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Domain Perturbations: The Higher Order Theory of Infinitesimal Water Waves

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1. Introduction

The higher order theory of infinitesimal water waves refers to a perturbation theory which represents solutions to problems in the theory of nonlinear water waves as a power series in the amplitude of the wave. The infinitesimal wave appears in this theory at zero$^{\text{th}}$ order. In this note I shall derive the higher order theory by my method of domain perturbations (see references [1], [2], [3] and [4] for other applications). This method can be used to find solutions to boundary value problems in which the perturbed domains are prescribed but the solutions are unknown, and to free surface problems, like the ones considered here, in which the shape of the domain is to be determined.

The aims of this note are best served by confining our attention to the easiest and most well unterstood of the water wave problems: the two-dimensional theory of progressive water waves. Mathematically rigorous demonstrations of the existence of two-dimensional progressive waves which are analytic in the amplitude of the wave have been given by LEVI-CIVITA [5], STRUIK [8], and by LITTMAN & NIRENBERG in STOKER's [7] book. These demonstrations rely heavily on the theory of analytic functions of a complex variable; the standard method is to write $x+iy=f(\phi+i\psi)$ which reduces the problem to a fixed domain. This type of Lagrangian formulation cannot be extended to three-dimensional wave problems or to problems of a still more general kind.

The "Lagrangian" formulation which is to be developed below does not rely on complex variables and applies equally to two- and three-dimensional wave problems and to more general problems. In this formulation one imagines a one-parameter family of domains on which the field equations and boundary conditions are to be solved. The solution is presumed known in some reference domain. The solution in the perturbed domain is developed in a power series in the parameter, the coefficients being substantial derivatives of the field variables evaluated in the reference domain. In [2] and later in [3] and [4] it was shown that the values of coefficients on the boundary may be computed by use of properties of the mapping on the boundary only, so that its nature in the interior need not be known. In this note I show that the interior values of the coefficients also depend only on boundary properties and not on interior properties of the mapping. It follows that the solution in the perturbed domain depends only on the boundary values of the perturbation.

2. The Equations Governing Progressive Irrotational Water Waves

We shall consider a channel of finite depth whose bottom is at $y=-d$. The equations which govern inviscid irrotational water waves in two dimensions are (see [7] or [9])

$$\boldsymbol{u}=\nabla\phi, \tag{2.1a}$$

$$\Delta\phi=0, \tag{2.1b}$$

$$\phi_y=0|_{y=-d}, \tag{2.1c}$$

$$\eta_t+\phi_x\eta_x-\phi_y=0, \tag{2.1d}$$

$$\phi_t+g\eta+\tfrac{1}{2}|\nabla\phi|^2=0|_{y=\eta(x,t)}. \tag{2.1e}$$

Here $\phi(x, y, t)$ is the velocity potential, $\eta(x, t)$ is the free surface and g is the constant value of the gravity.

For progressive waves, x and t appear always in the combination

$$s=x-ct, \tag{2.2}$$

so that

$$\phi=\phi(s, y), \qquad \eta=\eta(s). \tag{2.3}$$

For simplicity we shall confine our attention to periodic waves with period 2π:

$$\phi(s+2\pi, y)=\phi(s, y), \qquad \eta(s+2\pi)=\eta(s). \tag{2.4}$$

The amplitude of the wave which we seek is

$$\varepsilon^2=\langle\eta^2\rangle \tag{2.5}$$

where

$$\langle\circ\rangle=\frac{1}{2\pi}\int_0^{2\pi}\circ\, ds.$$

It is convenient to introduce the amplitude into the coefficients of the differential equations by changing variables. Thus, with

$$\eta=\varepsilon\mu, \qquad \phi=\varepsilon\psi, \qquad \boldsymbol{u}=\varepsilon\boldsymbol{v},$$

we get

$$\boldsymbol{v}=\nabla\psi, \tag{2.6a}$$

$$\Delta\psi=0, \tag{2.6b}$$

$$\psi_y=0\,|_{y=-d}, \tag{2.6c}$$

$$c\mu_s+\psi_y-\varepsilon\psi_s\mu_s=0 \tag{2.6d}$$

$$\left. g\mu-c\psi_s+\frac{\varepsilon}{2}|\nabla\psi|^2=0\right|_{y=\varepsilon\mu}, \tag{2.6e}$$

$$\text{periodicity}, \tag{2.6f}$$

$$1=\langle\mu^2\rangle, \tag{2.6g}$$

where

$$\Delta=\partial_{yy}^2+\partial_{ss}^2 \quad\text{and}\quad |\nabla\psi|^2=\psi_y^2+\psi_s^2.$$

