2). The adjoint eigenfunction ϕ_0^* and its derivatives for the same range of (α, R_c) values were computed as well.

Once the ϕ_0 problem had been solved, ϕ_0 was normalized according to condition (9.3). The solutions to the non-homogeneous systems (9.13) with (9.11*b*) and (9.14) with (9.12*b*) were obtained for the same given parameters c_r, α and R_c as were used in the solution of the ϕ_0 problem. These solutions of the non-homogeneous equations consist of a solution to the corresponding homogeneous equation plus a particular solution. The amplitude functions ϕ_{11} and ϕ_{12} and their derivatives follow directly from the numerical integration.

With the ϕ_0, ϕ_0^*, ϕ_{11} and ϕ_{12} solutions available, the expressions for $(D^2\psi_0, \psi_0^*)$, $(D^4\psi_0, \psi_0^*)$ and $[J_1]$ were next evaluated numerically. Finally, the values of λ_2 and ω_2 were calculated from (9.18) and (9.19). All the computations were done on an IBM 360/50 digital computer with double-precision arithmetic.

The numerical results for $\eta = 1/1{\cdot}01$, $\frac{1}{2}$ and $\frac{1}{3}$ were listed, respectively, in tables 1, 2 and 3. To maintain accuracy of the results, it was found that 100, 150 and 200 steps, respectively, were needed in the numerical integration over the interval $a \leqslant r \leqslant b$ with $b-a = 2$.

Inspection of tables 1, 2 and 3 shows that $\xi' = d\xi/d\lambda$ changes sign at the point of maximum wavenumber on the neutral-stability curves. Since the values of $[J_1]/(D^2\psi_0, \psi_0^*)$ are everywhere finite, it can be seen from (9.18) and (9.19) that λ_2 and ω_2 are both unbounded at that point on the neutral curve.

α	R_c	c_r	$\hat{\xi}'$	λ_2	ω_2	$\mathrm{Re}\left\{\frac{[J_1]}{(D^2\psi_0,\,\psi_0^*)}\right\}$
0·650	28254	0·2156	289·70	0·9512	2137·7	−275·57
0·700	20931	0·2369	236·64	1·1331	2005·7	−268·12
0·750	16268	0·2570	197·46	1·4469	2019·6	−285·70
0·800	13239	0·2756	167·31	1·9591	2186·8	−327·79
0·850	11293	0·2921	143·01	2·7795	2556·2	−397·51
0·900	10151	0·3058	122·09	4·1171	3265·5	−502·67
0·925	9860	0·3111	112·09	5·1195	3856·1	−573·83
0·946†	9775†	0·3146†	103·45†	6·2612†	4580·9†	−626·12†
0·950	9779	0·3152	101·74	6·5241	4755·3	−663·75
0·975	9990	0·3172	90·08	8·7029	6288·7	−783·99
1·000	10737	0·3160	74·61	12·9733	9646·0	−967·97
1·020	12541	0·3090	51·57	24·5959	19922·1	−1268·46
1·02702	16581	0·2933	11·22	161·787	154025·0	−1815·29
1·02081	21223	0·2789	−31·20	−76·6233	−83926·9	−2390·58
1·00669	27856	0·2631	−89·14	−35·6579	−45397·5	−3178·63
0·99491	33161	0·2532	−133·89	−28·3378	−39654·1	−3794·24
0·98101	39794	0·2431	−188·46	−24·1296	−37208·4	−4547·56
0·96824	46426	0·2347	−241·90	−21·8281	−36478·3	−5280·31
0·95659	53058	0·2276	−294·56	−20·3452	−36419·7	−5992·78

† Parameters evaluated at this minimum critical point are designated with a tilde.

TABLE 2. Parameter values at criticality for $\eta = \frac{1}{2}$

α	R_c	c_r	$\hat{\xi}'$	λ_2	ω_2	$\mathrm{Re}\left\{\frac{[J_1]}{(D^2\psi_0,\,\psi_0^*)}\right\}$
0·600	98802	0·1563	695·30	1·8257	10144·7	−1269·4
0·650	70529	0·1738	552·90	1·9853	9177·0	−1097·7
0·700	53245	0·1903	453·32	2·3726	8990·1	−1075·5
0·750	42386	0·2055	380·69	3·1574	9620·2	−1202·0
0·800	35577	0·2189	325·68	4·6162	11317·8	−1503·4
0·850	31611	0·2299	282·23	7·2143	14708·4	−2036·1
0·875	30535	0·2343	263·34	9·1776	17449·0	−2416·8
0.900	30091	0·2376	254·58	11·8047	21359·7	−2899·0
0·904†	30083†	0·2380†	242·78†	12·3052†	22135·0†	−2987·5†
0·925	30384	0·2396	228·03	15·4208	27203·3	−3516·4
0·950	31728	0·2399	209·11	20·7625	36833·0	−4341·7
0·975	35052	0·2373	184·53	30·2357	56513·4	−5579·3
0·99026	39506	0·2329	159·02	42·9279	86552·1	−6852·3
1·00320	52675	0·2206	92·65	107·140	261572	−9926·2
0·99549	79012	0·2025	−41·28	−367·223	−1168150	−15157·6
0·98008	105349	0·1899	−175·06	−113·517	−432583	−19871·8
0·96471	131687	0·1804	−305·29	−79·7726	−349581	−24353·4

† Parameters evaluated at this minimum critical point are designated with a tilde.

