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REMARKS ABOUT BIFURCATION AND STABILITY OF QUASI-PERIODIC SOLUTIONS WHICH BIFURCATE FROM PERIODIC SOLUTIONS OF THE NAVIER STOKES EQUATIONS

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SUMMARY

L. D. Landau (1944) and E. Hopf (1948) have conjectured that the transition to turbulence may be described as repeated branching of quasi-periodic solutions into quasi-periodic solutions with more frequencies. The simplest case is the bifurcation of periodic solutions from steady solutions. The next hardest problem is the bifurcation of quasi-periodic solutions from basic time periodic solutions of fixed frequency. This problem is treated in the lecture by a generalization of the Poincaré-Lindstedt perturbation which is successful in the simplest case. It is assumed that the Floquet exponents are simple eigenvalues of the spectral problem for the basic flow.

If the Floquet exponent is zero at criticality, the formal construction gives two bifurcating solutions of the same frequency as the basic flow. The small amplitude solutions which bifurcate supercritically are stable; subcritical solutions with small amplitudes are unstable.

When the Floquet exponents at criticality are complex and rationally independent of the fixed frequency, the solution of the spectral problem is quasi-periodic. The formal construction then gives the natural frequency of the bifurcating solution as a power series in the amplitude. The stability of the two-frequency power series is studied using a Floquet representation, generalizing a suggestion of Landau and using the Poincaré-Lindstedt method. Again the bifurcating solution is stable when supercritical and is unstable when subcritical.

When the Floquet exponents and the fixed frequency are rationally dependent at criticality, the perturbation problems whose solutions give the coefficients of

the Poincaré-Lindstedt series cannot be solved unless certain additional orthogonality conditions are satisfied. Though these "extra" conditions do not arise exactly when the frequencies are rationally independent, they may be viewed as limiting forms of conditions associated with small divisors. In general, a bounded inverse cannot be expected even in the quasi-periodic case.

§1. INTRODUCTION

This paper treats the problem of stability and bifurcation of periodic solutions of the Navier-Stokes equations. Such periodic motions may be assumed to be forced by periodic external conditions. For example, periodic fluid motion could arise from the rotation of gears in a gear box full of oil, or it could arise as an oscillating pipe flow driven by a time periodic pressure gradient. For sufficiently small Reynolds numbers these forced periodic solutions are both stable and unique.* Time periodic motions also arise from instability and bifurcation of steady motion. However, we shall concentrate here on the instability of the forced periodic motions.

Time dependent basic flows are currently a subject of active inquiry. The interested reader can find a bibliography on the linear aspects of this problem as well as derivation of formal perturbation theory involving amplitude equations of the Stuart-Watson type in the paper of Davis (1971).

Mathematically rigorous demonstrations of the existence of bifurcating solutions have been given by Sattinger (1971), Yudovich (1971), and Iooss (1972). The Poincaré-Lindstedt series for constructing the Floquet theory of stability of bifurcating periodic solutions given first by Joseph and Sattinger (1972) are important in the present work. In this paper we generalize (without justification) the series construction to study solutions which bifurcate from forced periodic motions. These series form the basis for generalizing to solutions bifurcating from periodic motions results known to hold for solutions bifurcating from steady motions (see Fig. 1) when (a) the bifurcating solution is also steady (Joseph (1971) hereafter

*Existence of forced periodic solutions at small R is proved by Serrin (1960) and at all R by Yudovich (1960), Prodi (1960) and Prouse (1963). Serrin shows that the forced periodic solution is stable and unique when R is small. Fojas (1962) has proved existence, stability and uniqueness of forced almost periodic solutions at small R.

called (J)) and (b) when the bifurcating solution is time periodic (Joseph and Sattinger (1972), hereafter called (JS)). In the context of present work, in which we consider bifurcation and instability of forced periodic motions, the steady motions appear as a special case in the limit of zero forcing frequency.

A precise summary of the results to be obtained from our formal analysis will now be given. Let $\tilde{\mathbf{u}}(\mathbf{x},t,R)$ and $\epsilon\mathbf{u}(\mathbf{x},t,R) + \tilde{\mathbf{u}}(\mathbf{x},t,R)$ be solutions of (1.1) below:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla\mathbf{V} - \frac{1}{R}\nabla^2\mathbf{V} + \nabla p + \mathbf{F}(\mathbf{x},t) = 0\big|_{\Omega(t)} \quad , \tag{1.1a}$$

$$\text{div } \mathbf{V} = 0\big|_{\Omega(t)} \quad , \tag{1.1b}$$

and

$$\mathbf{V} = \mathbf{f}(\mathbf{x},t)\big|_{\partial\Omega(t)} \quad , \tag{1.1c}$$

where $\mathbf{F}$, $\tilde{\mathbf{u}}$, p, $\mathbf{f}$ and $\Omega(t)$ are periodic functions of t of period $2\Pi/\tilde{\omega}$ and $\mathbf{u}$ is either a periodic function of the same period or an almost periodic function of t. Since the fluid is incompressible, the total volume of the (otherwise arbitrary but periodically varying) domain $\Omega(t)$ is constant in time. The symbol $\mathbf{F}$ stands for body force and $\mathbf{F}(\mathbf{x},t)$, $\mathbf{f}(\mathbf{x},t)$ and $\Omega(t)$ are preassigned periodic functions. The symbol p stands for the pressure.

We have already noted that the existence of at least one periodic solution of (1.1) for all R has been established by various authors. However, when R is sufficiently small there can be only one periodic solution of (1.1). We designate this basic solution as $\tilde{\mathbf{u}}$. The second solution $\tilde{\mathbf{u}} + \epsilon\mathbf{u}$ will be assumed to arise as an instability of $\tilde{\mathbf{u}}$ when R is increased past a critical value R_c obtained from a linearized theory of stability.

The disturbance $\epsilon\mathbf{u}$ satisfies equations (1.2) below:

$$\frac{\partial \mathbf{u}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\tilde{\mathbf{u}} + \epsilon\mathbf{u} \cdot \nabla\mathbf{u} - \frac{1}{R}\nabla^2\mathbf{u} + \nabla p = 0 \tag{1.2a}$$

$$\text{div } \mathbf{u} = 0 \ , \quad \mathbf{u} = 0\big|_{\partial\Omega(t)} \ . \tag{1.2b,c}$$

We now assume that the stability of the periodic solution $\tilde{\mathbf{u}}$, that is, the solution $\underset{\sim}{u} = p = 0$ of (1.2), can be studied by the method of Floquet (see Yudovich (1970 A,B)). We set $\epsilon = 0$ in (1.2) and introduce the Floquet representation

$$\underset{\sim}{u} = e^{-\gamma t}\underset{\sim}{\zeta}(\underset{\sim}{x}, t) \ , \quad p = e^{-\gamma t}\Pi(\underset{\sim}{x},t) \tag{1.3}$$

where $\underset{\sim}{\zeta}$ is a periodic function of t with period $2\Pi/\tilde{\omega}$ and

$$\gamma = \xi + i\eta$$

is the Floquet exponent. The exponent is determined as an eigenvalue of the spectral problem (1.4) below:

$$-\gamma\underset{\sim}{\zeta} + \frac{\partial\underset{\sim}{\zeta}}{\partial t} + L(\underset{\sim}{\tilde{u}},\lambda)\underset{\sim}{\zeta} + \nabla\Pi = 0 \ , \quad \text{div}\ \underset{\sim}{\zeta} = 0 \ , \quad \underset{\sim}{\zeta} = 0|_{\partial\Omega(t)} \tag{1.4a,b,c}$$

where $\lambda = 1/R$ and the i-th component of $L\underset{\sim}{\zeta}$ is defined as

$$\begin{aligned}(L(\underset{\sim}{\tilde{u}},\lambda)\underset{\sim}{\zeta})_i &= (L(\underset{\sim}{\tilde{u}}(\lambda,t),\lambda)\underset{\sim}{\zeta})_i = \underset{\sim}{\tilde{u}} \cdot \nabla\zeta_i + \underset{\sim}{\zeta} \cdot \nabla\tilde{u}_i - \lambda\Delta\zeta_i \\ &= \underset{\sim}{\tilde{u}} \cdot \nabla\zeta_i + \zeta_j\Omega_{ji}(\tilde{u}) + \zeta_j S_{ji}(\tilde{u}) - \lambda\Delta\zeta_i\end{aligned} \tag{1.5}$$

and $\partial_j\tilde{u}_i = \Omega_{ji} + S_{ji}$ is the resolution into the vorticity and the rate of strain tensor.

