*Offprint from "Archive for Rational Mechanics and Analysis",*
*Volume 49, Number 4, 1973, P. 241–269*


# *Quasilinear Dirichlet Problems Driven by Positive Sources*

D. D. JOSEPH & T. S. LUNDGREN

## Contents



## I. Introduction

We study the problem

$$(r^{n-1}u')'+\lambda r^{n-1}\mathcal{F}(u)=0, \qquad u(1)=u'(0)=0 \tag{I.1}$$

where $\mathcal{F}(u)>0$ when $u\geqq 0$. Our main concern is with functions $\mathcal{F}(u)=(1+\alpha u)^{\beta}$, $\alpha\beta>0$ and with $\mathcal{F}(u)=e^{u}$. The last section of the paper, however, deals with solutions of (I.1) when $\mathcal{F}(u)$ is Lipschitz continuous and $n$ is large.

Quasilinear problems of type (I.1) arise in the theory of nonlinear diffusion generated by nonlinear sources [1, 2, 3], in the theory of thermal ignition of a chemically active mixture of gases [4], in the theory of membrane buckling [5] and in the theory of gravitational equilibrium of polytropic stars [6, 7, 8], to mention just a few applications. (For others, see "references" in [3].)

A most striking feature of such problems is that positive solutions of (I.1) need not be unique. For example, the problem

$$(r^{n-1}u')'+\lambda r^{n-1}e^{u}=0, \qquad u(1)=u'(0)=0 \tag{I.2}$$

has the following uniqueness properties: There exists a finite positive value $\lambda_*$ depending on $n$, such that there are

(a) no solutions when $\lambda > \lambda_*$ $(n \geqq 1)$,
(b) one solution when $\lambda = \lambda_*$ $(n \geqq 1)$,
(c) two solutions when $0 < \lambda < \lambda_*$ $(n = 1, 2)$,
(d) an infinite number of solutions when $\lambda = 2$ $(n = 3)$, and
(e) a finite but large number of solutions when $|\lambda - 2| \neq 0$ is small $(n = 3)$.

This list is not complete. To it we shall add:

(f) an infinite number of solutions when $\lambda = 2(n-2)$ $(n < 10)$,
(g) a finite but large number of solutions when $|\lambda - 2(n-2)| \neq 0$ is small $(n < 10)$,
(h) one solution for each $\lambda < 2(n-2)$ $(n \geqq 10)$.

A similar list of uniqueness properties will be prepared for the problem

$$(r^{n-1} u')' + \lambda r^{n-1} (1 + \alpha u)^{\beta} = 0, \qquad u(1) = 0, \; u'(0) = 0, \tag{I.3}$$

for all real numbers $\alpha$ and $\beta$ such that $\alpha\beta > 0$. Problem (I.2) can be considered a limiting case of problem (I.3), for if $\alpha = 1/\beta$ then clearly

$$\lim_{\beta \to \pm\infty} \left(1 + \frac{1}{\beta} u\right)^{\beta} = e^{u}.$$

Problem (I.2) has been treated by I. M. GELFAND (for $n = 1, 2, 3$) in a study of the problem of self-ignition. Considerable mathematical interest accrues to his result (d above) concerning the change in the number (less than 3) of solutions of (I.2) when $n = 1, 2$ to an infinite number of solutions of problem (I.2) when $\lambda = 2$ and $n = 3$. An analogous result has recently been exhibited by CALLEGARI, REISS & KELLER [5] in a study of the case $(n = 4, \alpha = -1, \beta = -2)$ of problem (I.3) which governs the buckling of membranes. Our analysis of problem (I.3) generalizes and unifies these two results. Given GELFAND's result that there are an infinite number of solutions of (I.3) for $\lambda = 2$ when $n = 3$, it is perhaps to be expected that an infinite number of solutions can be found for some $\lambda$ when $n > 3$. But, as we shall see, this is not true when $n$ is large enough; for large $n$ the infinite multiplicity of solutions for some $\lambda$ gives way to uniqueness for all $\lambda < \lambda_*$.

Our study will be confined to real values $n > 2$. Problems (I.2) and (I.3) are already fairly well understood for $n = 1$ and $n = 2$. In problem (I.3) we shall allow all real values of $\alpha$ and $\beta$ such that $\alpha\beta > 0$.

Problems (I.2) and (I.3) can be regarded as members of a family of initial value problems in which the boundary condition $u(1) = 0$ is replaced with the initial condition $u(0) = A$. It is well to keep in mind the distinction between the boundary value problems (I.2) and (I.3) and the corresponding initial value problems.

The initial value problem for (I.2) and (I.3) can be studied by phase plane transformations of the type first used by EMDEN [6] in his celebrated study of the gravitational equilibrium of polytropic stars. EMDEN considers the case $\beta>1$ and $n=3$. EMDEN's studies have been supported and extended in the work of FOWLER [7], E. HOPF [9] and CHANDRASEKHAR [8]. The present work extends EMDEN's theory to all initial value problems of type (I.3) for which $\beta<0$, subject, of course, to our basic restrictions $\alpha\beta>0$ and $n>2$.

Our main concern is with the boundary value problems (I.2) and (I.3). Apart from two special cases we believe that our results are new. The special cases are: (a) problem (I.2) with $n=3$ which was discussed by GELFAND and (b) problem (I.3) with $n=4$, $\alpha=-1$, $\beta=-2$ which was treated by CALLEGARI, REISS & KELLER. A precise statement of the new results is given in Theorems 1 and 2.

CHANDRASEKHAR's monograph is a convenient reference for proofs and examples which are necessary to our work, and we shall make frequent reference to it. CHANDRASEKHAR's work is restricted to the case $n=3$. He does not consider the case $\beta<0$ and his analysis of the problem with $\beta>5$ is only local. Our global analysis of the initial-value problem (cf. Sections IX, X and XI) can also be used when $\beta>(2+n)/(n-2)$. Hence, even with $n=3$ the present analysis fills a gap; it extends EMDEN's analysis of the initial value problem to $\beta<0$ and gives a global result for $\beta>5$.*

We are also able to show that when $n$ is large the solutions of problem (I.1) (when $\mathscr{F}(u)$ is Lipschitz continuous and $\mathscr{F}(u)>0$ and $u\geq 0$) differ from the solutions of the first order problem

$$\frac{du}{dr}+\frac{\lambda}{n}r\mathscr{F}(u)=0, \qquad u(1)=0$$

by terms of $O(1/n)$, uniformly in $r$. A precise statement of this result is the content of Theorem 3.

The clarity of the exposition of the results given here profit from a careful and detailed review of an earlier draft of the paper by several persons. It is a pleasure to acknowledge particularly a most helpful review by Professor STUART HASTINGS.

## II. Equivalent Formulations of Problem I.1

By a regular solution of (I.1) we mean the twice continuously differentiable function of $r(0\leq r\leq 1)$, $u(r,\lambda)$ which for preassigned $\lambda>0$ satisfies (I.1). Consider positive solutions of (I.1) when $\mathscr{F}$ has the property that $\mathscr{F}(u)>0$ when

* The values $\beta<0$ and $\beta>5$ are not usually considered to be of astrophysical interest. Emden's problem is to find a stellar model characterized throughout by the relations

$$p=p(0)\theta^{1+\beta}, \qquad \rho=\rho(0)\theta^{\beta}$$

where $\theta$, $p$ and $\rho$ are a temperature, pressure and density (cf. [8]). The reason that Emden's problem retains interest is that it leads to qualitatively reasonable models. The limitation $\beta<5$ derives from the requirement that the relations hold all the way to the surface, where $\theta=0$. Professor N. LEBOVITZ (private communication) has observed, however, that if a modification of Emden's problem is considered wherein the relations above hold only up to some point (where $\theta$ is still positive) and some other kind of relation holds from that point to the surface, the condition $\beta<5$ can be relaxed.

$u \geqq 0$. Then $\lambda > 0$ and it is readily established [10] that

$$\lambda = \gamma_0 \int_\Omega \Psi u \, d\Omega \Big/ \int_\Omega \Psi \mathscr{F}(u) \, d\Omega = \gamma_0 \bar{u} / \mathscr{F}(\bar{u}) \tag{II.1}$$

where the weighted mean value $\bar{u}$ is defined by (II.1) and $\Psi$ and $\gamma_0$ are the principal eigenfunction and eigenvalue of $\Delta \Psi + \gamma_0 \Psi = 0$, $\Psi = 0|_{\partial\Omega}$. In fact, (II.1) holds for problems of the form $Lu + \lambda \mathscr{F}(u) = 0$, $u = 0|_{\partial\Omega}$ where $L$ is a uniformly elliptic operator and $\Omega$ an arbitrary closed domain. If $y/\mathscr{F}(y)$ is bounded for $y \geqq 0$, then there exists a least upper bound $\lambda_*$ such that no positive solutions exist when $\lambda > \lambda_*$ and positive solutions are not necessarily unique when $\lambda < \lambda_*$. Clearly if $\{\lambda\}$ is the set of positive numbers for which positive solutions of (I.1) exist, then

$$\lambda_* = \sup\{\lambda\}. \tag{II.2}$$

Positive solutions of (I.1) cannot have a minimum value in the interior $0 \leqq r < 1$. It follows from this that the maximum value of $u(r, \lambda)$ is at the origin

$$u(0, \lambda) = A.$$

In the analysis we shall find it convenient to regard $A$, instead of $\lambda$, as preassigned, because for a given $\lambda$ there can be more than one $u(r, \lambda)$ (see Figures 1a, b). Replacing (I.1) we have

$$(r^{n-1} u')' + \lambda r^{n-1} \mathscr{F}(u) = 0, \quad u(0) = A, \ u'(0) = 0 \text{ and } u(1) = 0. \tag{II.3a, b, c, d}$$

In (I.1) $\lambda$ is given and two side conditions are imposed. In (II.3) three side conditions are imposed and solutions exist only when $\lambda$ takes on special values $\lambda = \lambda(A)$.

Solutions of (I.1) are designated by the values $u(r, \lambda) \in C^2\,[0, 1]$. Solutions of (II.3) are designated by the pair

$$[u(r, A), \lambda(A)] \tag{II.4}$$

with values in $C^2[0, 1] \times R^1$.

Obviously every solution of (I.1) is also a solution of (II.3a, c, d) for some $A$. In a similar but less obvious way, we may also show that every bounded solution of (II.3a, b, d) is also a solution of (I.1). This means that even if $u'(0) = 0$ is not required of solutions of (II.3a, b, d) it will be a property of these solutions. This is a consequence of the following extension of a lemma by E. HOPF ([9], see also [8], p. 106).