TABLE 3. Parameter values at criticality for $\eta = \frac{1}{3}$

Several properties of the bifurcating solutions which are important will not be discussed here. We have already noted that in the present problem, as in the more general problem treated by Joseph & Sattinger (1972), the subcritical bifurcating solutions are unstable; disturbances which escape the domain of attraction of laminar Poiseuille flow cannot be attracted to the unstable bifurcating solution. Such disturbances snap through the bifurcating solution and are attracted to the turbulent flows lying on the friction-factor response curve (see figure 1*a*). It should also be noted that here, as in the plane Poiseuille flow problem treated by Chen & Joseph (1973), nonlinear periodic solutions exist for waves which have a smaller wavelength than any periodic wave which exists when $\epsilon = 0$. Unlike the plane Poiseuille flow problem, however, the periodic solutions with wavenumbers on the lower branch of the neutral-stability curve bifurcate subcritically when $\eta = \frac{1}{2}$ and $\frac{1}{3}$. We believe that supercritical bifurcation on this branch of the curve, which is evident when $\eta = 1/1{\cdot}01$, can also be found for ducts with $\eta = \frac{1}{2}$ and $\frac{1}{3}$ for wavenumbers smaller than the ones for which results were computed in the present study.

11. Comparison with experiments

The use of the friction-factor discrepancy $\epsilon^2 = f_T - f_L$ as the measure of the amplitude of the disturbances makes possible a clear interpretation of the results of studies of stability and bifurcation for the problem of transition and turbulence. Briefly stated, the reason why the linear theory of stability fails to describe experiments is that the mathematical solution which arises from the instability of laminar flow at $R = R_c$ bifurcates subcritically and is, therefore, itself unstable (see Joseph & Sattinger 1972). Theoretically, laminar flow is stable to small disturbances for some interval of $R < R_c$ but practically one does not observe the laminar flow because it is unstable to larger disturbances. Sufficiently small disturbances of laminar flow will decay though larger disturbances may persist. Disturbances which are not attracted to laminar flow when $R < R_c$ are attracted by other solutions which exist at the same R (with the same mass flux as laminar flow). In the present case the other solutions include at least the subcritical time-periodic bifurcating solution which we have calculated theoretically and the turbulent solutions which are observed in practice. Disturbances which escape the domain of attraction of laminar flow cannot be attracted by the bifurcating solution because this solution is itself unstable. Finite disturbances of laminar Poiseuille flow snap through the unstable time-periodic solution and are attracted to other solutions which, for want of a more precise description, are called stable turbulence.

In practice stable turbulence appears to have the property of consistent reproducibility on the average. By this we mean that for smooth pipes there appears to be a curve, which we have called a response curve, which defines a functional relation between the Reynolds number and friction factor (between the mass flux and pressure gradient). The existence of such a curve, which is widely accepted as natural even in elementary books, is actually a remarkable event since the curve is defined over a set of fluctuating turbulent flows each of

which differs from its neighbours. In this sense the response curve may be regarded as giving the steady average response of a fluctuating system subjected to steady external conditions. It is, therefore, natural that the response curve can be defined through analysis of functionals defined over statistically stationary solutions of the Navier–Stokes problem for flow in annular ducts (see Howard 1971).

The bifurcation diagram in the plane of the response curve is shown in figures 1(*a*), (*b*) and (*c*). The experimentally observed values are taken from the paper of Walker, Whan & Rothfus (1957). The co-ordinates Re_2 and f_2 used by these authors are related to R and f by

$$R = Re_2/4F_1(\eta), \quad f = 4f_2/F_1(\eta),$$

where

$$F_1(\eta) = \frac{1+(1-\eta^2)/(2\ln\eta)}{1-\eta}.$$

The response function for laminar flow is given by

$$f_L = (16/R)\,F_2(\eta), \tag{11.1}$$

where

$$F_2(\eta) = \frac{(1-\eta)^2}{(1+\eta)^2+(1-\eta^2)/\ln\eta} = \frac{1}{8b(\eta)}.$$

The response curve for laminar flow appears as a straight line

$$d\ln f_L/d\ln R = -1 \tag{11.2}$$

on a log–log plot. The response curve for the bifurcating solution near the point of bifurcation is given by

$$\epsilon^2 = f_T - f_L \cong (\lambda-\lambda_0)/\lambda_2 + O(\epsilon^4).$$

Hence

$$\ln f_T = \ln f_L + \ln\left[1+\frac{1-R/R_c}{16\lambda_2 F_2(\eta)}\right] + O(\epsilon^4)$$

and

$$\left.\frac{d\ln f_T}{d\ln R}\right|_{\tilde{R}_c} = \frac{d\ln f_L}{d\ln R} - \frac{1}{16\tilde{\lambda}_2 F_2(\eta)}. \tag{11.3}$$

Equation (11.3) holds for each fixed spatial period, i.e., for each value of α in tables 1, 2 and 3.