Stability in the linear approximation is determined by the sign of $\xi(R) =$ re $\gamma(R)$. When R is small we have stability and $\xi(R) > 0$; for $R = R_c$ we have the marginal case $\xi(R_c) = 0$; and when $R > R_c$ there is instability and $\xi(R) < 0$.

The assumption that

$$\gamma(R) = \xi(R) + i\eta(R)$$

is a simple discrete eigenvalue of (1.4) when $R - R_c$ is small is central to our analysis. This is not likely to be a restrictive assumption in bounded domains when $R - R_c$ is small.

We shall distinguish two cases: (a) the Floquet exponent is zero at criticality and (b) the Floquet exponent is not zero at criticality. In case (a) (Section 5), im $\gamma(R_c) = 0$ and is both simple and discrete. Therefore, the Floquet exponents $\gamma(R) = \xi(R)$ form a single curve through the origin of the complex γ plane.

No other eigenvalues of (1.4) lie in some circle centered at the origin. This case is the analogue for bifurcation from a real simple eigenvalue in the steady case (cf. J). We find that two periodic solutions of period $2\pi/\tilde{\omega}$ bifurcate from the basic periodic solution $\tilde{u}(\underset{\sim}{x},t\ R)$ at $R = R_c$. Assuming that the bifurcating solutions $\underset{\sim}{u}(\underset{\sim}{x},t,R(\epsilon),\epsilon)$, $p(\underset{\sim}{x},t,R(\epsilon),\epsilon)$ and $R(\epsilon)$ are smooth functions of ϵ, one finds that, in general, $dR(0)/d\epsilon \neq 0$ and a supercritical solution $(R(\epsilon) > R_c)$ and a subcritical solution $R(\epsilon) < R_c$ will branch out at $R(0) = R_c$. (Fig. 1 (c)).

A different set of bifurcation results may be associated with case (b) which is considered in Sections 3 and 4. In this case, $\eta(R_c) = \omega_0$. The formal analysis leads to a single quasi-periodic solution which bifurcates from the basic periodic solution when $R = R_c$. This quasi-periodic solution is a composition of functions with two frequencies*: the fixed and externally imposed frequency and a natural frequency $\omega(\epsilon)$ which arises from the instability of the basic motion and depends on the amplitude ϵ of the bifurcating motion. When the fixed forced frequency is zero, we return to the case of loss of stability of a steady solution to a bifurcating time periodic solution which was treated in (JS). Again we find that supercritical bifurcating solutions are stable and subcritical solutions are unstable when ϵ is small.

The techniques used here to study the bifurcation stability problem for the Navier-Stokes equations would appear to give a further mathematical structure (cf. JS) to previously unexplored aspects of Landau's conjecture about the transition to turbulence through repeated supercritical branching. However, the perturbation procedure which could be expected to give the quasi-periodic bifurcating solutions runs into solvability difficulties. The relation of these difficulties to the Landau-Hopf conjecture is discussed in the conclusion to Section 4.

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*Quasi-periodic functions are defined as the special class of almost periodic functions possessing only a finite basis of frequencies. In other words, we are studying oscillations containing finitely many (rationally independent) frequencies $\omega_1,\omega_2,\ldots,\omega_n$. For example, the function $f(t) = \cos t \cos\pi t$ is a quasi-periodic function with frequencies $\omega_1 = 2\pi$ and $\omega_2 = 2$. The value $f(t) = 1$ occurs when $t = 0$ but not again; though $f(t) < 0$ when $t \neq 0$, there is always $\tau(\epsilon) > 0$ such that $|f(\tau) - f(o)| < \epsilon$ for preassigned $\epsilon > 0$.

§2. THE SPECTRAL PROBLEM AND THE SOLVABILITY CONDITIONS

Ultimately we shall want to consider quasi-periodic vector fields $\underset{\sim}{q}(\underset{\sim}{x},s,t) = \tilde{\underset{\sim}{q}}(\underset{\sim}{x},\omega(\epsilon)t,\tilde{\omega}t)$ where $\tilde{q}$ is 2π periodic in the variables $s = \omega(\epsilon)t$ and $\tilde{\omega}t$, and $\tilde{\omega}$ are rationally independent.

We first need to define the operator

$$J = \frac{\partial}{\partial t} + L(\tilde{\underset{\sim}{u}},\lambda) , \tag{2.1}$$

whose domain is the set of smooth vector fields which are $2\pi/\tilde{\omega}$ periodic in t. For these periodic functions we define the scalar product

$$[\underset{\sim}{a},\underset{\sim}{b}] = \frac{2\pi}{\tilde{\omega}} \int_0^{2\pi/\tilde{\omega}} (\underset{\sim}{a},\underset{\sim}{b})dt ,$$

where (2.2)

$$(a,b) = \int_{\Omega(t)} \underset{\sim}{a} \cdot \bar{\underset{\sim}{b}}\, d\underset{\sim}{x}$$

and the overbar designates "complex conjugate".

By an admissible $2\pi/\tilde{\omega}$ periodic vector field, we mean a periodic field which is solenoidal and vanishes on $\partial\Omega(t)$.

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be arbitrary admissible vectors. We define the formal adjoint of J in the usual way*

$$[\underset{\sim}{a},J\underset{\sim}{b}] = [J^*\underset{\sim}{a},b] ,$$

and find that

$$J^* = -\frac{\partial}{\partial t} + L^*(\tilde{u};\lambda) , \tag{2.3}$$

where

$$(L^*(\tilde{\underset{\sim}{u}};\lambda)\underset{\sim}{a})_i = -\tilde{\underset{\sim}{u}} \cdot \nabla a_i - a_j\Omega_{ji}(\tilde{\underset{\sim}{u}}) + a_jS_{ji}(\tilde{\underset{\sim}{u}}) - \lambda\Delta a_i .$$

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*To obtain the adjoint we have made use of the Reynolds transport theorem in the form

$$\frac{d}{dt}\int_{\Omega(t)} \underset{\sim}{a}\cdot\bar{\underset{\sim}{b}}\, d\underset{\sim}{x} \int_{\Omega(t)} \partial(\underset{\sim}{a}\cdot\bar{\underset{\sim}{b}})/\partial t\, d\underset{\sim}{x} + \int_{\partial\Omega(t)} (\underset{\sim}{f}\cdot\underset{\sim}{n})\, \underset{\sim}{a}\cdot\bar{\underset{\sim}{b}}\, d\underset{\sim}{x} .$$

The boundary integral vanishes since $\underset{\sim}{a} \cdot \bar{\underset{\sim}{b}} = 0$ when $\underset{\sim}{x} \in \partial\Omega$.