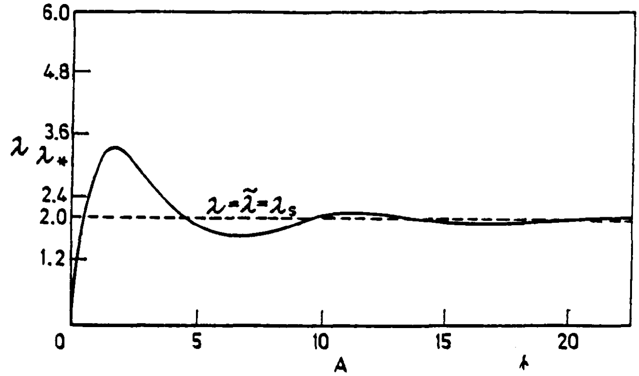
*Suppose $u(r)$ is any bounded solution of* (II.3a) *such that $r^{n-1} u'(r) \to 0$ as $r \to 0$ and $0 \leqq u(0) = A < \xi$ where $\xi$ is the first positive zero of $\mathscr{F}^{-1}(\xi)$ and $0 < \mathscr{F}(A) < \infty$. Then $u'(0) = 0$.*

**Proof.** We introduce $u = \chi / r^{\frac{n-1}{2}}$ and find that

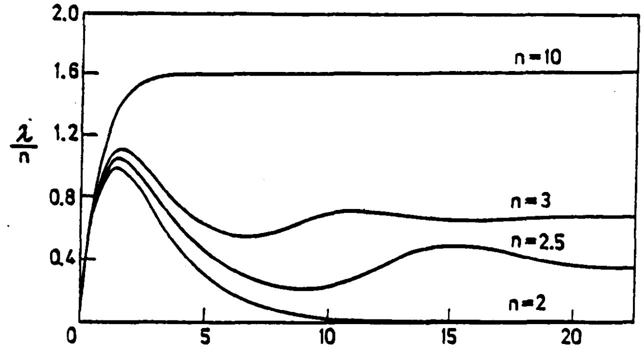
$$r^{(n-1)/2} \left\{ \chi'' - \left( \frac{n-1}{2} \right) \left( \frac{3-n}{2} \right) \frac{\chi}{r^2} \right\} + \lambda r^{n-1} \mathscr{F} \left( \frac{\chi}{r^{(n-1)/2}} \right) = 0$$

and

$$u' = \left[ r^{(n-1)/2} \chi' - \frac{(n-1)}{2} r^{(n-3)/2} \chi \right] \Big/ r^{n-1} \equiv A(r)/B(r),$$




a




b

Fig. 1 a. Bifurcation diagram for solutions of problem (II.3) with $n=3$. This curve was computed by numerical integration of (II.3). It gives the values of $\lambda=\lambda(A)$ for which solutions of (II.3) or the equivalent problem (I.2) are possible. For a fixed $\lambda$ it is possible to have different solutions $u(r, \lambda)$ having different values of $u(0, \lambda)=A$. When $\lambda>\lambda_* \approx 3.35$ there are no solutions. When $\lambda=\lambda_s=\tilde{\lambda}=2$ there are infinitely many solutions having different values of $A$.

Fig. 1 b. Bifurcation diagrams for the solutions of problem (II.3). This figure is constructed by numerical integration of (II.3) for different values of $n$. When $n=2$ and $\lambda \neq \lambda_*$ there are either two solutions ($\lambda<\lambda_*$) or no solutions $\lambda>\lambda^*$. When $2<n<10$ there are infinitely many solutions for $\lambda=\tilde{\lambda}=\lambda_s$. When $n \geqq 10$ and $\lambda<\lambda_*=\lambda_s$ the solutions are unique.

which is an indeterminate form (0/0) as $r \to 0$ if $r^{n-1}u' \to 0$. It follows that

$$\begin{aligned} u'(0) &= \lim_{r\to 0}\left(A'(r)/B'(r)\right) \\ &= \frac{1}{n-1}\lim_{r\to 0}\left(r^{\frac{n-1}{2}}\left\{\chi'' - \frac{(n-1)}{2}\left(\frac{3-n}{2}\right)\frac{\chi}{r^2}\right\}\Big/ r^{n-1}\right) \\ &= \frac{-\lambda}{n-1}\lim_{r\to 0} r\mathcal{F}(u) = 0. \end{aligned}$$

The key to the study of uniqueness of solutions $u(r, \lambda)$ of (I.1) is the function $\lambda(A)$. If $\lambda(A)$ is not single-valued, regular solutions of (I.1) cannot be unique.

The curves $\lambda(A)$ are called bifurcation diagrams. Bifurcation diagrams for the problem (I.2) have been computed by numerical integration and are displayed in Figures 1a, b. The existence-uniqueness theory for (I.2) is most easily understood in a preliminary way by careful study of these two figures.

### III. Limiting Singular Solutions and Singular Similarity Solutions of (I.3)

In Section III through X we shall consider problem (I.3). The results obtained for (I.3) are extended to (I.2) in Section XI. To treat (I.3) we shall need to define a limiting singular solution. In this definition we use the value $\xi$ which has been defined as the first infinity of the function $\mathscr{F}$. For $\mathscr{F}=(1+\alpha u)^{\beta}$ with $\alpha\beta>0$ there are the cases:

(a) Here $\beta>0$ and $\mathscr{F}$ is finite and not zero for all values of $0\leqq u<\infty$;

(b) Here $\beta<0$ and $\mathscr{F}=1/(1-|\alpha|u)^{|\beta|}$ is finite and not zero for all $0\leqq u\leqq 1/|\alpha|$.

The singular limit is designated as

$$A\to\xi\begin{cases}\text{(a)}: & \xi=\infty\\ \text{(b)}: & \xi=1/|\alpha|.\end{cases}$$

**Definition.** *A limiting singular solution* $[\tilde{u}(r),\tilde{\lambda}]$ *is a limit of regular solutions* $[u(r,A),\lambda(A)]$ *of* (I.3) *as* $A\to\xi$, *which itself satisfies* (I.3) *on* $0<r\leqq 1$.

We shall next identify a set of candidates for limiting singular solutions. The candidates are called singular similarity solutions and are defined by

$$u_s=\frac{1}{\alpha}[r^{-\tau}-1] \tag{III.1a}$$

and

$$\lambda_s=\frac{\tau}{\alpha}(n-2-\tau) \tag{III.1b}$$

where

$$\tau=2/(\beta-1). \tag{III.1c}$$

The pair $[u_s,\lambda_s]$ is a solution of the differential equation (I.3) and the boundary condition $u(1)=0$, but it is not a regular solution at the origin.* The singular similarity solution is a limiting solution if, as $A\to\xi$,

$$[u(r,A),\lambda(A)]\to[u_s,\lambda_s]. \tag{III.2}$$

It is important to know the values of $\alpha$, $\beta$, $n$ for which (III.2) holds. All of the solutions of (I.3) map into one curve in the phase plane, and the singular similarity solution maps into a critical point. When (III.2) holds the critical point terminates the solution curve, and the properties of the critical point determine the properties of the solution.

* The singular similarity solution need not be unbounded. It is only necessary that the Laplacian of $u(x)$ be unbounded at $x=0$. For example, in the expressions (III.1a) is exhibited a singular solution which (a) is unbounded at $r=0$ when $\beta>(2+n)/(n-2)$, (b) is bounded when $\beta>0$, but has unbounded first derivatives at $r=0$ when $\beta<-1$ and (c) is bounded along with its first derivative when $-1<\beta<0$. This solution cannot, however, have a second derivative at $r=0$ and cannot be a regular solution of (I.3).

A criterion for determining the values of the parameters for which (III.2) holds is given in Section IV. It is convenient to eliminate the values $0 \leqq \beta \leqq 1$ from consideration at the outset.

When $0 \leqq \beta \leqq 1$ only unique solutions of problem (I.3) exist (the case $\beta = 0$ which can be integrated explicitly is typical of $0 \leqq \beta < 1$). For the case $0 \leqq \beta < 1$, $\lambda$ tends to infinity monotonically with $A$ and existence of unique solutions of $\Delta u + \lambda(1+\alpha u)^{\beta} = 0$, $u = 0$ on the boundary, is guaranteed by the monotone convergence of the Keller-Cohen iterates, Theorems 3.2 and 4.2. When $\beta = 1$, the problem (I.3) is linear, $\lambda \to \gamma_0$ where $\gamma_0$ is as in (II.1), and there is just one positive solution $u$ for each $0 < \lambda \leqq \gamma_0$.

## IV. Emden's Problem and a Criterion to Determine when a Singular Similarity Solution is a Limit of Regular Solutions

The boundary value problem (I.3) (or the equivalent problem (II.3)) is most conveniently treated by a type of shooting argument which is greatly simplified by the fact that the corresponding initial value problem (II.3 a, b, c) may be studied by Emden's method in the phase plane.

In preparation for the phase plane we shall change variables to bring (II.3 a, b, c) into Emden's form. We then show that the set of all solutions (called $E$) of Emden's problem contains the set of all solutions of the boundary value problems (I.3) or the equivalent problem (II.3 a, b, c, d). Finally, we give a necessary condition for determining if the singular similarity solution is a limiting form of a sequence of regular solutions.

The change of variables which was just mentioned is

$$r = x\sqrt{\tau(n-2-\tau)/\alpha\lambda(1+\alpha A)^{\beta-1}} \tag{IV.1}$$

and

$$1+\alpha u = (1+\alpha A)v. \tag{IV.2}$$

Cast in $(x, v)$ variables, (II.3) may be written as

$$(x^{n-1}v')' + \tau(n-2-\tau)x^{n-1}v^{\beta} = 0, \tag{IV.3a}$$

$$v(0) = 1, \qquad v'(0) = 0, \tag{IV.3b}$$

and

$$v\left(\sqrt{\frac{\alpha\lambda(1+\alpha A)^{\beta-1}}{\tau(n-2-\tau)}}\right) = (1+\alpha A)^{-1}. \tag{IV.3c}$$

The problem (IV.3 a, b) is the initial value problem $E$. The solution of $E$ is uniquely determined by the initial values (cf. remark following Lemma 3). This unique solution is called the $E$ solution. It can be found from a single forward integration on a computer.

Given the $E$ solution, we find the solution of (IV.3 a, b, c) when $A$ is given by choosing $\lambda = \lambda(A)$ to satisfy (IV.3 c). Of course, every solution of (IV.3 a, b, c) is also a solution of (I.3). The transformation given below enables one to convert the $E$ solution which has $v(0) = 1$, $v'(0) = 0$ into a solution $u(r)$ of (IV.3 a) with

$u(0)=A$ and $u'(0)=0$. It is exactly this elementary invariance property which allows one to solve (I.3) with the procedure mentioned in the previous paragraph.