Solutions of (2.6) are the functions

$$\mu(s;\varepsilon), \qquad -\infty<s<\infty,$$

$$\psi(s,y;\varepsilon), \qquad -d<y<\varepsilon\mu,$$

and the wave speeds $c(\varepsilon)$. The solution is to be constructed as a power series in the amplitude parameter ε.

When $\varepsilon=0$, the problem (2.6) reduces to

$$\begin{aligned}
&\boldsymbol{v}^{\langle 0\rangle}=\nabla\psi^{\langle 0\rangle}, \quad \Delta\psi^{\langle 0\rangle}=0,\\
&\psi_y^{\langle 0\rangle}=0\,|_{y=-d}, \quad c^{\langle 0\rangle}\mu_s^{\langle 0\rangle}+\psi_y^{\langle 0\rangle}=0\,|_{y=0},\\
&g\mu^{\langle 0\rangle}-c^{\langle 0\rangle}\psi_s^{\langle 0\rangle}=0\,|_{y=0}, \quad \text{periodicity},\\
&1=\langle\mu^{\langle 0\rangle 2}\rangle.
\end{aligned} \tag{2.7}$$

This problem (2.7) is tractable because it is linear and is posed on a domain of simple configuration $-d\leqq y_0\leqq 0$.

3. Analytic Mappings onto the Reference Configuration

The problem (2.6) is nearly intractable because it is nonlinear and is posed on an unknown domain of complicated configuration $-d\leqq y\leqq\varepsilon\mu(s;\varepsilon)$. To study the problem (2.6), it is advantageous to map it onto the simple domain $-d\leqq y\leqq 0$.

The mapping we seek is in the form $\boldsymbol{x}=\boldsymbol{i}s+\boldsymbol{j}y(y_0;\varepsilon)$,

$$s=s_0, \qquad -\infty<s<\infty, \tag{3.1a}$$

and

$$y=y(y_0;\varepsilon)^{\star}, \qquad -d\leqq y\leqq\varepsilon\mu, \tag{3.1b}$$

where

$$y_0=y(y_0;0), \qquad -d\leqq y_0\leqq 0. \tag{3.2}$$

⋆ The dependence of y on s_0 (*cf.* 3.6) has been suppressed.

The mapping is to be one to one, carrying boundary points into boundary points

$$\varepsilon\mu = y(0;\varepsilon)$$

and

$$-d = y(-d;\varepsilon). \tag{3.3}$$

We also want the mapping to be analytic in ε and uniquely invertible with

$$y_0 = y_0(y;\varepsilon). \tag{3.4}$$

Associated with the mapping (3.1) is the mapping velocity

$$\boldsymbol{U}(y,\varepsilon) = \frac{d\boldsymbol{x}}{d\varepsilon} = \boldsymbol{j}\,\frac{dy(y_0;\varepsilon)}{d\varepsilon}. \tag{3.5}$$

The requirements (3.2), (3.3) and (3.4) do not suffice to determine a unique mapping. An example of an acceptable mapping is (3.1a) and

$$y = y_0(1+\varepsilon\mu/d) + \varepsilon\mu(s;\varepsilon). \tag{3.6}$$

The associated velocity field is

$$\boldsymbol{U} = \boldsymbol{j}\,[\mu y_0/d + \mu + \varepsilon y_0 \mu_\varepsilon/d + \varepsilon\mu_\varepsilon]. \tag{3.7}$$

When $\varepsilon = 0$, equation (3.7) becomes

$$\boldsymbol{U}^{\langle 0\rangle} = \boldsymbol{j}\,\mu^{\langle 0\rangle}(y_0/d+1) = \boldsymbol{j}\,\frac{dy(y_0;0)}{d\varepsilon} \equiv \boldsymbol{j}\,y^{[1]}. \tag{3.8}$$