The adjoint eigenvalue problem is

$$-\bar{\gamma}\underset{\sim}{\zeta}^* + J^*\underset{\sim}{\zeta}^* + \nabla\Pi^* = 0 \ , \quad \text{div}\ \underset{\sim}{\zeta}^* = 0 \ , \quad \underset{\sim}{\zeta}^* = 0\big|_{\partial\Omega(t)} \quad . \tag{2.4}$$

Let $\underset{\sim}{\varphi}$ be any periodic admissible vector. Then

$$-\gamma[\underset{\sim}{\varphi},\underset{\sim}{\zeta}] + [\underset{\sim}{\varphi},J\underset{\sim}{\zeta}] = 0 \tag{2.5a}$$

and

$$-\bar{\gamma}[\underset{\sim}{\varphi},\underset{\sim}{\zeta}^*] + [\underset{\sim}{\varphi},J^*\underset{\sim}{\zeta}^*] = 0 \quad . \tag{2.5b}$$

Choose $\underset{\sim}{\varphi} = \underset{\sim}{\zeta}$ in (2.5b) and $\underset{\sim}{\varphi} = \bar{\underset{\sim}{\zeta}}^*$ in (2.5a) and use (2.2) to find that

$$[\underset{\sim}{\zeta},\bar{\underset{\sim}{\zeta}}^*] = 0 = [\underset{\sim}{\zeta},\underset{\sim}{\zeta}^*] \quad . \tag{2.6a}$$

By normalization

$$[\underset{\sim}{\zeta},\underset{\sim}{\zeta}^*] = [\bar{\underset{\sim}{\zeta}},\bar{\underset{\sim}{\zeta}}^*] = 1 \quad . \tag{2.6b}$$

When γ is real (2.6a) does not hold. Then it is sufficient to consider real-valued fields $\underset{\sim}{\zeta}$. Of course, we may always normalize as in (2.6b).

We next demonstrate that

$$[L_\lambda\underset{\sim}{\zeta},\underset{\sim}{\zeta}^*] = \frac{d\gamma}{d\lambda} \quad , \tag{2.7}$$

where

$$L_\lambda = \frac{d}{d\lambda}J = \frac{d}{d\lambda}L(\tilde{\underset{\sim}{u}}(\lambda),\lambda)$$

is a differential operator of order two. Let $\gamma' = \partial\gamma/\partial\lambda$. Then

$$-\gamma\underset{\sim}{\zeta}_\xi + J\underset{\sim}{\zeta}_\xi - \gamma'\underset{\sim}{\zeta} + L_\lambda\underset{\sim}{\zeta} + \nabla\Pi_\xi = 0$$

where $\underset{\sim}{\zeta}_\xi$ is solenoidal and vanishes on the boundary. Since

$$[\underset{\sim}{\zeta}^*,J\underset{\sim}{\zeta}_\xi] = [J^*\underset{\sim}{\zeta}^*,\underset{\sim}{\zeta}_\xi] = \gamma[\underset{\sim}{\zeta}^*,\underset{\sim}{\zeta}_\xi]$$

using (2.6), we find (2.7).

When $\omega_0 \neq 0$ the solutions of the linearized problem (1.2 with $\varepsilon = 0$) are not necessarily periodic with period $2\Pi/\tilde{\omega}$ in t . If, at criticality, $i\omega_0$ is a simple eigenvalue, there are two solutions of the linearized problem,

$$\underset{\sim}{Z}_1 = e^{-is}\underset{\sim}{\zeta}(\underset{\sim}{x},t,R_c) \ , \qquad \underset{\sim}{Z}_2 = \bar{Z}_1 \tag{2.8a}$$

with $s = \omega_o t$, and two solutions of the adjoint problem

$$\underset{\sim}{Z}_1^* = e^{-is}\underset{\sim}{\zeta}^*(\underset{\sim}{x},t,R_c) \ , \qquad \underset{\sim}{Z}_2^* = \bar{\underset{\sim}{Z}}_1^* \quad . \tag{2.8b}$$

These solutions are quasi-periodic with two frequencies.

For the quasi-periodic functions it is useful to introduce two times s and t corresponding to the two frequencies: the fixed forced frequency $\tilde{\omega}$ and the natural frequency $\omega(\epsilon)$. Associated with these quasi-periodic functions of two variables is the operator

$$\hat{J} = \frac{\partial}{\partial t}\Big|_{\underset{\sim}{x}} + L(\tilde{\underset{\sim}{u}},\lambda) = \omega\frac{\partial}{\partial s}\Big|_{t,x} + \frac{\partial}{\partial t}\Big|_{s,\underset{\sim}{x}} + L(\tilde{u},\lambda) \tag{2.9}$$

and the scalar product

$$[\![\underset{\sim}{a},\underset{\sim}{b}]\!] = \frac{1}{2\pi}\int_0^{2\pi} [\underset{\sim}{a},\underset{\sim}{b}]ds \quad . \tag{2.10}$$

When the domains of the operators $\hat{J}$, $[\![\cdot]\!]$ are restricted to the $2\pi/\tilde{\omega}$ periodic functions, these operators reduce to J and $[\cdot]$. The operator

$$\hat{J}^* = -\omega\frac{\partial}{\partial s} - \frac{\partial}{\partial t} + L^*(\tilde{\underset{\sim}{u}},\lambda)$$

is the adjoint to (2.9) relative to (2.10) over solenoidal vector fields vanishing on the boundary.

At criticality $\lambda = \lambda_o$, $\omega = \omega_o$, $J = J_o$ and $\hat{J} = \hat{J}_o$. It is readily verified that

$$\hat{J}_o\underset{\sim}{Z}_1 + e^{-is}\nabla\Pi = 0 \tag{2.11a}$$

and

$$\hat{J}_o^*\underset{\sim}{Z}_1^* + e^{-is}\nabla\Pi^* = 0 \quad , \tag{2.11b}$$

so that

$$[\![\underset{\sim}{Z}_i,\underset{\sim}{Z}_j^*]\!] = \delta_{ij} \qquad (i,j = 1,2) \quad .$$

As a notational convenience we define

$$\| \underset{\sim}{a} \|_i \equiv \| \underset{\sim}{a}, \bar{Z}_i^* \| \tag{2.12}$$

for doubly periodic vector fields of s and t and

$$[\underset{\sim}{b}] \equiv [\underset{\sim}{b}, \bar{\underset{\sim}{\zeta}}^*] \tag{2.13}$$

for singly periodic vector fields $b(\underset{\sim}{x},t)$ of t. If $\underset{\sim}{a}$ is a real-valued field then $\overline{\| \underset{\sim}{a} \|}_1 = \| \underset{\sim}{a} \|_2$. For real-valued fields we shall take $\| \cdot \| \equiv \| \cdot \|_1$.