**Lemma 1.** *If* $v(x)$ *is a solution of* (IV.3a, b), *then*

$$\hat{v}(\rho x)=\rho^{-\tau}v(x) \tag{IV.4}$$

*is also a solution of* (IV.3a, b) *for any real numbers* $\rho$ *and* $\tau=2/(\beta-1)$.

The proof of (IV.4) is by direct calculation. The numbers $\rho$ are called homology constants. Clearly, by choosing a suitable homology constant, one can find a solution (IV.3a) with a zero slope and any preassigned finite value at the origin.

Now we are prepared to give a criterion for the case where the singular similarity solution (III.1) is a limit of a sequence of regular solutions. Recalling that this limit is defined by equation (III.2), we have the following lemma:

**Lemma 2.** *Equation* (III.2) *holds for those values of* $\alpha$, $\beta$ *and* $n$ *for which the* $E$ *solutions of* (IV.3a, b) *are positive for all* $0<x<\infty$ *and for which* $v\to x^{-\tau}$ *as* $x\to\infty$.

It will be shown in later sections that this asymptotic behavior is a property of the $E$ solution for a certain range of $\alpha$, $\beta$, $n$.

The proof of this lemma follows from (IV.1, 2) written as

$$1+\alpha u(r,A)=(1+\alpha A)\,v\left(\left(\frac{\alpha\lambda(1+\alpha A)^{\beta-1}}{\tau(n-2-\tau)}\right)^{1/2} r\right) \tag{IV.5}$$

where $v$ is the solution of (IV.3a, b) and $\lambda$ is any fixed positive number. Consider the limit as $A\to\xi$ with $r$ fixed but not zero. In this limit $(1+\alpha A)^{\beta-1}\to\infty$ (since $\xi=\infty$ for $\beta>1$, $\xi=-\dfrac{1}{\alpha}$ for $\beta<0$); hence the argument of $v(x)$ becomes large. Since $v(x)\to x^{-\tau}$ by assumption, (IV.5) gives

$$1+\alpha\lim_{A\to\xi}u(r,A)=\left(\frac{\alpha\lambda}{\tau(n-2-\tau)}\right)^{-\tau/2} r^{-\tau}. \tag{IV.6}$$

The choice $\lambda=\tau(n-2-\tau)/\alpha=\lambda_s$ makes the limit function $u(r,\xi)$ satisfy the boundary condition $u=0$ at $r=1$. Therefore, the limit solution is the singular similarity solution $u_s$ as given by (III.1a, b).

The limit as $A\to\xi$ with $r$ fixed is called an outer limit in the nomenclature of matched asymptotic expansions (e.g., COLE [11]) and $r$ is an outer variable. The $x=r/\varepsilon$, with the small parameter

$$\varepsilon=\left(\frac{\tau(n-2-\tau)}{\alpha\lambda(1+\alpha A)^{\beta-1}}\right)^{1/2},$$

is an inner or boundary layer variable with $\varepsilon$ the boundary layer "thickness." The outer limit (here the limiting singular solution) is corrected for large but finite $(1+\alpha A)^{\beta}$ over distances of order $\varepsilon$ near $r=0$). While the problem is natural for the method of matched asymptotic expansions, this method is not needed here because the "inner expansion" contains the "outer expansion." The simple homologous transformation permits a change of variables which eliminates $\varepsilon$

from the problem. This would not be the case with a more complicated source function. For instance, a polynomial source instead of a single power would not possess such a simple change of variables. In this case, the inner expansion would be the same as the $E$ solution for the largest power (with some restrictions), and this would have to be "matched" with the outer expansion but would not contain it. Further, there would be higher order terms in the expansions.

Lemma 2 does *not* hold when there exists a finite termination point $x_T$ such that forward integration of the $E$ problem (IV.3a, b) gives

$$v(x_T)=0 \quad \text{when } \beta>1, \tag{IV.7}$$

or

$$v(x_T)=\infty \quad \text{when } \beta<0. \tag{IV.8}$$

Suppose there is a finite $x_T$. Further, let $A \to \xi$. Then, from (IV.5) it is clear that $\lambda \to 0$ as $(1+\alpha A)^{-(\beta-1)}$ in order to keep the argument of $v$ less than $x_T$. Further, when $\beta>1$ so that $A \to \xi=\infty$, (IV.5) shows that $u \to \infty$ as $A \to \infty$ for any $r$ such that the argument of $v$ is smaller than $x_T$. We shall see (Section IX, Lemma 6) that finite $x_T$ cannot occur when $\beta<0$.

To determine the values of $\alpha$, $\beta$ and $n$ for which limiting singular solutions do exist, we shall next study $E$ solutions in the phase plane.

## V. Termination Points for $E$ Solutions in the Phase Plane

Introduce the variables $z$, $y$ and $t$ through the transformations

$$z=x^{\tau}v(x), \qquad x=e^{-t} \tag{V.1}$$

and

$$y=dz/dt.$$

In these new variables, we may rewrite (IV.3a) as

$$y\frac{dy}{dz}-(n-2-2\tau)y+\tau(n-\tau-2)(z^{\beta}-z)=0. \tag{V.2a}$$

The initial conditions which replace (IV.3b) are

$$z \to x^{\tau} \tag{V.2b}$$

and

$$y=\frac{dz}{dt}\,\frac{d}{dx}(x^{\tau}v(x)) \to -\tau x^{\tau} \tag{V.2c}$$

as $x \to 0$ or $t \to \infty$. When $\beta>1$, $\tau>0$, the image of the $E$ solution at the point $x=0$ in the phase plane is $(z, y)=(0, 0)$. When $\beta<0$, $\tau<0$, the image of the $E$ solution at the point $x=0$ in the phase plane is $(z, y)=(\infty, \infty)$. It will be observed that at the two points which were just mentioned, the value of $dy/dz$ cannot be determined from (V.2a). That is, (0, 0) when $\beta>1$ and $(\infty, \infty)$ when $\beta<0$ are singular points of (V.2a).

The point $(z, y)=(1, 0)$ is another singular point of (V.2a) and there are no other singular points. We shall refer to the singular points as $0_1$ and $0_2$:

(*i*) $0_1$: $(z, y)=(0, 0)$ when $\beta>1$,
$(z, y)=(\infty, \infty)$ when $\beta<0$;

(*ii*) $0_2$: $(z, y)=(1, 0)$.

About $0_1$ we may prove the following lemma:

**Lemma 3.** *The E solution is the only solution curve of* (V.2a) *which is tangent to the line*

$$y+\tau z=0 \tag{V.3}$$

*at the point* $0_1$. *Only one solution of* (V.2a) *satisfies* (V.3).

**Proof.** Inspection of (V.2b, c) shows that the $E$ solution must satisfy (V.3). The uniqueness proof is a copy of the one which was first given by HOPF [9] when $n=3$ and $1<\beta\leqq 3$. The program given there works equally well when $n>2$ and $\beta>3$ and when $n>2$ and $\beta<0$ (to construct the proof, copy the work on pp. 109–110 of reference [8]).

Lemma 3 tells how to start the phase plane integration. We next show how to terminate the integration. The integration is said to be terminated at the finite or infinite value $x=x_T>0$ at which we first have $v(x_T)=0$ for $\beta>1$ or $v(x_T)=\infty$ for $\beta<0$. The *point of termination* is the pair $v(x_T)=0$ or $v(x_T)=\infty$.

Our next task is to classify the points of termination. This classification will give the values of $n>2$ and $\beta$ for which limiting singular solutions of (I.3) or the equivalent problem (IV.3a, b) exist. The singular solution $x^{-\tau}$ maps into the point $0_2$. The singular solution is not an $E$ solution; it is a limiting form of an $E$ solution when the point of termination is $0_2$.

**Lemma 4.** *An E solution starts at* $0_1$ *tangent to the line* $y+\tau z=0$. *A singular harmonic solution G terminates at* $0_1$ *tangent to the line* $y+(\tau-n+2)z=0$. *A solution cannot terminate at* $0_1$ *unless it is a G solution.* If it terminates there, then* $v\sim C/x^{n-2}$ *as* $x\to\infty$.

To determine the $G$ direction, set $y=dz/dt$ *in* (V.2a),

$$\frac{d^2 z}{dt^2}-(n-2-2\tau)\frac{dz}{dt}+\tau(n-\tau-2)(z^\beta-z)=0, \tag{V.4}$$

and note that near $0_1$, $z^\beta$ is small relative to $z$ (when $\beta>1$, $z\to 0$ and when $\beta<0$, $z\to\infty$ at $0_1$). Then, in the neighborhood of $0_1$, we seek the solution of the linearized equation

$$\frac{d^2 z}{dt^2}-(n-2-2\tau)\frac{dz}{dt}-\tau(n-\tau-2)z=0. \tag{V.5}$$

The general solution (V.5) is a linear combination of $(z_1, z_2)=(c_1 e^{m_1 t}, c_2 e^{m_1 t})$ where $m_1$ and $m_2$ are the roots of

$$m^2-(n-2-2\tau)m-\tau(n-2-\tau)=0.$$

* This direction $G$ is called $X$ in the monograph [8] (cf. pp. 110, 111). The $G$ solution $v\sim C/x^{n-2}$ corresponds to $u=c/r^{n-2}$ which is the fundamental singularity for Laplace's equation in the $n$ dimensional case (the fundamental harmonic singularity).

One finds that

$$m_1 = -\tau, \qquad m_2 = -\tau + (n-2). \tag{V.6}$$

The root $m_1$ corresponds to the starting slope of the $E$ solution. For $m_2$ we have

$$z = x^\tau v_2 \sim C_2 e^{-m_2 \log x} \text{ as } x \to 0.$$

Hence,

$$v_2(x) \sim C_2 x^{-m_2-\tau} = C_2/x^{n-2}, \tag{V.7}$$

when $x$ is small. Equation (V.7) is the fundamental harmonic singularity.

There are three possible ways to have the solution terminate:

(a) at a regular point $z=0$, $y \neq 0$ and $0 < x_T < \infty$,

(b) at the singular point $0_1$ along $G$,

(c) at the singular point $0_2$.

**Lemma 5.** *The points of termination of the E solution are*:

(a) *a regular point when* $1 < \beta < (2+n)/(n-2)$,

(b) $0_1$ *when* $\beta = (2+n)/(n-2)$,

(c) $0_2$ *when* $\beta > (2+n)/(n-2)$ *and when* $\beta < 0$.

**Proof.** We note that the Bendixson theorem [12] rules out the possibility of nontermination in a limit cycle if

$$\frac{\partial}{\partial z}\left(\frac{dz}{dt}\right) + \frac{\partial}{\partial y}\left(\frac{dy}{dt}\right) = 2\tau + 2 - n \neq 0.$$

In the exceptional case, $\beta = (2+n)/(n-2)$, the solution does terminate (at $0_1$), but the solution curve is indeed closed (see Figure 2). This is proved, along with part (b) of the lemma, in Section VI.