4. Substantial Derivatives and Partial Derivatives of Field Variables with Respect to ε at $\varepsilon = 0$

In this calculation we shall use the following notation*:

$$\psi^{[n]} = d^n\psi/d\varepsilon^n|_{\varepsilon=0} \tag{4.1}$$

and

$$\psi^{\langle n\rangle} = \partial^n\psi/\partial\varepsilon^n|_{\varepsilon=0}. \tag{4.2}$$

* The use of the substantial derivative is a computational convenience. We are considering functions of the form

$$\psi(s,y;\varepsilon), \qquad \psi(s_0, y(y_0;\varepsilon);\varepsilon) \equiv \hat{\psi}(s_0,y_0;\varepsilon).$$

Of course, the reference coordiantes s_0 and y_0 do not depend on ε:

$$\frac{\partial\hat{\psi}}{\partial\varepsilon} = \frac{d\psi}{d\varepsilon} = \frac{\partial\psi}{\partial\varepsilon} + \frac{dy}{d\varepsilon}\,\partial_y\psi$$

and

$$\begin{aligned}\frac{d^2\psi}{d\varepsilon^2} &= \frac{d}{d\varepsilon}\frac{\partial\psi}{\partial\varepsilon} + \frac{d^2y}{d\varepsilon^2}\,\partial_y\psi + \frac{dy}{d\varepsilon}\frac{d}{d\varepsilon}\,\partial_y\psi \\ &= \frac{\partial^2\psi}{\partial\varepsilon^2} + 2\frac{dy}{d\varepsilon}\,\partial_y\frac{\partial\psi}{\partial\varepsilon} + \left(\frac{dy}{d\varepsilon}\right)^2\partial^2_{yy}\psi + \frac{d^2y}{d\varepsilon^2}\,\partial_y\psi.\end{aligned}$$

The superscript $[n]$ is a substantial derivative following the mapping; $\langle n\rangle$ means "differentiate" holding $y(y_0;\varepsilon)$ fixed:

$$\psi^{[0]}=\psi^{\langle 0\rangle}, \tag{4.3a}$$

$$\psi^{[1]}=\psi^{\langle 1\rangle}+y^{[1]}\psi_y^{\langle 0\rangle}, \tag{4.3b}$$

$$\psi^{[2]}=\psi^{\langle 2\rangle}+2y^{[1]}\psi_y^{\langle 1\rangle}+y^{[2]}\psi_y^{\langle 0\rangle}+y^{[1]^2}\psi_{yy}^{\langle 0\rangle}, \tag{4.3c}$$

$$\begin{aligned}\psi^{[3]}=\psi^{\langle 3\rangle}&+3y^{[2]}\psi_y^{\langle 1\rangle}+3y^{[1]}\psi_y^{\langle 2\rangle}+y^{[3]}\psi_y^{\langle 0\rangle}\\&+3y^{[1]^2}\psi_{yy}^{\langle 1\rangle}+3y^{[1]}y^{[2]}\psi_{yy}^{\langle 0\rangle}+y^{[1]^3}\psi_{yyy}^{\langle 0\rangle},\end{aligned} \tag{4.3d}$$

etc.

At $y=-d=y_0$,

$$y^{[j]}=0, \quad j=1,2,\dots. \tag{4.4}$$

At $y=\varepsilon\mu$ (or $y_0=0$) we have

$$y^{[j]}=d^j(\varepsilon\mu)/d\varepsilon^j|_{\varepsilon=0}.$$

Since $\mu=\mu(s;\varepsilon)$, we have

$$y^{[j]}=(\varepsilon\mu)^{\langle j\rangle}|_{\varepsilon=0}.$$

For example, at $y_0=0$

$$\begin{aligned}&y^{[1]}=\mu^{\langle 0\rangle},\\&y^{[2]}=2\mu^{\langle 1\rangle},\\&y^{[3]}=3\mu^{\langle 2\rangle},\\&\text{etc.}\end{aligned} \tag{4.5}$$

A major simplification in the domain perturbation is possible because the field equations (2.6a, b) are identities in y. For example, consider (2.6b). This is an identity in ε; hence,

$$d^n\Delta\psi/d\varepsilon^n=0. \tag{4.6}$$

Since it is also an identity in s and $y(y_0;\varepsilon)$ $(-d\leqq y\leqq\varepsilon\mu)$, we have that for each and every integer $n\geqq 0$,

$$\Delta(\partial^n\psi/\partial\varepsilon^n)=d^n\Delta\psi/d\varepsilon^n. \tag{4.7}$$