<u>Lemma 1</u>. <u>Let</u> $\underset{\sim}{b}(\underset{\sim}{x},s,t)$ <u>be a doubly periodic vector field with period</u> 2π <u>in</u> s <u>and</u> $2\pi/\tilde{\omega}$ <u>in</u> t. <u>The problem</u>

$$\hat{J}_o \underset{\sim}{q} + \nabla p = \underset{\sim}{b}, \quad \operatorname{div} \underset{\sim}{q} = 0, \quad \underset{\sim}{q} = 0|_{\partial\Omega(t)} \tag{2.14a}$$

<u>can have doubly periodic solutions only if</u>

$$\| \underset{\sim}{b} \|_i = 0 \quad . \tag{2.14b}$$

<u>Proof</u>.

$$\| \hat{J}_o \underset{\sim}{q} \|_i = \| \hat{J}_o \underset{\sim}{q}, \bar{Z}_i^* \| = \| \underset{\sim}{q}, \hat{J}_o^* \bar{Z}_i^* \| = 0 \quad .$$

The solvability Lemma 1 may be reduced to a solvability lemma involving only the $2\pi/\tilde{\omega}$ periodic functions by expanding $\underset{\sim}{b}$, $\underset{\sim}{q}$, and p into Fourier series

$$\begin{bmatrix} \underset{\sim}{b} \\ \underset{\sim}{q} \\ p \end{bmatrix} = \sum_{-\infty}^{\infty} e^{i\ell s} \begin{bmatrix} \underset{\sim}{b}^{(\ell)} \\ \underset{\sim}{q}^{(\ell)} \\ p^{(\ell)} \end{bmatrix} \quad . \tag{2.15}$$

The Fourier coefficients are $2\pi/\tilde{\omega}$ periodic functions of t. This expansion is always possible when s and t are independent; s and t are independent when $\omega(\epsilon)$ and $\tilde{\omega}$ are rationally independent.* Using the Fourier series and assuming

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*Our basic decomposition into two times and the Fourier decomposition is not valid when $\omega(0)/\tilde{\omega}$ is rationally dependent.

rational independence, we find from (2.14a) that

$$i\ell\omega_o \underset{\sim}{q}^{(\ell)} + J_o \underset{\sim}{q}^{(\ell)} + \nabla p^{(\ell)} = \underset{\sim}{b}^{(\ell)} , \quad \text{div } \underset{\sim}{q}^{(\ell)} = 0 \quad , \tag{2.15a,b}$$

$$q^{(\ell)} = 0_{|\partial\Omega(t)} , \quad q^{(\ell)}(\underset{\sim}{x},t) = q^{(\ell)}(\underset{\sim}{x}, t + 2\pi/\tilde{\omega}) \quad , \tag{2.15c,d}$$

and from (2.14b) that

$$[\underset{\sim}{b}^{(1)}] = \underset{\sim}{b}^{(-1)}] = 0 \quad . \tag{2.15e}$$

To find more general solvability conditions than (2.15e) which will also clearly bring out the problem of small divisors, we note that the eigenvalue problem for J_o implies that

$$-i(\omega_o - n\tilde{\omega})\underset{\sim}{f}_n + J_o \underset{\sim}{f}_n + e^{i\tilde{\omega}nt}\nabla\Pi = 0 \quad ,$$

where

$$\underset{\sim}{f}_n = e^{-in\tilde{\omega}t}\underset{\sim}{\zeta}(\underset{\sim}{x},t) \; , \qquad n = 0, \pm 1, \pm 2, \ldots \quad ,$$

and $\underset{\sim}{f}_n$ satisfies (2.15b,c,d). The adjoint eigenfunction is

$$\underset{\sim}{f}_n^* = e^{-in\tilde{\omega}t}\underset{\sim}{\zeta}^*(\underset{\sim}{x},t) \quad ,$$

and

$$i(\omega_o - n\tilde{\omega})\underset{\sim}{f}_n^* + J_o^* \underset{\sim}{f}_n^* + e^{-i\tilde{\omega}nt}\nabla\Pi^* = 0 \quad . \tag{2.16}$$

Comparison of (2.16) and its complex conjugate with (2.15) shows that

$$[\underset{\sim}{b}^{(\ell)}, \underset{\sim}{f}_n^*] = i(\ell\omega_o + \omega_o - n\tilde{\omega})[\underset{\sim}{q}^{(\ell)}, \underset{\sim}{f}_n^*] \tag{2.17a}$$

and

$$[\underset{\sim}{b}^{(\ell)}, \bar{\underset{\sim}{f}}_n^*] = i(\ell\omega_o - \omega_o + n\tilde{\omega})[\underset{\sim}{q}^{(\ell)}, \bar{\underset{\sim}{f}}_n^*] \quad . \tag{2.17b}$$

Equations (2.17a) and (2.17b) may be regarded as conditions on the amplitude of $\underset{\sim}{q}^{(\ell)}$. Given non-zero values on the left side, the ℓth iterate $\underset{\sim}{q}^{(\ell)}$ will be unbounded when one of the numbers $\ell\omega_o \pm \omega_o \mp n\tilde{\omega} \to 0$.

Lemma 2. Suppose that $\omega_o \neq 0$ and

$$\ell\omega_o \pm \omega_o \mp n\tilde{\omega} = 0 \quad , \tag{2.18}$$

for some ℓ and n where

$$\ell \text{ or } n = 0, \pm 1, \pm 2, \ldots \quad .$$

Then (2.15a,b,c,d) cannot be solved unless

$$[\underset{\sim}{b}^{(\ell)}, e^{-in\tilde{\omega}t}\bar{\underset{\sim}{\zeta}}^*] = 0 \quad , \tag{2.19a}$$

and

$$[\underset{\sim}{b}^{(\ell)}, e^{-in\tilde{\omega}t}\underset{\sim}{\zeta}^*] = 0 \quad . \tag{2.19b}$$

Remark. In our construction we shall consider real-valued vector fields $\underset{\sim}{b}$. For these

$$\underset{\sim}{b}^{(\ell)} = \bar{\underset{\sim}{b}}^{(-\ell)}$$

and (2.18) and (2.19a) imply (2.19b).

When $\tilde{\omega} = 0$ (2.19a) reduces to

$$[\underset{\sim}{b}^{(1)}, \bar{\underset{\sim}{\zeta}}^*] = 0 \quad .$$

This was the case treated in (JS), and (2.19) is then a sufficient as well as a necessary condition for the solvability of (2.15).

Equation (2.18) cannot be satisfied when ω_o and $\tilde{\omega}$ are rationally independent. However, in this case we may always choose integers ℓ and n to make $\ell\omega_o \pm \omega_o \mp n\tilde{\omega}$ arbitrarily small. This is the small divisor problem, and a bounded inverse for all values of ℓ is not to be expected.

Only slight changes of interpretation of Lemmas 1 and 2 are needed when $\omega_o = 0$. In this case $\gamma(R_c) = 0$, and we may assume that the eigenfunctions of (1.4) and (2.4) are real-valued. Thus,

$$\underset{\sim}{Z}_1 = \bar{\underset{\sim}{Z}}_1 = \underset{\sim}{\zeta}\ , \quad \underset{\sim}{Z}_2 = \bar{\underset{\sim}{Z}}_2 = \underset{\sim}{\zeta}^* \quad .$$

When $\omega_o = 0$ we define

$$[\underset{\sim}{u}] = [\underset{\sim}{u}, \underset{\sim}{\zeta}^*]$$

and find that (2.6), (2.7) and (2.13) hold with the new understanding.

§3. QUASI-PERIODIC BIFURCATIONS OF PERIODIC SOLUTIONS

It will be convenient to treat the more general problem of almost periodic bifurcations (case (b) first. The simpler problem (a) in which the bifurcating solution is time periodic is considered in Section 5.