The proof of part (a) of Lemma 5 can be constructed copying proofs given for $\beta > 1$ when $n=3$ in [8] for the case $\beta > 1, n > 2$. Two cases are to be distinguished here: (*i*) $1 < \beta \leq n/(n-2)$. In this range $\lambda_s = \frac{\tau}{\alpha}[n-2-\tau]$ is negative, and the singular solution could not be a proper limiting solution of true solutions which necessarily have $\lambda > 0$. The proof that the termination point is a regular point is due to Hopf (reported in [8] as Lemmas 1 and 2 on pp. 115 and 116). Hopf's proofs are for $n=3$, $1 < \beta \leqq 3$ but are generalized easily by declaration. Solution curves for this case are shown in Figures 6 and 7 of [8]; (*ii*) $n/(n-2) \leqq \beta < (2+n)/(n-2)$. Though a singular solution ($0_2$) exists, it is not a limiting form for Emden's solution. In this case, solutions can leave $0_2$ as $x \to \infty$ but they cannot terminate there (the proof is given in Section VIII). The proof that the $E$ solution terminates at a regular point is the generalization by declaration of the lemma of p. 127 in [8]. Solution curves for this case are shown in Figures 9 and 10 in [8].

The proof of part (c) has been given in part in No. 20 and No. 21 of [8] (see Figure 13 of [8]). Rigorous proof (c) will be given in Section IX for the case $\beta < 0$ where the problem has not previously been considered. The method and results of IX apply equally when $\beta > (2+n)/(n-2)$ with slight and obvious changes of argument.

The arrangement of $E$ solutions mentioned in Lemma 5 is shown for representative cases in Figure 2 ($\beta > 1$) and Figure 4 ($\beta < 0$). It will be noted that the only possible crossing of the line $y=0$ for solutions of (V.2a) have $|dy/dz| = \infty$. The largest value of $z$ for which $y(z)=0$ is called $z_*$ when $\beta > 1$. When $\beta < 0$, $z = z_*$ at the smallest zero of $y(z)$.

At this point we wish to draw the reader's attention to the summarizing statement given below which explains how one may determine the uniqueness properties of solutions of (I.3) from the phase plane analysis. We started by transforming boundary value problem (II.3a, b, c, d) for $[u(r, A), (A)]$ into the $v(x)$ boundary value problem (VI.3a, b, c). Dropping the outer boundary condition (IV.3c), we defined Emden's initial value problem (which is independent of $A$). The second order Emden problem was then transformed into the first order equation (V.2a) for $y(z)$ determining an $E$ curve. Each $(z, y(z))$ pair on the $E$ curve gives a solution of (IV.3). Since $A$ does not appear in the $y, z$ equation (V.2a), the uniqueness results hold because, for a given $z$, different values of $y$ on the $E$ solution correspond to different values of $x$ and, therefore, to different values of $A$, while $\lambda$ is determined only by $z$. Let $(z_1, y(z_1))$ be a point on the $E$ curve which corresponds to the point $x_1 = [\alpha\lambda(1+\alpha A)^{\beta-1}/\tau(n-2-\tau)]^{1/2}$ where $v(x_1) = (1+\alpha A)^{-1}$ satisfies the outer boundary condition (IV.3c). By (V.1) $z_1 = x_1^\tau v(x_1)$. Hence,

$$\lambda = \frac{\tau}{\alpha}(n-2-\tau)\, z_1^{2/\tau} \tag{V.8}$$

is determined when $z_1$ is known.

The number of solutions for a given $z$ (or $\lambda$) is just the number of values $y$ on the $E$ curve at a given $z$ (or $\lambda$). The value $\lambda = \lambda_*$ is the largest value for which solution of (IV.3) can exist. $\lambda_*$ corresponds to the largest crossing $z = z_*$ when $\beta > 0$. Similarly, the smallest crossing gives the largest $\lambda = \lambda_*$ when $\beta < 0$.

When $n$ is large enough, there are no crossings at $y=0$. Then $\lambda_* = \bar{\lambda} = \lambda_s$.

## VI. An Exact Solution when $\beta = (2+n)/(n-2)$

In the $E$ solution literature, this is sometimes called the Schuster-Emden-Fowler solution (e.g., [7], p. 263). When $\beta = (2+n)/(n-2)$, equation (V.2a) may be written

$$y\frac{dy}{dz} + \frac{4}{(n-2)^2}\left(z^{(2+n)/(n-2)} - z\right) = 0. \tag{VI.1}$$

This equation is readily integrated subject to the $E$ condition $y + \frac{n-2}{2} z = 0$. The result is a closed curve in the phase plane (see Figure 2). Setting $\alpha = 1$, we may transform $v(x)$ variables back into $u(r)$ variables using (IV.1), that is,

$$x = (n-2)(1+A)^{2/(n-2)} \lambda^{1/2} r/2 \tag{VI.2}$$

and

$$v = \frac{1+u}{1+A} = 1/\{1 + [(1+A)^{2/(n-2)} - 1]\, r^2\}^{(n-2)/2}, \tag{VI.3}$$

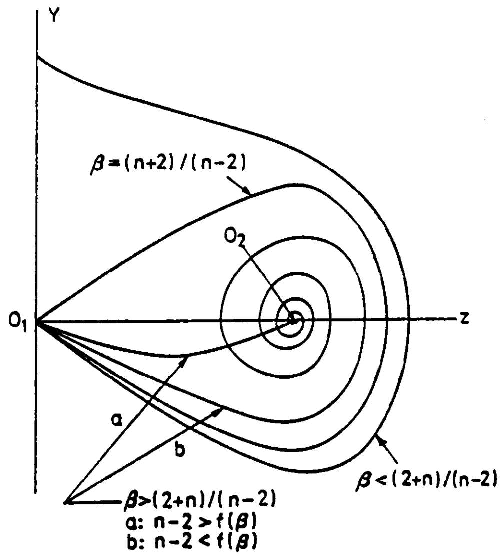



**Fig. 2.** Arrangement of $E$ solutions in the phase plane ($\beta > 1$, $n < 2$)

where, using the condition (IV.3c), we have determined that

$$\lambda = n(n-2)\left[(1+A)^{2/(n-2)} - 1\right]/(1+A)^{4/(n-2)}. \qquad \text{(VI.4)}$$

Equation (VI.4) gives two values of $A$ for each value $\lambda$. Consider the singular limit $A \to \xi = \infty$. We have

$$x \to rA^{1/(n-2)}\left[n(n-2)^2\right]^{1/2},$$

$$\lambda \to n(n-2)/A^{2/(n-2)},$$

$$v \to 1/Ar^{n-2} = 1/[n(n-2)]^{n-2}x^{n-2},$$

and

$$u \to 1/r^{n-2}.$$

The limiting form of the Schuster-Emden-Fowler solution is the fundamental harmonic singularity.

## VII. Local Analysis of Singular Solutions

It will be recalled that the singular solution $x^{-\tau}$ maps into the point $0_2$ of the phase plane. We have shown (Lemma 2) that if the $E$ solution terminates at $0_2$, then the singular solution is a proper limiting solution of the regular solution of (I.3).

In this section, we study the "stability" of $0_2$. The analysis gives the values of $\beta$ and $n$ for which $0_2$ is a terminal point of the integration as $x \to \infty$ or $t \to -\infty$. It also gives the discriminating criterion for solutions which approach $0_2$ in a definite direction and those which reach $0_2$ as a limit of a solution spiraling around

$0_2$. The conclusion of the uniqueness theorem of Section X is implied already by the results of this section, but to prove the theorem the global analysis of Section IX is required.

Suppose that some solution curve does terminate at $0_2$. Then in the neighborhood of $0_2$, this solution must be a small deviation $w(x)$ from the singular solution, that is,

$$v(x) = w(x) + x^{-\tau}, \tag{VII.1}$$

where, as $x \to \infty$,

$$x^{\tau} w(x) \to 0. \tag{VII.2}$$

From (VII.1, 2) and (IV.3a) we obtain the variational (linearized) equation

$$(x^{n-1}\hat{w}')' + \beta\tau(n-\tau-2)x^{n-3}\hat{w} = 0 \tag{VII.3}$$

where $w(x) \to \hat{w}(x)$ as $x \to \infty$. Solutions of (VII.2, 3) are necessarily of the form

$$\hat{w}(x) = Ax^{-l_+} + Bx^{-l_-}, \tag{VII.4}$$

where $l_+$ and $l_-$ are the roots of

$$l^2 - (n-2)l + \frac{2\beta}{\beta-1}(n-2-2/(\beta-1)) = 0.$$

We have

$$l_{\pm} = (n-2)/2 \pm \sqrt{(n-2)^2/4 - 2\beta[n-2-2/(\beta-1)]/(\beta-1)}. \tag{VII.5}$$

These roots are complex when

$$2 + \frac{4\beta}{\beta-1} - 4\sqrt{\frac{\beta}{\beta-1}} < n < 2 + \frac{4\beta}{\beta-1} + 4\sqrt{\frac{\beta}{\beta-1}}. \tag{VII.6}$$

Solutions which have complex roots and terminate at $0_2$ spiral around $0_2$ an infinite number of times.

Not all solutions for which $l_{\pm}$ are complex terminate at $0_2$. Suppose (VII.6) holds. Then condition (VII.2) can hold if and only if as $x \to \infty$,

$$x^{\tau - \left(\frac{n-2}{2}\right)}(Ax^{-b} + Bx^{b}) \to 0. \tag{VII.7}$$

Since $b$ is imaginary, we must have

$$\frac{n-2}{2} > \tau = 2/(\beta-1). \tag{VII.8}$$

The inequality (VII.8) always holds when $\beta < 0$, but it holds for $\beta > 0$ if and only if $\beta > (2+n)/(n-2)$. Hence, when $b$ is imaginary, $0_2$ is a terminal point only when $\beta < 0$ or $\beta > (2+n)/(n-2)$.

Now consider the situation when (VII.6) does not hold, that is, when $b$ is positive and $\beta > 1$. Then (VII.7) shows that $0_2$ *cannot* be a terminal point when

$$\tau - \frac{n-2}{2} - b = \frac{2}{\beta-1} - \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - \frac{2\beta}{\beta-1}\left(n-2-\frac{2}{\beta-1}\right)} > 0. \tag{VII.9}$$

This inequality requires $2/(\beta-1)>(n-2)/2$. Noting this and transposing and squaring the root in (VII.9), we find that if

$$\frac{n}{n-2}<\beta<\frac{2+n}{n-2}, \text{ that is, if } \frac{2\beta}{\beta-1}<n<\frac{2(1+\beta)}{\beta-1},$$

then (VII.8) holds and $0_2$ is not a terminal point. This eliminates the possibility that a terminal point $0_2$ with $b>0$ exists on the left side of the interval excluded by (VII.6), i.e., the interval

$$\frac{2\beta}{\beta-1}<n<2+\frac{4\beta}{\beta-1}-4\sqrt{\frac{\beta}{\beta-1}}\leqq\frac{2(1+\beta)}{\beta-1}.$$

When $n<2\beta/(\beta-1)$, the terminal point is a regular point and not $0_2$ (cf. point (*i*) of the proof of Lemma 5).