The slope of the response curve for the wavenumber which gives the smallest critical value $R = \tilde{R}_c$,

$$\tilde{R}_c = R_c(\tilde{\alpha}) = \min_{\alpha} R_c(\alpha),$$

is obtained from tables 1, 2 and 3. These slopes are given in table 4. They are very close to the slope of the laminar response curve. Inspection of the tables 1, 2 and 3 shows that the solutions which bifurcate with α slightly larger than $\tilde{\alpha}$ lie even closer to the laminar response curve. It may be interesting to determine the envelope of two-dimensional bifurcating solutions.

All of the features which we have just summarized are evident in figures 1(*a*), (*b*) and (*c*). The circles there represent experimental observations; solid lines are stable solutions and dashed lines are unstable solutions. It is clear from the values of $d\ln f_T/d\ln R$ given in the figures that the slope of the response curve for

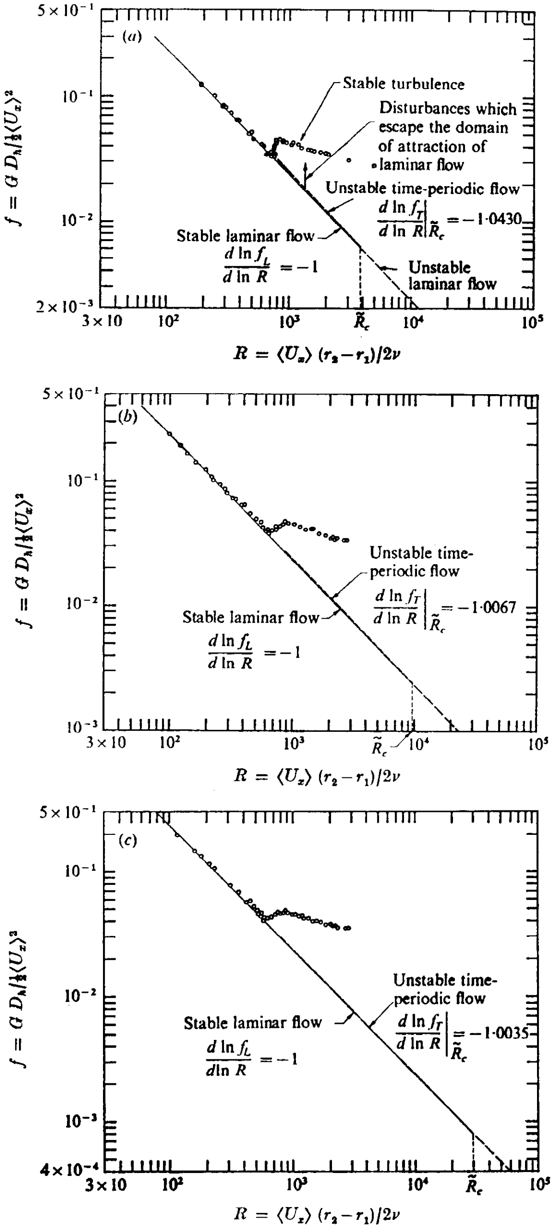

FIGURE 1. Bifurcation diagram in the plane of the response curve. (a) $\eta = 1/1{\cdot}01$. (b) $\eta = \frac{1}{2}$. (c) $\eta = \frac{1}{3}$.

η	$1/1{\cdot}01$	$\frac{1}{2}$	$\frac{1}{3}$
$1/[16\tilde{\lambda}_2 F_2(\eta)]$	0·0430	0·0067	0·0035
$[d \ln f_T/d \ln R]_{\tilde{R}}$	−1·0430	−1·0067	−1·0035

TABLE 4. Values of the slopes of the response curve at $R = \tilde{R}_c$

time-periodic flow at $R = \tilde{R}_c$ differs by only a small amount from the slope (−1) for laminar flow. Of course, the heavy dashed line in the figures only represents the slope of the response curve; the computation of the actual curve would require computations of the higher-order derivatives of $\lambda(\epsilon)$ at least. It is natural to wonder if the envelope of response curves for the bifurcating solutions which continue the dashed line would merge smoothly with the response curve defined by experiments (figure 1). It is possible that the continuation of the envelope of bifurcation response curves lies near to the line $d \ln f_L/d \ln R = -1$ for values of R between $\tilde{R}_c$ and the transition value ($R \sim 600$ or in the conventional notation $2\langle U_x\rangle(r_2 - r_1)/\nu \sim 2400$). In the neighbourhood of this transition value the value of R on the envelope of response curves for the bifurcating solutions might attain a minimum. The envelope of bifurcating solutions on the upper branch above the minimum could possibly regain some stability and appear as a curve through the experimental data shown in figure 1. The computation of higher-order terms in general, and of λ_4, in particular, is one way to test further these conjectures.

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