The solution of the bifurcation problem (1.2) which we are going to construct can be called a quasi-periodic solution with two frequencies: the fixed forced frequency $\tilde{\omega}$ and the natural frequency $\omega(\epsilon)$. The natural frequency is introduced at the point of instability and bifurcation of the forced periodic solution $\underset{\sim}{u}(\underset{\sim}{x}, t)$. When $\epsilon = 0$ and $\lambda = \lambda_o = 1/R_c$, we get an infinitesimal almost periodic solution

$$\epsilon \underset{\sim}{u}_o = 2\epsilon \, \mathrm{re}(\underset{\sim}{Z}_1) = \epsilon(e^{-is}\underset{\sim}{\zeta}(\underset{\sim}{x}, t) + e^{+is}\bar{\underset{\sim}{\zeta}}(\underset{\sim}{x}, t)) \quad . \tag{3.1}$$

A crucial part of our construction of the quasi-periodic bifurcating solution for small $\epsilon \neq 0$ is the Poincare mapping

$$\omega(\epsilon)t = s \quad . \tag{3.2}$$

To understand the role of this mapping, we consider the case $\tilde{\omega} = 0$. This leads to time periodic bifurcations of steady solutions, a problem which is thoroughly treated in (JS). In the case $\tilde{\omega} = 0$, one constructs a power series solution of the following type:

$$\begin{bmatrix} \underset{\sim}{u}(\underset{\sim}{x}, s, \epsilon) \\ p(\underset{\sim}{x}, s, \epsilon) \\ \lambda(\epsilon) \\ \omega(\epsilon) \end{bmatrix} = \sum_{n=0} \epsilon^n \begin{bmatrix} \underset{\sim}{u}_n(\underset{\sim}{x}, s) \\ p_n(\underset{\sim}{x}, s) \\ \lambda_n \\ \omega_n \end{bmatrix} \tag{3.3}$$

where, for example,

$$\underset{\sim}{u}_n(x,s) = \sum_{\ell=-N}^{N} \underset{\sim}{u}_{n\ell}(x)\, e^{i\ell s} \tag{3.4}$$

is a 2π periodic Fourier polynomial. The Poincare mapping is crucial here because in terms of the variable t,

$$\underset{\sim}{u}_n(\underset{\sim}{x},\epsilon,t) = \sum_{\ell=-N}^{N} \underset{\sim}{u}_{n\ell}(\underset{\sim}{x})\, e^{i\ell t \Sigma \epsilon^n \omega_n} \tag{3.5}$$

cannot be approximated by its power series in ϵ uniformly in time.

Now we shall formally construct a quasi-periodic bifurcating series solution with two frequencies. The solution to be constructed is in the form

$$\begin{bmatrix} \underset{\sim}{u}(\underset{\sim}{x},s,t,\epsilon) \\ p(x,s,t,\epsilon) \\ \lambda(\epsilon) \\ \omega(\epsilon) \end{bmatrix} = \sum_{n=0} \epsilon^n \begin{bmatrix} \underset{\sim}{u}_n(\underset{\sim}{x},s,t) \\ p_n(x,s,t) \\ \lambda_n \\ \omega_n \end{bmatrix} \tag{3.6}$$

where (3.2) holds and ϵ is defined by the projection

$$\epsilon = [\epsilon \underset{\sim}{u}]$$

or

$$[\underset{\sim}{u}] = [\underset{\sim}{u}, \underset{\sim}{Z}_1^*(\underset{\sim}{x},s,t)] = 1 \quad . \tag{3.7}$$

When $\epsilon = 0$ and $\lambda = \lambda_o = 1/R_c$, we get an infinitesimal almost periodic solution

$$\epsilon \underset{\sim}{u}_o = 2\epsilon\, \mathrm{re}(\underset{\sim}{Z}_1) = \epsilon(e^{-is}\underset{\sim}{\zeta}(\underset{\sim}{x},t) + e^{+is}\bar{\underset{\sim}{\zeta}}(\underset{\sim}{x},t)) \quad . \tag{3.8}$$

We shall find the $\underset{\sim}{u}_n(x,s,t)$ (and p_n) in the form

$$\underset{\sim}{u}_n = \sum_{\ell=-N_n}^{N_n} e^{i\ell s} \underset{\sim}{u}_{n\ell}(\underset{\sim}{x},t) \quad , \tag{3.9}$$

where $\underset{\sim}{u}_{n\ell} = \bar{\underset{\sim}{u}}_{n(-\ell)}$ and $\underset{\sim}{u}_{n\ell}$ is $2\pi/\tilde{\omega}$ periodic in t.

There are two times in this problem: t with period $2\pi/\tilde{\omega}$ and $s = \omega(\epsilon)t$ with period 2π. We use the notion of two times because it is essential that the period of the natural oscillation $2\pi/\omega(\epsilon)$ be mapped into the fixed 2π periodic (in s) domain, whereas the period $2\pi/\tilde{\omega}$ is given externally and is fixed.

With these preliminaries aside, we are ready to construct the series (3.5) solution of

$$\hat{J}\underset{\sim}{u} + \epsilon \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} + \nabla p = 0 \,, \quad \text{div}\, \underset{\sim}{u} = 0 \,, \quad \underset{\sim}{u} = 0_{\partial\Omega} \,, \quad [\![\underset{\sim}{u}]\!] = 1 \quad . \tag{3.10}$$

When $\epsilon = 0$ we have $\hat{J}_o \underset{\sim}{u}_o + \nabla p = 0$, $[\![\underset{\sim}{u}_o]\!] = 1$ and the other conditions. The solution is given by (3.8). At order m we must solve the problem

$$\hat{J}_o \underset{\sim}{u}_m + \omega_m \frac{\partial \underset{\sim}{u}_o}{\partial s} + L_m \underset{\sim}{u}_o + \underset{\sim}{F}_m + \nabla p_m = 0 \quad , \tag{3.11a}$$

$$\text{div}\, \underset{\sim}{u}_m = 0 \,, \quad \underset{\sim}{u}_m = 0_{\partial\Omega(t)} \quad \text{and} \quad [\underset{\sim}{u}_m] = 0 \quad . \tag{3.11b,c,d}$$

Here,

$$L_m = \frac{1}{m!}\frac{\partial^m}{\partial\lambda^m} L = (\lambda_1 L_\lambda)_{m-1} = \sum_{\ell+\nu=m-1} \lambda_{\ell+1}(L_\lambda)_\nu \quad ,$$

$$\underset{\sim}{F}_m = \sum_{\ell+j=m-1} (\underset{\sim}{u}_\ell \cdot \nabla)\underset{\sim}{u}_j + \widetilde{\sum_{\ell+j=m}} L_\ell \underset{\sim}{u}_j + \widetilde{\sum_{\ell+j=m}} \omega_\ell \frac{\partial \underset{\sim}{u}_j}{\partial s} \quad , \tag{3.12}$$

and the summation with the tilda overbar denotes a sum over all non-negative integers $\ell + j = m$ minus terms of order m. The unknowns of problem (3.11) are $\underset{\sim}{u}_m$, p_m, λ_m and ω_m. The vector $\underset{\sim}{F}_m$ contains only terms known from lower order computation (order $\ell < m$).