When $b$ is not negative and $\beta<0$, the more positive exponent in (VII.7) is negative. Thus

$$\begin{aligned}\tau+b-(n-2)/2&=2/(\beta-1)-(n-2)/2\\&+\sqrt{(n-2)^2/4-2\beta[n-2-2(\beta-1)]/(\beta-1)}<0.\end{aligned} \tag{VII.10}$$

The inequality (VII.10) also holds when $b\geqq 0$, i.e.,

$$n\geqq 2+\frac{4\beta}{\beta-1}+4\sqrt{\frac{\beta}{\beta-1}}>\frac{2(1+\beta)}{\beta-1}, \tag{VII.11}$$

and $\beta>1$. In both cases (VII.7) holds and $0_2$ can be a terminal point at which $x\to\infty$.

We have shown that $0_2$ can be a terminal point when $\beta<0$ and $\beta>(2+n)/(n-2)$ and not otherwise. It remains to show that $0_2$ is the terminal point of $E$ solutions (Section IX). In Section VIII we shall assume that $0_2$ is the terminal point of the $E$ solution.

## VIII. Asymptotic Formulas for Limiting Singular Solutions

The plan of this section is: (*i*) form the $E$ solution (IV.3a, b) from the asymptotic formula (VII.1); (*ii*) evaluate $\lambda$ from the requirement that the asymptotic solution satisfy the outer boundary condition (IV.3c).

The asymptotic (large $x$) form of the $E$ solution when $0_2$ is a terminal point is

$$v(x)-x^{-\tau}\sim\begin{cases}A_1 x^{-(n-2)/2}\cos(b\ln x+A_2) & \text{when } 0<n-2<f(\beta)\\ A_3 x^{b-(n-2)/2}+A_4 x^{-b-(n-2)/2} & \text{when } n-2\geqq f(\beta),\end{cases} \tag{VIII.1}$$

where

$$f(\beta)=\frac{4\beta}{\beta-1}+4\sqrt{\frac{\beta}{\beta-1}},$$

for both $\beta<0$ and $\beta>(2+n)/(n-2)$. Here, the constants $A_i$ are independent of $x$ and depend only on $\beta$ and $n$. These constants are determined uniquely by the initial value problem for the $E$ solution and can be regarded as known. If $A_3\neq 0$,

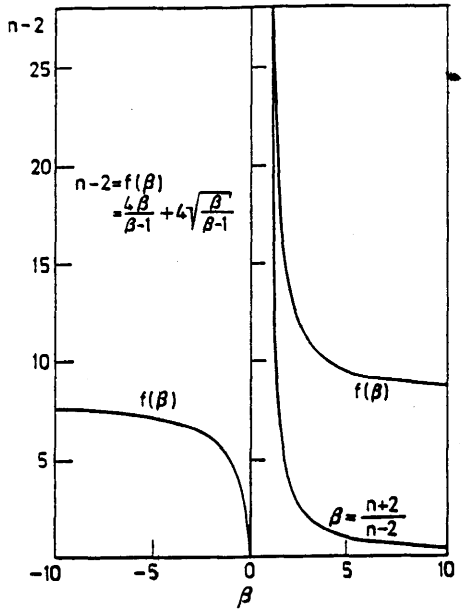



Fig. 3

then at large $x$ with $n-2 \geqq f(\beta)$, the $E$ solution is

$$v(x)-x^{-\tau} \sim A_3 x^{b-(n-2)/2}. \tag{VIII.2}$$

Step (*ii*) of the plan is the application of (IV.3c) to (VIII.1, 2). This gives

$$\frac{1}{1+\alpha A}=v(x_0)=x_0^{-\tau}+w(x_0), \tag{VIII.3}$$

where

$$x_0=(1+A)^{1/\tau}\alpha\lambda/\tau(n-2-\tau).$$

It is convenient to write (VIII.3) as

$$\left[\frac{x_0^\tau}{1+\alpha A}\right]^{2/\tau}=[1+x_0^\tau w(x_0)]^{2/\tau}. \tag{VIII.4}$$

We find from (V.8), (V.1), (VIII.3) and (VIII.1) that when $n-2<f(\beta)$

$$\lambda \sim \frac{\tau}{\alpha}(n-2-\tau)\left[1+A_1 x_0^{\tau-(n-2)/2}\cos(b\ln x_0+A_2)\right]^{2/\tau} \tag{VIII.5}$$

as $x_0 \to \infty$, and when $n-2 \geqq f(\beta)$,

$$\lambda \sim \frac{\tau}{\alpha}(n-2-\tau)\left[1+A_3 x_0^{\tau+b-(n-2)/2}\right]^{2/\tau} \tag{VIII.6}$$

as $x_0 \to \infty$. To first order

$$\lambda=\frac{\tau}{\alpha}(n-2-\tau), \qquad x_0=(1+\alpha A)^{1/\tau}.$$

The second term in each bracket of (VIII.5, 6) is small when

$$\beta>(2+n)/(n-2) \quad \text{and} \quad A\to\xi=\infty$$

and when ($\alpha<0$)

$$\beta<0 \quad \text{and} \quad A\to\xi=1/|\alpha|.$$

In both cases

$$\lambda-\frac{\tau}{\alpha}(n-2-\tau) \sim\frac{2}{\tau}\begin{cases} C_1(1+\alpha A)^{1-(n-2)/4\tau}\cos\left[\frac{b}{\tau}\ln(1+\alpha A)+A_2\right], & n-2<f(\beta) \quad \text{(VIII.7)}\\ C_3(1+\alpha A)^{1+b/2-(n-2)/4\tau}, & n-2>f(\beta). \quad \text{(VIII.8)}\end{cases}$$

Formula (VIII.7) already shows that there are an infinite number of roots $A$ for which $\lambda(A)-\frac{\tau}{\alpha}(n-2-\tau)=0$ and explains the shape of the bifurcation diagrams shown in Figure 1.

The formula (VIII.8) indicates that when $n-2>f(\beta)$, the approach of $\lambda(A)$ to $\frac{\tau}{\alpha}(n-2-\tau)$ is monotone. In fact, in Section XII we shall show (equation (XII.6b)) that when $n$ is very large,

$$\frac{2C_3}{\tau}\to-\frac{2}{\alpha(\beta-1)},$$

and then

$$\lambda-\frac{\tau}{\alpha}(n-2-\tau)\sim-\frac{2}{\alpha(\beta-1)}(1+\alpha A)^{1+b/2-(n-2)/4\tau}$$

approaches its limiting value from below.

In Section XI we shall show that the results just given persist in the limit

$$\alpha=1/\beta, \quad |\beta|\to\infty, \quad \mathscr{F}(u)=(1+\alpha u)^{\beta}\to e^{u}.$$

In this limit, the discriminating boundary tends to a limiting value

$$n-2=f(\beta)\to 8.$$

Hence, when $n<10$,

$$\lambda\sim 2(n-2)+B_1\,e^{-(n-2)A/4}\cos(b\,A/2+B_2)$$

where $B_1$ and $B_2$ depend only on $n$. When $n\geqq 10$

$$\lambda\sim 2(n-2)+B_3\,e^{\frac{A}{2}[b-(n-2)/2]},$$

which for very large $n$ may be written as (see (XII.7b))

$$\lambda\sim 2(n-2)-2e^{\frac{A}{2}[b-(n-2)/2]}.$$

Solution properties which are implied already by the local section have a rigorous foundation in the global analysis of the next section.

## IX. Phase Plane Analysis when $\beta<0$

A nearly identical analysis with a similar result can be carried out when $\beta>(n+2)/(n-2)$. In fact, some of the analysis for this is partly available in [8]. We shall restrict our attention to $\beta<0$. The two kinds of $E$ curves which can exist when $\beta<0$ are shown in Figure 4.

**Lemma 6.** *The $E$ solution starts (at $x=0$ or $t=\infty$) along the line $y+\tau z=0$. It drops and stays below this line. Then it either* (a) *cuts the $z$ axis vertically at $z_0$, $0<z_0<1$, or* (b) *terminates at $0_2$ after cycling around $0_2$ an infinite number of times.*

The last sentence of Lemma 6 eliminates the possibility that the solution can terminate at a finite point $x_T$ when $\beta<0$.

**Proof.** The steps of the proof are: (*i*) uniqueness of the $E$ solution, (*ii*) the $E$ curve is under the line $y+\tau z=0$ and (*iii*) the $E$ curve terminates at $0_2$.

When $\beta<0$ then $\tau<0$ and the singular solution $x^{-\tau}$ tends to infinity with $x$. The $E$ solution then satisfies

$$y\frac{dy}{dz}-(n-2-2\tau)y+\tau(n-2-\tau)(z^{\beta}-z)=0 \tag{IX.1}$$

and has the property

$$y+\tau z\to 0 \quad\text{as}\quad z\to\infty. \tag{IX.2}$$

(*i*) Proof of uniqueness of the $E$ solution is almost a word-for-word extension of one given by HOPF [9]. (The reader may follow his proof on p. 109 of [8]. In the present case $\beta<0$ it is only required that one change $z\to 0$ to $z\to\infty$ on p. 109.)

(*ii*) If the $E$ curve is to rise above the line $y+\tau z=0$, it must first cross the line $y+(1+\varepsilon)\tau z=0$ ($\varepsilon>0$ is small) from the underside. This it cannot do because the direction field of (IX.1) on the line $y+(1+\varepsilon)\tau z=0$,

$$\begin{aligned}\frac{dy}{dz}&=(n-2-2\tau)+\frac{(n-2-\tau)}{(1+\varepsilon)}\left(\frac{1}{z^{|\beta|+1}}-1\right)\\&=(n-2)\varepsilon-\tau(1+\varepsilon)+(n-2-\tau)(1+\varepsilon)^{-1}/z^{|\beta|+1}>-\tau(1+\varepsilon),\end{aligned} \tag{IX.3}$$

forces the solution to cross the line $y+(1+\varepsilon)\tau z=0$ only from above. Set $\varepsilon=0$. The inequality (IX.3) holds, and it shows that the solution drops below the line $y+\tau z=0$.