To prove (4.7), we shall show that if (4.7) holds for some integer $l=n$, then it also holds when $l=n+1$:

$$\begin{aligned}\frac{d}{d\varepsilon}\Delta(\partial^n\psi/\partial\varepsilon^n)&=\Delta(\partial^{n+1}\psi/\partial\varepsilon^{n+1})+\frac{dy}{d\varepsilon}\partial_y\Delta(\partial^n\psi/\partial\varepsilon^n)\\&=\Delta(\partial^{n+1}\psi/\partial\varepsilon^{n+1})=0.\end{aligned} \tag{4.8}$$

Since (4.7) holds when $n=1$, the induction (4.8) proves that (4.7) holds for all n.

The simplification (4.8) which arises because the field equations are identities in y, $-d\leqq y\leqq\varepsilon\mu$, as well as in ε does not apply to the equations (2.6c, d, e) which give the conditions which must be satisfied at the boundary $y=\varepsilon\mu$.

5. Expansion of the Solution in a Series of Powers of ε

The solution of problem (2.6) may be found in the form of a power series in the parameter ε. Thus,

$$\begin{Bmatrix} \psi(s,y;\varepsilon) \\ \mu(s;\varepsilon) \\ c(\varepsilon) \end{Bmatrix} = \sum_{n=0} \frac{1}{n!} \begin{Bmatrix} \psi^{[n]}(s_0,y_0) \\ \mu^{[n]}(s_0) \\ c^{[n]} \end{Bmatrix} \varepsilon^n. \tag{5.1 a, b, c}$$

To compute this series, we generate linear boundary value problems for the Taylor coefficients. The zero$^{\text{th}}$-order problem has

$$\psi^{[0]} = \psi^{\langle 0\rangle}, \quad \mu^{[0]} = \mu^{\langle 0\rangle}, \quad \text{and} \quad c^{[0]} = c^{\langle 0\rangle}$$

(*cf.* 2.7). To calculate n^{th} derivatives, we differentiate equations (2.6) with respect to ε at $\varepsilon=0$ and find that $c^{[n]} \equiv c^{\langle n\rangle}$,

$$\Delta\psi^{\langle n\rangle} = 0, \tag{5.2a}$$

$$\psi_y^{\langle n\rangle} = 0|_{y_0=-d}, \tag{5.2b}$$

$$(c\mu_s + \psi_y - \varepsilon\psi_s\mu_s)^{[n]} = 0 \tag{5.2c}$$

$$\left.\left(g\mu - c\psi_s + \frac{\varepsilon}{2}|\nabla\psi|^2\right)^{[n]} = 0\right|_{y_0=0}, \tag{5.2d}$$

periodicity, (5.2e)

$$0 = \langle(\mu^2)^{\langle n\rangle}\rangle. \tag{5.2f}$$

When $n=1$, equations (5.2c) and (5.2d) may be written as

$$c^{\langle 0\rangle}\mu_s^{\langle 1\rangle} + \psi_y^{\langle 1\rangle} = \psi_s^{\langle 0\rangle}\mu_s^{\langle 0\rangle} - \mu^{\langle 0\rangle}\psi_{yy}^{\langle 0\rangle} - c^{\langle 1\rangle}\mu_s^{\langle 0\rangle} \tag{5.3c}$$

and

$$g\mu^{\langle 1\rangle} - c^{\langle 0\rangle}\psi_s^{\langle 1\rangle} = c^{\langle 1\rangle}\psi_s^{\langle 0\rangle} - \tfrac{1}{2}(\nabla\psi^{\langle 0\rangle})^2 + c^{\langle 0\rangle}\mu^{\langle 0\rangle}\psi_{sy}^{\langle 0\rangle}. \tag{5.3d}$$