The unknown coefficients λ_m and ω_m are determined by application of the solvability condition of Lemma 2. First, we note that (see 2.7)

$$[\![L_\lambda \underset{\sim}{u}_o]\!] = [\![L_\lambda \underset{\sim}{\zeta}, \underset{\sim}{\zeta}^*]\!] = \gamma' \quad .$$

The condition $\xi(\lambda_o) > 0$ means that the periodic solution loses stability as R is increased past R_c. Noting now that

$$[\![\frac{\partial \underset{\sim}{u}_o}{\partial s}]\!] = -i$$

and using (2.4b), we have

$$-i\omega_m + [\![L_m \underset{\sim}{u}_o]\!] + [\![\underset{\sim}{F}_m]\!] = 0 \tag{3.13}$$

where

$$\|L_m \underset{\sim}{u}_o\| = \lambda_m \gamma' + \tilde{\sum_{\ell+\nu=m-1}} \lambda_{\ell+1} \|(L_\lambda)_\nu \underset{\sim}{u}_o\| \quad . \tag{3.14}$$

We see that the real and imaginary parts of (3.13) determine the values λ_m and ω_m :

$$-\lambda_m \xi' = \mathrm{re}\{\|\underset{\sim}{F}_m\| + \tilde{\sum_{\ell+\nu=m-1}} \lambda_{\ell+1} \|(L_\lambda)_\nu \underset{\sim}{u}_o\|\}$$

and

$$\omega_m - \lambda_m \;\mathrm{im}\; \gamma' = \mathrm{im}\{\|\underset{\sim}{F}_m\| + \tilde{\sum_{\ell+\nu=m-1}} \lambda_{\ell+1} \|(L_\lambda)_\nu \underset{\sim}{u}_o\|\} \quad .$$

For $m = 1$ we find

$$\underset{\sim}{F}_1 = \underset{\sim}{u}_o \cdot \nabla \underset{\sim}{u}_o$$

and

$$\lambda_1 = \omega_1 = 0 \quad . \tag{3.15}$$

When $m = 2$ we have

$$\lambda_2 = -\mathrm{re}\|\underset{\sim}{F}_2\|/\xi' \tag{3.16a}$$

and

$$\omega_2 - \lambda_2 \;\mathrm{im}\; \gamma' = \mathrm{im}\,\|F_2\| \quad . \tag{3.16b}$$

We next want to show that quasi-periodic solutions of (3.12) are in the form (3.9) and also to give conditions under which the $\underset{\sim}{u}_{n\ell}(\underset{\sim}{x},t)$ may be uniquely determined. We may write (3.12) as

$$\hat{J}_o \underset{\sim}{a} + \nabla p = \underset{\sim}{b} \;, \quad \mathrm{div}\; \underset{\sim}{a} = 0 \;, \quad \underset{\sim}{a} = 0|_{\partial\Omega} \;, \quad \|\underset{\sim}{a}\| = 0 \quad . \tag{3.17a,b,c,d}$$

We argue by induction. Assuming that (3.8) holds for $\underset{\sim}{u}_m = \underset{\sim}{a}$ when $m < n$, we have $\underset{\sim}{b}$ in the form

$$\underset{\sim}{b} = \Sigma\, e^{i\ell s} \underset{\sim}{b}_\ell(\underset{\sim}{x},t) \quad ,$$

where the summation is over a finite number of terms and $\underset{\sim}{b}_\ell(x,t)$ is $2\pi/\tilde{\omega}$ periodic in t . The induction may be started because $\underset{\sim}{b}_1 = \underset{\sim}{u}_o \cdot \nabla \underset{\sim}{u}_o$ is in the required form. Since (3.17) is a linear problem, the full solution is obtained as a sum of the solutions of problems with

$$\underset{\sim}{b} = e^{i\ell s} \underset{\sim}{b}_\ell(\underset{\sim}{x},t) \quad . \tag{3.18}$$

Noting that through $L(\underset{\sim}{\tilde{u}};\lambda)$ the operator $\hat{\underset{\sim}{J}}_o$ is $2\pi/\omega$ periodic in t we seek quasi-periodic solutions of (3.17) with $\underset{\sim}{b}$ in the form (3.18) as Fourier polynomials

$$\underset{\sim}{a} = \Sigma\, e^{i\ell s} \underset{\sim}{a}_\ell(\underset{\sim}{x},t) \ , \quad p = \Sigma\, e^{i\ell s} p_\ell(\underset{\sim}{x},t)$$

where a_ℓ, p_ℓ are $2\pi/\tilde{\omega}$ periodic in t . Substituting (3.18) into (3.17), we arrive at the system of equations considered in Lemma 2.

$$[i\ell\omega + \frac{\partial}{\partial t} + L(\underset{\sim}{\tilde{u}};\lambda)]\ \underset{\sim}{a}_\ell + \nabla p = \underset{\sim}{b} \quad , \tag{3.19a}$$

$$\text{div}\ \underset{\sim}{a}_\ell = 0 \ , \quad \underset{\sim}{a}_\ell = 0|_{\partial\Omega} \ , \quad [\underset{\sim}{a}_\ell]_i = 0 \quad . \tag{3.19b,c,d}$$

We note that the reduction of the quasi-periodic problem (3.17) to the periodic problem (3.19) can be carried out even when $\omega(\epsilon)$ and ω are rationally dependent. Then, for a dense set of values ϵ near $\epsilon = 0$, our quasi-periodic solution is periodic with period $2\pi/\omega$. What is crucial here, however, is that $\omega(0) = \omega_o$, and ω should be rationally independent.

Assuming the validity of the series construction, we can prove the following lemma:

<u>Lemma 3</u>. $\lambda(\epsilon)$ <u>and</u> $\omega(\epsilon)$ <u>are even functions</u>.

It follows from this that in the case of a simple eigenvalue, the bifurcation of periodic solutions into quasi-periodic solutions with two frequencies is one-sided. To prove the lemma we shall consider polynomials of Floquet type

$$\underset{\sim}{G} = \sum_\ell e^{i\ell s} \underset{\sim}{g}_\ell(\underset{\sim}{x},t) \quad ,$$

where $\underset{\sim}{g}_\ell(\underset{\sim}{x},t)$ is periodic in t with period $2\pi/\tilde{\omega}$ and $\omega/\tilde{\omega}(\epsilon)$ is not rational. We call $\underset{\sim}{G}$ an even (odd) polynomial if the integers are even (odd). If $\underset{\sim}{G}$ is an even polynomial, then

$$[\underset{\sim}{G}] = 0 \quad .$$

By the nature of the perturbation construction, $\underset{\sim}{u}_m$ is a polynomial of the Floquet type. The hypothesis of the induction is that

$$\underset{\sim}{u}_{2\ell} \text{ is an odd polynomial} \quad ,$$
$$\underset{\sim}{u}_{2\ell+1} \text{ is an even polynomial} \quad , \tag{3.20a}$$

and

$$\lambda_{2\ell+1} = \omega_{2\ell+1} = 0 \tag{3.20b}$$

when $\ell < m$. When $\ell = 0$ we find that $\underset{\sim}{u}_o = e^{-is}\underset{\sim}{\zeta} + e^{is}\bar{\underset{\sim}{\zeta}}$ is an odd polynomial. Since $\underset{\sim}{F}_1 = \underset{\sim}{u}_o \cdot \nabla \underset{\sim}{u}_o$ is an even polynomial, $[\underset{\sim}{F}_1] = 0$ and by (3.15) $\lambda_1 = \omega_1 = 0$. Then $L_1 = 0$ and the only inhomogeneous term in (3.20a) when $m = 1$ is the even polynomial $\underset{\sim}{F}_1$. It follows that $\underset{\sim}{u}_1$ is an even polynomial and the hypotheses (3.20) of the induction are true when $m = 1$. Noting now that