(*iii*) To show that the solution terminates at $0_2$, divide the phase plane into the four regions where the direction field has one sign (see Figure 4). The regions are bounded by the coordinate lines and the line

$$Y(z)=\frac{\tau(n-2-\tau)}{n-2-2\tau}(z^{\beta}-z), \tag{IX.4}$$

which is the locus of points on which $dy/dz=0$. If the solution is to cross the $z$ axis at a point $z\neq 1$, it must have a vertical tangent there. The solution which starts in the region ($A$) of the phase plane drops below the line $y+\tau z=0$. Since

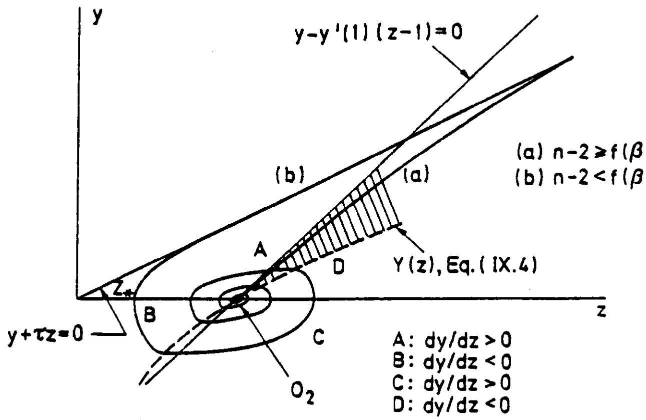



Fig. 4. Arrangement of $E$ solutions in the phase plane ($\beta<0$, $n>2$)

$dy/dz>0$ it can leave ($A$) by ($a$) termination at $0_2$ or ($b$) entering ($B$) vertically at $z<1$, $y=0$. Suppose ($b$) holds. In ($B$) the solution cannot avoid the line (IX.4) and must pass into ($C$). It cannot continue to infinity since $dy/dz>0$ there. Hence, the solution enters region ($D$) and, once again, cannot avoid the line (IX.4). Now the solution must circle around $0_2$ a finite or infinite number of times, approaching closer to $0_2$ with each cycle since it cannot cross itself or approach a limit cycle. The Bendixson theorem (cf. Lemma 5) rules out the limit cycle. Now, since the solution must terminate at $0_2$, the local analysis of the last section is valid for the $E$ solution. When $n-2<f(\beta)$ the number of cycles is infinite. When $n-2\geqq f(\beta)$ there are real directions of approach to $0_2$.

Consider the possible ways in which the $E$ solution can approach $0_2$ when $n-2\geqq f(\beta)$. At $0_2$, $dy/dz=y'(1)$ and

$$\begin{aligned} y'(1)&=(n-2-2\tau)-\tau(n-\tau-2)\left.\frac{z^{\beta}-z}{y}\right|_{z=1} \\ &=(n-2-2\tau)-\tau(n-\tau-2)(\beta-1)/y'(1), \end{aligned} \tag{IX.5}$$

where we have used L'Hospital's rule. This is a quadratic equation in $y'(1)$ and

$$y'(1)=\frac{1}{2}(n-2-2\tau)\pm\sqrt{\frac{(n-2)^2}{4}-\beta\tau(n-2-\tau)}. \tag{IX.6}$$

Since $n-2\geqq f(\beta)$, the smallest root $y'_-(1)$ is real and positive. We shall show that the $E$ solution must lie in the region bounded by (IX.4) and the line $y'_-(1)\,(z-1)$ (see Figure 4).

First we shall show that

$$y'_-(1)>y'(\infty)=-\tau. \tag{IX.7}$$

The proof is that

$$y'_-(1)+\tau=(n-2)/2-\sqrt{\frac{(n-2)^2}{4}-\beta\tau(n-2-\tau)}>0.$$

We next show that every solution curve $y(z)$ on the line

$$\bar{y}(z)=y'_-(1)(z-1) \tag{IX.8}$$

has

$$\frac{dy}{dz}\geqq y'_-(1), \tag{IX.9}$$

with equality only when $z=1$. To prove (IX.9) we note from (IX.1) that on the line (IX.8) the direction of solution curves is

$$\frac{dy}{dz}=(n-2-2\tau)-\frac{\tau(n-2-\tau)(z^\beta-z)}{y'_-(1)(z-1)}. \tag{IX.10}$$

At $z=1$, (IX.5) holds for $y'_-(1)$, i.e.,

$$(n-2-2\tau)=y'_-(1)-\tau(n-\tau-2)(|\beta|+1)/y'_-(1). \tag{IX.11}$$

On combining (IX.10, 11), we find that

$$\frac{dy}{dz}-y'_-(1)=\frac{-\tau(n-\tau-2)}{y'_-(1)}F(z,|\beta|) \tag{IX.12}$$

where

$$F(z,|\beta|)=|\beta|+1+(z^{-|\beta|}-z)/(z-1), \qquad F(1,|\beta|)=0,$$

and

$$\frac{dF}{dz}=|\beta|(1-1/z^{|\beta|+1})>0.$$

Hence, $F$ is positive when $z>1$ and (IX.12) implies (IX.9).

Equation (IX.7) shows that the $E$ solution starts with a smaller positive slope $(-\tau)$ than $y'_-(1)$. Equation (IX.9) shows that the $E$ solution could not cross the line $y'_-(1)(z-1)$ since it would have to cut this line from the underside and (IX.9) allows only cutting from above. Hence the $E$ curve must remain in the shaded region of Figure 4.

The situation which prevails when $n-2\geqq f(\beta)$ is rather more simple than one might first anticipate. In fact, for very large $n$ the $E$ curve can be given explicitly as $y(z)\to\tau(z^\beta-z)$. This result is left as an exercise since more far reaching results implying this one will be obtained in Section XII.

## X. Uniqueness Properties for Positive Solutions of $(r^{n-1}u')'+\lambda r^{n-1}(1+\alpha u)^\beta=0,\ u(1)=0,\ u'(0)=0$

This is the original problem (I.3). Recall that $\alpha\beta>0$.

**Theorem 1.** *Let $n-2>0$ and $\lambda>0$ be preassigned. Positive solutions $u(r,\lambda)$ of* (I.2) *have the following properties:*

(*i*) *When $0\leqq\beta<1$, solutions are unique.*

(*ii*) *When $\beta<0$ and $\beta>1$ there exists a value $\lambda_*>0$ such that positive solutions do not exist when $\lambda>\lambda_*$. A unique positive solution exists when $\lambda=\lambda_*$ and $\tilde{\lambda}<\lambda_*$.*

(*iii*) *When $1<\beta\leqq(2+n)/(n-2)$ and $\lambda<\lambda_*$ there are just two solutions, one with large $A=u(0)$ and one with small $A$.*

*(iv) When* $\beta<0$ *or* $\beta>(2+n)/(n-2)$, $n-2<f(\beta)$, $\lambda_*>\bar{\lambda}=\frac{\tau}{\alpha}(n-2-\tau)$ *and* $\lambda=\bar{\lambda}$ *there are a countably infinite number of solutions. For* $\lambda$ *sufficiently close to* $\bar{\lambda}$ *there are a large but finite number of positive solutions.*

*(v) When* $\beta<0$ *or* $\beta>(2+n)/(n-2)$, $n-2\geqq f(\beta)$ *and* $\lambda<\lambda_*=\bar{\lambda}$ *there is one and only one positive solution.*

**Proof.** The proof follows from the lemmas which establish the "arrangement of $E$ solutions," Figures 2 and 4. For this we use Lemma 6, its equivalent when $\beta>(2+n)/(n-2)$ and the generalization by declaration of the three lemmas, pp. 115, 116, HOPF [9] and p. 127 in [8]. Solutions of (I.3) are just rescaled solutions of (IV.3). As was noted in (V), the number of solutions of (IV.3) is just the number of $y$ values of an $E$ solution at given $z$. Here at the given $z$ we may evaluate $\lambda$ from the outer boundary condition (IV.3c) as $\lambda=\frac{\tau}{\alpha}(n-2-\tau)z$.

(*i*) This is a special case of Theorem 4.2 of [2].

(*ii*) The existence of $\lambda_*$ when $\beta<0$ and $\beta>1$ is guaranteed by equation (II.1) and the fact that $y/(1+\alpha y)^\beta$ is a bounded function when $\alpha\beta>0$. There is only one $z_*$ on an $E$ solution.

(*iii*) For each $0<z<z_*$ there are just two values of $y$ (see Figure 2).

(*iv*) When $z=1$ there are a denumerably infinite number of values $y(1)$ on the $E$ solution. For these $\lambda=\bar{\lambda}=\frac{\tau}{\alpha}(n-2-\tau)$.

(*v*) When $n-2\geqq f(\beta)$ the solution is single-valued. Then at $z=1$, $\lambda=\lambda_*=\bar{\lambda}=\frac{\tau}{\alpha}(n-2-\tau)$.

## XI. Uniqueness Properties for Positive Solutions of $(r^{n-1}u')'+\lambda r^{n-1}e^u=0$, $u(1)=0$, $u'(0)=0$

In the introduction and again at the end of Section VIII, we asserted that the solution story for $\mathcal{F}(u)=e^u$ can be obtained as a limiting case of Theorem 2 in the limits in which $(1+\alpha u)^\beta\to e^u$. In this section we wish to specify the form of interesting limiting quantities, to justify assertions about the validity of limiting forms and to establish Theorem 2.

**Theorem 2.** *Let* $n-2>0$ *and* $\lambda>0$ *be preassigned. Positive solutions* $u(r,\lambda)$ *of* (I.2) *have the following properties*:

*(i) There exists a value* $\lambda_*>0$ *such that positive solutions do not exist when* $\lambda>\lambda_*$. *When* $n\geqq 10$, $\lambda_*=\bar{\lambda}=2(n-2)$.

*(ii) When* $n\geqq 10$ *there is a unique positive solution for each* $\lambda$, $0<\lambda<\lambda_*$.

*(iii) When* $n<10$ *and* $\lambda=\lambda_*$ *there is a unique positive solution.*

*(iv) When* $n<10$ *and* $\lambda=\bar{\lambda}=2(n-2)$, *there are an infinite number of positive solutions.*

*(v) When* $n<10$ *and* $|\lambda-2(n-2)|\neq 0$ *is small, there are a large but finite number of solutions.*

Theorem 2 is, of course, indirectly implied by Theorem 1 in the limits $\beta \to \pm\infty$. We are going to prove this theorem, however, by repeating the analysis which was given when $\mathscr{F}=(1+\alpha u)^\beta$ for the case in which $\mathscr{F}(u)=e^u$. The exponential problem is considerably simpler because there are no parameters other than $n$ and $\lambda$ and because the singular similarity solution is always a limiting form of a sequence of regular solutions.