When $n=2$, (5.2c) and (5.2d) may be written as

$$\begin{aligned} c^{\langle 0\rangle}\mu_s^{\langle 2\rangle} + \psi_y^{\langle 2\rangle} = {} & 2(\psi_s^{\langle 1\rangle}\mu_s^{\langle 0\rangle} + \psi_s^{\langle 0\rangle}\mu_s^{\langle 1\rangle}) - c^{\langle 2\rangle}\mu_s^{\langle 0\rangle} - 2\mu^{\langle 1\rangle}\psi_{yy}^{\langle 0\rangle} \\ & - 2\mu^{\langle 0\rangle}\psi_{yy}^{\langle 1\rangle} - 2c^{\langle 1\rangle}\mu_s^{\langle 1\rangle} - (\mu^{\langle 0\rangle})^2\psi_{yyy}^{\langle 0\rangle} \\ & + 2\mu^{\langle 0\rangle}\mu_s^{\langle 0\rangle}\psi_{sy}^{\langle 0\rangle} \end{aligned} \tag{5.4c}$$

and

$$\begin{aligned} g\mu^{\langle 2\rangle} - c^{\langle 0\rangle}\psi_s^{\langle 2\rangle} = {} & c^{\langle 2\rangle}\psi_s^{\langle 0\rangle} + 2c^{\langle 1\rangle}\psi_s^{\langle 1\rangle} + 2c^{\langle 1\rangle}\mu^{\langle 0\rangle}\psi_{sy}^{\langle 0\rangle} \\ & + 2c^{\langle 0\rangle}\mu^{\langle 0\rangle}\psi_{sy}^{\langle 1\rangle} + 2c^{\langle 0\rangle}\mu^{\langle 1\rangle}\psi_{sy}^{\langle 0\rangle} \\ & + c^{\langle 0\rangle}(\mu^{\langle 0\rangle})^2\psi_{syy}^{\langle 0\rangle} - 2\mu^{\langle 0\rangle}\nabla\psi_y^{\langle 0\rangle}\cdot\nabla\psi^{\langle 0\rangle} \\ & - 2\nabla\psi^{\langle 0\rangle}\cdot\nabla\psi^{\langle 1\rangle}. \end{aligned} \tag{5.4d}$$

Similar problems may be generated for the higher-order derivatives with respect to ε.

In closing this section we draw attention to the fact that the functions on the left of (5.1) are defined on the wavy domain and the functions on the right of (5.1) are defined on the flat domain. The functions on the flat domain are to be obtained from the solution of the perturbation problems (2.7) and (5.2). The flat and wavy domains are related through the transformations (3.1 a, b).

6. The Interior Fields are Independent of the Mapping onto the Reference Configuration

Interior values of the mapping (3.1) do not enter into the determination of the shape of the free surface. To form the required derivatives, we need only to know the boundary data (3.1 b) for the mapping.

This same independence from interior values of the mapping (3.1) also holds for the interior field $\psi(s, y; \varepsilon)$; in fact, an even stronger result holds:

$$\psi(s, y; \varepsilon) = \sum_{n=0} \frac{1}{n!} \psi^{[n]}(s_0, y_0)\,\varepsilon^n = \sum_{n=0} \frac{1}{n!} \psi^{\langle n\rangle}(s, y)\,\varepsilon^n. \tag{6.1 a, b}$$

It follows from (6.1) that the power series development of $\psi(s, y; \varepsilon)$ in powers of ε when the wavy domain variables (s, y) are fixed may be obtained directly from the partial derivative $\psi^{\langle n\rangle}(s_0, y_0)$ by replacing the flat domain variables (s_0, y_0) with wavy domain variables (s, y). This form for the independence of the final fields on the mapping was conjectured by JOSEPH & FOSDICK (p. 338).

The proof of (6.1), assuming convergence, follows directly from combining (4.3) with middle member of (6.1) followed by a rearrangement of the resulting series:

$$\begin{aligned}
&\psi^{[0]}(s_0, y_0; 0) + \varepsilon\psi^{[1]} + \varepsilon^2\psi^{[2]}/2 + \varepsilon^3\psi^{[3]}/3! + \cdots \\
&\quad = \psi^{\langle 0\rangle}(s_0, y_0; 0) + \varepsilon y^{[1]}\psi_y^{\langle 0\rangle} + \varepsilon^2(y^{[2]}\psi_y^{\langle 0\rangle} + y^{[1]^2}\psi_{yy}^{\langle 0\rangle})/2! \\
&\qquad + \varepsilon^3(y^{[3]}\psi_y^{\langle 0\rangle} + 3y^{[1]}y^{[2]}\psi_{yy}^{\langle 0\rangle} + y^{[1]^3}\psi_{yyy}^{\langle 0\rangle})/3! + \cdots \\
&\qquad + \varepsilon\{\psi^{\langle 1\rangle}(s_0, y_0; 0) + \varepsilon y^{[1]}\psi_y^{[1]} + \varepsilon^2(y^{[2]}\psi_y^{\langle 1\rangle} + y^{[1]^2}\psi_{yy}^{\langle 1\rangle})/2! + \cdots\} \\
&\qquad + \frac{\varepsilon^2}{2}\{\psi^{\langle 2\rangle}(s_0, y_0; 0) + \varepsilon y^{[1]}\psi_y^{\langle 2\rangle} + \cdots\} + \cdots \qquad (6.2) \\
&\quad = \psi^{\langle 0\rangle}(s, y(y_0; \varepsilon); 0) + \varepsilon\psi^{\langle 1\rangle}(s, y(y_0; \varepsilon); 0) \\
&\qquad + \frac{\varepsilon^2}{2}\psi^{\langle 2\rangle}(s, y(y_0; \varepsilon); 0) + \cdots \\
&\quad = \psi(s, y(y_0; \varepsilon); \varepsilon).
\end{aligned}$$

The Lagrangian formulation of the domain perturbation theory gives an unambiguous computational procedure for determining perturbation fields in perturbed domains. The functions $\psi^{\langle n\rangle}(s_0, y_0; 0)$, $\mu^{\langle n\rangle}(s_0; 0)$ and the numbers $c^{\langle n\rangle}$ are to be determined from (5.2). The functions $\psi^{\langle n\rangle}(s_0, y_0; 0)$ are originally defined on the flat domain $y_0 \leqq 0$. Given an ε analytic solution $\psi(s, y(y_0; \varepsilon); \varepsilon)$ of the water wave problem, the existence of functions $\psi^{\langle n\rangle}(s, y(y_0; \varepsilon); 0)$ follows immediately (see 6.2). These functions are defined for $y \leqq \eta$ and $\psi^{\langle n\rangle}(s, y; 0) = \psi^{\langle n\rangle}(s_0, y_0; 0)$ whenever $y = y_0$. The final solution of the wave problem is given in series form by (6.1) and (5.1 b, c).

7. Eulerian Formulation of the Higher Order Theory

The theory which I have just derived using the (Lagrangian) method of domain mappings can also be obtained from an Eulerian formulation. The Eulerian theory can be traced back to STOKES [6]; popular derivations of this now traditional theory are given by STOKER [7] and by WEHAUSEN & LAITONE [9].

In the traditional Eulerian derivation of the higher order theory (see [7] or [9]) one substitutes the series

$$\begin{bmatrix} \phi(s,y;\varepsilon) \\ \eta(s;\varepsilon) \\ c(\varepsilon)-c_0 \end{bmatrix} = \sum_{l=1} \begin{bmatrix} \phi^{(l)} \\ \eta^{(l)} \\ c_l \end{bmatrix} \varepsilon^l \tag{7.1}$$

into (2.1)–(2.4); this leads directly to our problem (5.1) with $n!\,\eta^{(n+1)}=\mu^{\langle n\rangle}(y_0;0)$, $n!\,\phi^{(n+1)}=\psi^{\langle n\rangle}(s_0,y_0;0)$ and $c_n=c^{\langle n\rangle}$.

The traditional derivation of the higher order wave theory is rational given the understandings achieved in this note, but I have not been able to understand the previously published accounts of this derivation. The basic difficulty is that it does not seem possible to expand the function $\phi(s,y;\varepsilon)$, which is initially defined only where there is water ($y\leqq\eta$), into a series of functions $\phi^{(l)}(s_0,y_0)$, which are defined only under the flat surface $y_0=0$. This difficulty seems not to have been discussed in the popular literature, but experts in the subject believe that the difficulty can be overcome by assuming that the functions $\phi^{(l)}$ in the expansion (7.1) are defined on both the flat and wavy domains. These functions are determined as solutions of Laplace's equation in the flat domain, and it is assumed that they can be continued analytically into that part of the wavy domain outside the flat domain. I have here shown that the analytic continuation which is required is just that which is specified by equations (6.1b) and (6.2); the existence of an ε analytic solution implies the existence of the continuation required for a rational Eulerian derivation of the higher order theory.

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