$$L_{2\ell+1} = (\lambda_1 L_\lambda)_{2\ell} = \lambda_{2\ell+1} L_\lambda + \lambda_{2\ell}\lambda_1 L_{\lambda\lambda} + --- \quad ,$$

we verify that if $\lambda_{2\ell+1} = 0$ for $\ell \leq m$, $L_{2\ell+1} = 0$ for $\ell \leq m$. Given (3.20a,b) and $L_{2\ell+1}$ for $\ell \leq m$, we verify that $\underset{\sim}{F}_{2m}$ is odd. Returning now to (3.12a) we note that the inhomogeneous terms in the equation governing $\underset{\sim}{u}_{2m}$ are odd polynomials; hence $\underset{\sim}{u}_{2m}$ is odd. Now evaluating (3.13) with $m = 2m+1$, $[\underset{\sim}{F}_{2m+1}] = 0$, and

$$[L_{2m+1}\underset{\sim}{u}_o] = \lambda_{2m+1}[L_\lambda \underset{\sim}{u}_o] = \lambda_{2m+1}\left(\frac{d\lambda}{d\xi}\right)^{-1} \quad ,$$

we get

$$\omega_{2m+1} = \lambda_{2m+1} = 0 \quad .$$

Returning again to (3.12a) with m replaced by $2m+1$, we note that the inhomogeneous terms are even; hence $\underset{\sim}{u}_{2m+1}$ is even.

§4. STABILITY OF THE QUASI-PERIODIC BIFURCATING SOLUTION WITH TWO FREQUENCIES

In trying to determine the stability of the almost periodic motion with two frequencies, we are led to a generalization of Floquet theory which develops an idea suggested by Landau.* Let $\underset{\sim}{v}(\underset{\sim}{x},s,t;\epsilon)$ be a small disturbance of $\epsilon\underset{\sim}{u}$. Then

$$\omega\frac{\partial \underset{\sim}{v}}{\partial s} + \frac{\partial \underset{\sim}{v}}{\partial t} + L(\lambda;t)\underset{\sim}{v} + \epsilon(\underset{\sim}{u}\cdot\nabla\underset{\sim}{v} + \underset{\sim}{v}\nabla\underset{\sim}{u}) + \nabla p' = 0 \tag{4.1a}$$

$$\operatorname{div} \underset{\sim}{v} = 0 , \quad \underset{\sim}{v} = 0|_{\partial\Omega} \qquad . \tag{4.1b,c}$$

The coefficients of (4.1a) are almost periodic functions with two frequencies. Landau (1944; 1959, p. 106) suggests that we look for solutions of (4.1) of the form

$$\underset{\sim}{u} = e^{-\sigma s}\underset{\sim}{\Gamma}(\underset{\sim}{x},s,t;\epsilon) , \quad p' = e^{-\sigma s}p(x,s,t;\epsilon) \tag{4.2}$$

where $\sigma = \sigma(\lambda)$ is a (possibly) complex number, called the Floquet exponent, and $\underset{\sim}{\Gamma}$ and p' are 2π periodic in $s = \omega(\epsilon)t$ and $2\pi/\tilde{\omega}$ periodic in t. With the "two" times so introduced, the rest of the argument is essentially that given in Section 5 of (JS).

Substitution of (4.2) into (4.1) leads to

$$-\sigma\omega(\epsilon)\underset{\sim}{\Gamma} + \omega\frac{\partial\underset{\sim}{\Gamma}}{\partial s} + \frac{\partial\underset{\sim}{\Gamma}}{\partial t} + L\,\underset{\sim}{\Gamma} + \epsilon(\underset{\sim}{u}\cdot\nabla\underset{\sim}{\Gamma} + \underset{\sim}{\Gamma}\cdot\nabla\underset{\sim}{u}) + \nabla p = 0 \qquad . \tag{4.3}$$

When $\epsilon = 0$, $\omega(0) = \omega_o$, $\sigma = \sigma_o$, $\lambda = \lambda_o$, $L = L_o$ and two solutions of (4.3) are given by

$$\sigma_o = 0 \text{ and } \Gamma = e^{-is}\zeta(x,t) , \quad \text{or } \bar{\Gamma} = e^{is}\bar{\zeta}(x,t) \qquad .$$

Thus for $\epsilon = 0$, there is a double Floquet exponent at the origin. We get all the Floquet exponents by looking for solutions of the form $\underset{\sim}{\Gamma}_o(\underset{\sim}{x},s,t) = e^{iks}\underset{\sim}{\Phi}(x,t)$, where k is an integer and $\underset{\sim}{\Phi}$ is $2\pi/\tilde{\omega}$ periodic in t. This leads us to the eigenvalue problem

$$(-\sigma\omega_o + ik)\underset{\sim}{\Phi} + J_o\underset{\sim}{\Phi} + \nabla p = 0 \qquad . \tag{4.4}$$

*See the extended quote by Landau at the end of this section.

Since all eigenvalues $\gamma = \xi \pm i\omega_o$, except the two for which $\xi = 0$ are positive ($\xi > 0$) at criticality and

$$-\gamma\underset{\sim}{\zeta} + J_o\underset{\sim}{\zeta} + \nabla p = 0 \quad ,$$

we have

$$\sigma = ik/\omega_o + \gamma \quad , \tag{4.5}$$

and all other Floquet exponents have positive real parts and lead to stability.

Now consider the case $\epsilon = 0$. Assuming that the perturbation theory holds for the Floquet exponents, we have only to worry about the critical ones, namely, those at the origin. One solution of (4.1)

$$\underset{\sim}{v} = \frac{\partial \underset{\sim}{u}}{\partial s} , \quad \sigma = 0$$

always exists even with $\epsilon \neq 0$. To check this, differentiate (3.10) with respect to s (holding t constant)

$$\omega\frac{\partial^2 \underset{\sim}{u}}{\partial s^2} + L(\tilde{\underset{\sim}{u}},\lambda)\frac{\partial \underset{\sim}{u}}{\partial s} + \epsilon\underset{\sim}{u} \cdot \nabla\frac{\partial \underset{\sim}{u}}{\partial s} + \epsilon\frac{\partial \underset{\sim}{u}}{\partial s} \cdot \nabla \underset{\sim}{u} + \nabla\frac{\partial p}{\partial s} = 0 \quad . \tag{4.6}$$

Therefore, of the two eigenvalues $\sigma = 0$ at the origin when $\epsilon = 0$, one remains at the origin even when $\epsilon \neq 0$.