To prove Theorem 2 we use various geometric properties of $E$ solutions (Lemma 7) so as to form an analogue of Lemma 6 in the exponential case.* Lemma 7, which will be stated and proved in due course, is about the $E$ solutions of the phase plane problem

$$\frac{y}{n-2}\frac{dy}{dz}-y+2e^z-2=0 \tag{XI.1}$$

where

$$y=dz/dt$$

and

$$z\to-\infty,\quad y\to-2,\qquad dy/dz\to 0 \tag{XI.2}$$

as $\tau\to\infty$. The point $(z, y)=(0, 0)$ is the only critical point of the finite $(z, y)$ plane.

The problem (XI.1, 2) arises from the initial value problem

$$(x^{n-1}\phi')'+2(n-2)x^{n-1}e^\phi=0, \tag{XI.3a}$$

$$\phi(0)=\phi'(0)=0 \tag{XI.3b}$$

under the change of variables

$$\phi=z+\log(1/x^2),\qquad x=e^{-\tau}. \tag{XI.4}$$

The *boundary value* problem (IX.3a, b, c), where the outer boundary condition is

$$\phi([\lambda e^A/2(n-2)]^{1/2})=-A, \tag{XI.3c}$$

can be obtained from the "given $A$" form

$$(r^{n-1}u')'+\lambda r^{n-1}e^u=0,\qquad u(0)=A,\ u'(0)=0,\ u(1)=0 \quad \text{(XI.5a, b, c, d)}$$

of (I.2) by the change of variables

$$\phi+A=u,\qquad r=x[2(n-2)/\lambda e^A]^{1/2}. \quad \text{(XI.6a, b)}$$

Each of the above equations arises as a limiting equation of the problem (I.3). For example, problem (XI.5) arises from (II.3) in the exponential limit $\alpha=1/\beta$, $|\beta|\to\infty$. The variables $\phi$ and $v$ of the problem (IV.3) are related by

$$v^\beta=(1+\alpha\phi)^\beta=(1+\alpha u)^\beta/(1+\alpha A)^\beta. \tag{XI.7}$$

In the exponential limit we find (XI.6a) from (XI.7) and (XI.6b) from (IV.2a), etc. The singular similarity solution in the exponential case

$$u_s=\log 1/r^2,\qquad \lambda_s=2(n-2)$$

* The exponential case in three dimensions is considered in [8, pp. 161–168]. In EMDEN's analysis the exponential case is the one appropriate for isothermal stars.

satisfies (XI.5d). It may be obtained from (III.1a) and (III.1b) in the exponential limit.

Replacing Lemma 1 we have: *If* $\phi(x)$ *is a solution of* (XI.3a, b) *then*

$$\tilde{\phi}(\rho x) = -2\log\rho + \phi(x) \tag{XI.8}$$

*is also a solution* (XI.3a, b) *for any real number* $\rho$.

Like (IV.4), this lemma guarantees that one can find, from a given solution $\phi(x)$, a family of solutions with a zero slope and any finite value at the origin. This homology arises from (IV.4) by setting $v=(1+\alpha\phi)$. We pass to the exponential limit in the expression

$$\tilde{\phi}(\rho x) = \frac{1}{\alpha}(\rho^{-\tau}-1) - \rho^{-\tau}\phi(x).$$

Replacing Lemma 2 we have: *Equation* (III.2) *can hold only for those values of* $n$ *for which the* $E$ *solutions of* (XI.3a, b) *have* $\phi<0$ *for all* $0<x<\infty$ *and* $\phi\to\log(1/x^2)$ *as* $x\to\infty$.

This lemma gives conditions which must hold if the singular similarity solution $[u_s, \lambda_s]$ is the $A\to\infty$ limit of regular solutions. In this case, however, $[u_s, \lambda_s]$ is always a limiting form of regular solutions (cf. Lemma 7).

Replacing Lemma 3 we have the following result: *There cannot be two different* ($E$) *solutions which are both asymptotic to the line* $y=-z$ *as* $z\to-\infty$. The proof is due to Hopf [9] and can also be found in [8, p. 162].

Lemmas 4 and 5 are not relevant in the present case, with the exception that the critical point $0_2(z, y)=(0, 0)$ is a termination point in the present case.

The local analysis of the singular similarity solution which was given in Sections VII and VIII can be constructed directly from (XI.2) by linearizing (XI.2a) around the singular solution $\phi=1/x^2$. This is equivalent to studying the stability of the critical point in the phase plane. The analysis is straightforward; it is given in [8, pp. 163–168] for the case $n=3$, and the results for $n-2>0$ are just those asserted in the penultimate paragraph of Section VIII. The local analysis shows that the critical point is an attractor.

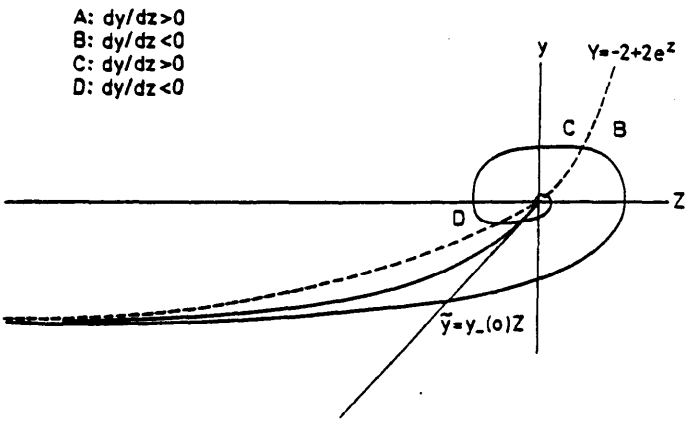



Fig. 5. Arrangements of $E$ solutions in the phase plane ($\mathcal{F}(u)=e^u$)

The analysis of solutions (XI.1, 2) is simple because there is only one critical point (0, 0) in the phase plane and it is an attractor. The lemma asserted two paragraphs before this one shows that there is only one $E$ curve which can begin asymptotic to $y=-z$ as $z\to-\infty$. The solution remains unique and either terminates at $\infty$ or at the critical point. The possibility of escape to infinity can be eliminated by inspecting the direction field for (XI.1). The nature of this direction field can be determined by inspection of Figure 5. The phase plane divides into four regions where the direction field has one sign. These regions are bounded by lines $y=0$ and the line

$$Y=-2+2e^{z} \tag{XI.9}$$

which is the locus of points on which $dy/dz=0$. The solution has a vertical tangent in the finite $(z, y)$ plane only on the line $y=0$. The unique $E$ solution which starts asymptotic to the line $y=-2$ as $z=-\infty$ (as $x\to 0$ or $\tau\to\infty$) must increase in the region $(A)$, but it must stay below the line $Y(z)$. Since $dy/dz>0$ in $(A)$, it can leave $(A)$ by terminating at the critical point (0, 0) or $(b)$ by entering $(B)$ vertically from below. The $E$ solution could not escape to $\infty$ in $(A)$ since the direction field forces it to the line $y=0$ where it has a vertical tangent. In $(B)$ the solution cannot avoid the line (XI.9) and it must pass into $(C)$. It cannot escape to infinity in $(C)$ for the same reason that it could not escape in $(A)$, and, after crossing $y=0$ vertically into $(D)$, it cannot avoid the line (XI.9). Now the solution must cycle around the critical point (0, 0) a finite or infinite number of times, approaching closer to (0, 0) with each cycle since it cannot cross itself or approach a limit cycle (this possibility is excluded by the Bendixson theorem). Since the solution must terminate at (0, 0), the local analysis of $E$ solutions must apply. The results of this analysis are at the end of Section VIII. When $n<10$ the number of cycles is infinite. When $n\geqq 10$ there are real directions of approach to (0, 0).

Consider the possible ways in which the $E$ solution can approach (0, 0) when $n\geqq 10$. At (0, 0) we have

$$\begin{aligned} y'(0)&=(n-2)-(n-2)(2e^{z}-2)/y \\ &=(n-2)\{1-2/y'(0)\} \end{aligned} \tag{XI.10}$$

where we have used L'Hospital's rule. This is a quadratic equation in $y'(0)$ with roots

$$y'(0)=\frac{n-2}{2}\pm\sqrt{(n-2)^2/4-2(n-2)}. \tag{XI.11}$$

Since $n\geqq 10$ the roots are real and positive.

To show that the $E$ solution ($(a)$ in Figure 5) is single-valued, we shall show that the $E$ solution must lie in the region bounded by the line (XI.9) and the line

$$\tilde{y}(z)=y'_{-}(0)\,z. \tag{XI.12}$$

Equation (XI.12) shows that $y'_{-}(0)>0$. Towards this end we first show that every solution curve $y(z)$ on the line $y(z)$ has $dy/dz\leqq y'_{-}(0)$ with equality only when $z=0$. To prove this we note that on the line $y(z)$ we have

$$\frac{1}{(n-2)}\frac{dy}{dz}=1-\frac{2e^{z}-2}{y'_{-}(0)\,z}. \tag{XI.13}$$

On the other hand from (XI.10)

$$\frac{1}{n-2}\, y'_-(0)=1-\frac{2}{y'_-(0)}. \tag{XI.14}$$

Hence, when $z<0$

$$\frac{1}{n-2}\,\frac{dy}{dz}-y'_-(0)=-\frac{2}{y'_-(0)}\{(e^z-1)/z-1\}\geqq 0$$

with equality at $z=0$ and

$$\frac{dy}{dz}\geqq y'_-(0) \tag{XI.15}$$

at $\tilde{y}=y(z)$ for any $E$ solution.

A solution curve cannot cross the straight line $\tilde{y}=y'_-(0)z$ because to do so it would have to cut this line from above and (XI.5) allows only crossing from below.

The results which we have just proved are summarized in Lemma 7.

**Lemma 7.** *As $x$ increases from $x=0$ ($t$ decreases from $\infty$) the $E$ solution rises above the line $y=-2$ as $z$ increases from $-\infty$.* (a) *If $n<10$ it then crosses the $z$ axis vertically from below at a point $z>0$ and cycles around* (0, 0) *an infinite number of times.* (b) *If $n\geqq 10$ it terminates at* (0, 0) *on a single-valued solution curve $y(z)$.*

With this lemma now established, the proof of Theorem 2 follows from the geography of the phase plane as in the proof of Theorem 1.

## XII. Large $n$ Solutions of $(r^{n-1}u')'+\lambda r^{n-1}\mathscr{F}(u)=0,\ u(0)=A,\ u'(0)=0,\ u(1)=0$

This is the "$A$ is given, find $u(r, A)$, $\lambda(A)$" version of problem (I.1). It will be assumed that $\mathscr{F}(u)>0$ when $u\geqq 0$ and that $\mathscr{F}(u)$ satisfies a Lipschitz condition with constant $k$. Problem (I.1), defined in this way, is called the title problem. When $n$ is large the title problem can be written as

$$\varepsilon r u''+u'+\mu r\mathscr{F}(u)=0,\qquad u(0)=A,\ u'(0)=0,\ u(1)=0, \tag{XII.1}$$

where $\varepsilon=1/(n-1)$ and $\mu=\lambda/(n-1)$.

The form of (XII.1) suggests a singular perturbation problem, but the perturbation is not singular because the solution $u$ of (XII.2) below has a bounded second derivative for all $r$, $0\leqq r\leqq 1$.

**Theorem 3.** *Suppose $u(r, A, n)$ and $\lambda(A, n)$ are positive solutions of the title problem. Let $u(r, A)$ and $\mu(A)$ solve the problem*

$$\frac{du}{dr}+\mu r\mathscr{F}(u)=0,\qquad u(0)=A,\ u(1)=0. \tag{XII.2}$$

*Then when $A$ is fixed and $n$ is sufficiently large,*

$$|u(r, A, n)-u(r, A)|_\infty\leqq K_1(A)/n$$

*and*

$$|\mu(A)-\lambda/n|\leqq K_2(A)/n$$

*where* $K_1$ *and* $K_2$ *are independent of* $n$.

**Proof.** The title problem can be converted into an integral equation, that is,

$$u(r,A)=\lambda(A)\int G(r,r_0)\mathscr{F}(u(r_0,A))\,d\Omega_0, \tag{XII.3}$$

where $\lambda(A)$ is given by (XII.3) with $r=0$ and $u(0,A)=A$. Here $G(r,r_0)$ is the Green's function for the Dirichlet problem in the $n$ dimensional sphere

$$G(r,r_0)=\begin{cases}\dfrac{1}{n-2}\left[\dfrac{1}{r^{n-2}}-1\right] & \text{when } r>r_0\\[2ex] \dfrac{1}{n-2}\left[\dfrac{1}{r_0^{n-2}}-1\right] & \text{when } r<r_0,\end{cases}$$

and

$$\frac{u(r)}{A}=\left[\left(\frac{1}{r^{n-2}}-1\right)\int_0^r r_0^{n-1}\mathscr{F}(u(r_0))\,dr_0 + \int_r^1 (r_0-r_0^{n-1})\mathscr{F}(u(r_0))\,dr_0\right]\Big/\int_0^1 (r_0-r_0^{n-1})\mathscr{F}(u(r_0))\,dr_0, \tag{XII.4a}$$

with

$$A=\frac{\lambda}{n-2}\int_0^1 (r_0-r_0^n)\mathscr{F}(u(r_0))\,dr_0. \tag{XII.4b}$$

Note that

$$\int_a^b \mathscr{F}[u(r)]\,r^{n-1}\,dr=\bar{\mathscr{F}}\int_a^b r_0^{n-1}\,dr_0=[r^n]_a^b\,\bar{\mathscr{F}}_{ab}/n,$$

where $\bar{\mathscr{F}}_{ab}$ is a mean value. Let $\mu=\lambda/(n-2)$. From (XII.4a) we find that

$$\frac{u(r)}{\mu}=\int_r^1 r_0\mathscr{F}(u)\,dr_0+\frac{F(r)}{n} \tag{XII.5a}$$

where

$$\frac{A}{\mu}=\int_0^1 r_0\mathscr{F}(u)\,dr_0+\frac{F(0)}{n} \tag{XII.5b}$$

and

$$F(r)=r^2\,\bar{\mathscr{F}}_{0r}-\bar{\mathscr{F}}_{01}$$

is a bounded quantity independent of $n$.

Similarly

$$\frac{\hat{u}(r)}{\hat{\mu}}=\int_r^1 r_0\mathscr{F}(\hat{u})\,dr_0, \tag{XII.6a}$$

where

$$\frac{A}{\hat{\mu}}=\int_0^1 r_0\mathscr{F}(\hat{u})\,dr_0. \tag{XII.6b}$$

From (XII.5b) and (XII.6b) we find that

$$\mu-\hat{\mu}=A\left\{\left(\int_0^1 r_0\mathscr{F}(u)\,dr_0+\frac{F(0)}{n}\right)^{-1}-\left(\int_0^1 r_0\mathscr{F}(\hat{u})\,dr_0\right)^{-1}\right\} \tag{XII.7}$$
$$=O(1/n).$$

Then, using (XII.5a), (XII.5b) and (XII.7) we have

$$u(r)-\hat{u}(r)=\hat{\mu}\int_r^1[\mathscr{F}(u)-\mathscr{F}(\hat{u})]\,r_0\,dr_0+O(1/n). \tag{XII.8}$$

We note that

$$\begin{aligned}\int_r^1(\mathscr{F}(u)-\mathscr{F}(\hat{u}))\eta\,d\eta &\leqq k\int_r^1|u-\hat{u}|\,\eta\,d\eta\\ &\leqq\frac{k}{2}(1-r^2)\max_{r\leqq\eta\leqq 1}|u-\hat{u}|\end{aligned} \tag{XII.9}$$

where $k$ is the Lipschitz constant. Combining (XII.8) and (XII.9), we find that

$$\left\{1-\frac{\hat{\mu}k}{2}(1-r^2)\right\}\max_{r\leqq\eta\leqq 1}|u-\hat{u}|\leqq O(1/n).$$

Hence $|u-\hat{u}|\to 0$ for each $r\in[0,1]$ with an error proportional to $1/n$.

The $O(1/n)$ correction of the large $n$ solution can also be easily obtained. Thus solutions of (XII.5) satisfy

$$\frac{u}{\mu}=\int_r^1 r_0\,\mathscr{F}(u)\,dr_0+[r^2\,\mathscr{F}(u(r))-\mathscr{F}(0)]/n+O(1/n^2)$$

and

$$\frac{A}{\mu}=\int_0^1 r_0\,\mathscr{F}(u)\,dr_0-\mathscr{F}(0)/n+O(1/n^2).$$

The proof of these formulas follows from (XII.5a, b) and the observation that

$$\lim_{n\to\infty}\bar{\mathscr{F}}_{0r}=\lim_{n\to\infty}\frac{n}{r^n}\int_0^r\mathscr{F}(u(r_0))\,r_0^{n-1}\,dr_0=\mathscr{F}(u(r)).$$

It is of some interest to compare the large $n$ solutions to the known limiting singular solutions of the problem with $\mathscr{F}(u)=(1+\alpha u)^\beta$. We find from (XII.2) that $\hat{u}$ and $\hat{\mu}$ are given by

$$(1+\alpha\hat{u})^{\beta-1}=(1+\alpha A)^{\beta-1}/[(1-r^2)+r^2(1+\alpha A)^{\beta-1}]$$

and

$$\hat{\mu}=2[1-(1+\alpha A)^{1-\beta}]/\alpha(\beta-1).$$

In the exponential limit $\alpha=1/\beta$, $|\beta|\to\infty$

$$e^{\hat{u}}=e^A/[(1-r^2)+e^{-A}r^2],$$

and

$$\hat{\mu}=2[1-e^{-A}].$$

Two other special values of $\beta$ are of interest. The first is $\beta=1$. This is the limiting value of $(2+n)/(n-2)$; since proper singular solutions exist only for positive $\beta$ greater than this value, the limit $n\to\infty$ could not commute with $A\to\infty$ (since the latter limit does not exist). At finite $A$ with $\beta=1$, one can find the handsome expression

$$1+\hat{u}(r)=(1+A)^{1-r^2}$$

and

$$\hat{\mu}=2\log(1+A).$$

This solution also appears as the limiting form for $n \to \infty$, at fixed $A$, of the exact solution of (VI.2, 3) for $\beta = (2+n)/(n-2)$. The second interesting value $\beta = 0$ is also on the border separating the proper singular solutions ($\beta < 0$) from the others. Here, we have from Theorem 3

$$\hat{u} = A(1-r^2), \qquad \hat{\lambda}/2n = A,$$

and this is also the exact solution of the original second order problem for all $n$ and all $A$.

## XIII. Closing Remarks

The uniqueness results proved in this paper show a most striking dependence of the uniqueness of solutions on the number of space dimensions, and it is natural to wonder about the extent to which these results carry over into arbitrary non-spherical domains. To study this problem in the context of partial differential equations, it would first be necessary to determine if the form of the singular similarity solution for the $n$ dimensional sphere is also the form of a limiting singularity

$$\Delta u + \lambda \mathscr{F}(u) = 0, \qquad u = 0|_{\Omega}$$

where $\Omega$ is an arbitrary closed domain and $\mathscr{F}(u)$ tends to $(1+\alpha u)^{\beta}$ or $e^u$ for large $u$.

It is not hard to show that every spherically symmetric singularity must have the form of the singular similarity solution. However, it remains to answer whether a non-spherically symmetric singularity is possible when $\Omega$ is not a sphere. A tentative answer "no" is suggested by a formal perturbation analysis of the singular similarity solution to small but otherwise arbitrary deformations of the sphere. This problem can be solved explicitly and the result shows that the singular similarity solution remains spherically symmetric under such small, arbitrary deformations.

This work was supported in part by NSF Grant GK-12500.

## References

1. JOSEPH, D. D., Non-linear heat generation and the stability of the temperature distribution in conducting solids. Int. J. Heat Mass Transfer 8, 281–288 (1965).
2. KELLER, H., & D. COHEN, Some positone problems suggested by nonlinear heat generation. J. Math. Mech. 16, 1361–1376 (1967).
3. JOSEPH, D. D., & E. M. SPARROW, Nonlinear diffusion induced by nonlinear sources. Quart. Appl. Math. 28, 327–342 (1970).
4. GELFAND, I. M., Some problems in the theory of quasilinear equations. Amer. Math. Soc. Transl. (2) 29, 295–381 (1963).
5. CALLEGARI, A. J., E. L. REISS, & H. KELLER, Membrane buckling. Comm. Pure Appl. Math. 24, 499–527 (1971).
6. EMDEN, V. R., Gaskugeln, 1907.
7. FOWLER, R. H., Further studies of Emden's and similar differential equations. Quart. J. Math. (Oxford series) 2, 259–288 (1931).
8. CHANDRASEKHAR, S., An Introduction to the Study of Stellar Structure New York: Dover 1957.

9. Hopf, E., On Emden's differential equation. M.N.R.A.S. **91**, 653–663 (1931).
10. Joseph, D. D., Bounds on $\lambda$ for positive solutions of $\Delta\psi+\lambda f(r)(\psi+G(\psi))=0$. Q. Appl. Math. **23**, 349–354 (1966).
11. Cole, J. D., Perturbation Methods in Applied Mathematics. Blaisdell Publishing Co. 1968.
12. Bendixson, I., Sur les courbes définies par des équations différentiells. Acta. Math. **24**, 1–88 (1901).

Department of Aerospace Engineering and Mechanics
University of Minnesota
Minneapolis

*(Received December, 1970; revised June, 1972)*

Universitätsdruckerei H. Stürtz AG Würzburg
Printed in Germany