The problem now is to calculate the second Floquet exponent. We look for an eigenfunction of (4.3) in the form

$$\underset{\sim}{\Gamma}(\underset{\sim}{x},s,\epsilon) = a(\epsilon)\frac{\partial \underset{\sim}{u}}{\partial s} + \underset{\sim}{\gamma}(\underset{\sim}{x},s,\epsilon) \quad , \tag{4.7}$$

where

$$\underset{\sim}{\gamma}(\underset{\sim}{x},s,\epsilon) = \underset{\sim}{u}_o(\underset{\sim}{x},s) + \epsilon\underset{\sim}{\gamma}_1(\underset{\sim}{x},s,\epsilon)$$

and $a(\epsilon)$ is a coefficient to be determined. Substituting (4.7) into (4.3) and using (4.6), we get for a, σ, and $\underset{\sim}{\gamma}$ the equation

$$\omega\frac{\partial \underset{\sim}{\gamma}}{\partial s} - \sigma\omega\underset{\sim}{\gamma} + \frac{\partial \underset{\sim}{\gamma}}{\partial t} + L\underset{\sim}{\gamma} + \epsilon(\underset{\sim}{u} \cdot \nabla\underset{\sim}{\gamma} + \underset{\sim}{\gamma} \cdot \nabla\underset{\sim}{u}) - \sigma\omega a\frac{\partial \underset{\sim}{u}}{\partial s} + \nabla p = 0 \quad . \tag{4.8}$$

We again seek a solution in series

$$\begin{bmatrix} \underset{\sim}{\gamma}(\underset{\sim}{x},s,\epsilon) \\ p(\underset{\sim}{x},s,\epsilon) \\ \sigma(\epsilon) \\ a(\epsilon) \end{bmatrix} = \sum_{\ell=0} \epsilon^{\ell} \begin{bmatrix} \underset{\sim}{\gamma}_{\ell}(\underset{\sim}{x},s) \\ p_{\ell}(\underset{\sim}{x},s) \\ \sigma_{\ell} \\ a_{\ell} \end{bmatrix} .$$

We shall require, as in the construction of the quasi-periodic solutions, that

$$[\underset{\sim}{\gamma}_o] = 1 , \quad [\underset{\sim}{\gamma}_{\ell}] = 0 , \quad \ell > 0 .$$

As in Section 3, we get

$$-\sigma_o\omega_o(a_o\frac{\partial \underset{\sim}{u}_o}{\partial s} + \underset{\sim}{\gamma}_o) + J_o\underset{\sim}{\gamma}_o + \nabla p_o = 0 ,$$

$$\text{div } \underset{\sim}{\gamma}_o = 0 , \quad \underset{\sim}{\gamma}_o = 0|_{\partial\Omega} .$$

Using (3.11) and (4.7), we get

$$-\sigma_o\omega_o(-ia_o + 1) = 0 . \tag{4.9}$$

Hence, $\sigma_o = 0$ and $\underset{\sim}{\gamma}_o = \underset{\sim}{u}_o$.

At first order we must solve the problem

$$-\sigma_o\omega_o(a_o\frac{\partial \underset{\sim}{u}_o}{\partial s} + \underset{\sim}{u}_o) + J_o\underset{\sim}{\gamma}_1 + 2\underset{\sim}{u}_o \cdot \nabla\underset{\sim}{u}_o + \nabla p_1 = 0 \tag{4.10}$$

and

$$\text{div } \underset{\sim}{\gamma}_1 = 0 , \quad \underset{\sim}{\gamma}_1 = 0|_{\partial\Omega} \text{ and } [\underset{\sim}{\gamma}_1] = 0 .$$

Applying the orthogonality conditions to (4.10), we get $\sigma_1 = 0$ since $[\underset{\sim}{u}_o \cdot \nabla\underset{\sim}{u}_o] = 0$. Now, comparing (4.10) for $\underset{\sim}{\gamma}_1$ with (3.12) and (3.15) for $\underset{\sim}{u}_1$, we see that $\underset{\sim}{\gamma}_1 = 2\underset{\sim}{u}_1$, since $[\underset{\sim}{u}_1] = [\underset{\sim}{\gamma}_1] = 0$. This relation enables us to establish the stability of supercritical bifurcation and the instability of subcritical bifurcation.

At second order

$$J_o\underset{\sim}{\gamma}_2 - \sigma_o\omega_o(a_o\frac{\partial \underset{\sim}{u}_o}{\partial s} + \underset{\sim}{u}_o) + \omega_2\frac{\partial \underset{\sim}{u}_o}{\partial s} - \lambda_2\Delta\underset{\sim}{u}_o + \underset{\sim}{T}_2 + \nabla p_2 = 0 , \tag{4.11}$$

$$\text{div } \underset{\sim}{\gamma}_2 = 0 \quad \underset{\sim}{\gamma}_2 = 0|_{\partial\Omega} \text{ and } [\underset{\sim}{\gamma}_2] = 0 ,$$

where, since $\underset{\sim}{\gamma}_1 = \underset{\sim}{u}_1$

$$\underset{\sim}{T}_2 = 3(\underset{\sim}{u}_1 \cdot \nabla \underset{\sim}{u}_o + \underset{\sim}{u}_o \cdot \nabla \underset{\sim}{u}_1) = 3\underset{\sim}{F}_2 \quad ,$$

where $\underset{\sim}{F}_2$ is given by (3.13) using (3.15).

Applying the solvability conditions to (4.11) using (3.14)

$$-\sigma_o \omega_o (1 - i a_o) - i\omega_2 + \lambda_2 \gamma' + 3[\underset{\sim}{F}_2] = 0 \quad .$$

Taking real and imaginary parts, we get using (3.16a,b)

$$\sigma_2 \omega_o + 2\lambda_2 \xi' = 0 \quad ,$$

$$\sigma_2 \omega_o a_o + 2\omega_2 - \lambda_2 \, \text{im} \, \gamma' = 0 \quad .$$

Hence,

$$\sigma_2 = -\frac{2\lambda_2 \xi'}{\omega_o} \tag{4.12}$$

and assuming $\lambda_2 \neq 0$,

$$a_o = -\frac{2\omega_2 - \lambda_2 \text{im}\gamma'}{\sigma_2} \quad .$$

Equation (4.12) contains an important result. Recall that solutions of the linearized equation decay when $\sigma > 0$ and that $\xi' > 0$.

Subcritical quasi-periodic motions with two frequencies ($\lambda_2 > 0$) are unstable ($\sigma_2 > 0$) and supercritical motions ($\lambda_2 > 0$) are stable ($\sigma_2 > 0$) in the linearized theory.

The same formal perturbation construction which was just used for quasi-periodic bifurcations of periodic solutions could be applied to the problem of bifurcation of solutions with n frequencies into solutions with $n + 1$ frequencies. Our construction is close to Landau's sketch of the way repeated bifurcation leads to turbulence. In this process new frequencies are introduced by instability when the Floquet exponents are complex. In Landau's view, a steady motion loses its stability to time periodic motion. (In the words of Landau (1944, p. 343),

"As Re is further increased, this periodic motion, too eventually becomes unsteady. The investigation of its unsteadiness should be conducted in a manner analogous to that described above. The role of the principal motion is now played by the periodic motion $\underset{\sim}{v}_o(\underset{\sim}{x},t)$ of frequency ω_1. Substituting $\underset{\sim}{v} = \underset{\sim}{v}_o + \underset{\sim}{v}_2$ with small $\underset{\sim}{v}_2$ into the equation of motion, we shall again obtain for $\underset{\sim}{v}_2$ a linear equation, but this time the coefficients of this equation are not only functions of the coordinates, but of time also; with respect to time, they are are periodic functions with a period $2\pi/\omega_1$. The solution of such an equation should be sought in the form $\underset{\sim}{v}_2 = \underset{\sim}{\Pi}(\underset{\sim}{x},t)e^{-i\Omega t}$ where $\underset{\sim}{\Pi}(\underset{\sim}{x},t)$ is a periodical function of time (with a period $2\pi/\omega_1$). Unsteadiness sets in again when the frequency $\Omega_2 = \omega_2 + i\gamma_2$ turns up whose imaginary part γ_2 is positive and the corresponding real part ω_2 determines then the newly appearing frequency."

"The result is a quasi-periodic motion characterized by two different periods. It involves two arbitrary quantities (phases), i.e., has two degrees of freedom.

"In course of the further increase of the Reynolds number, new and new periods appear in succession, and the motion assumes an involved character typical of a developed turbulence. For every value of Re the motion has a definite number of degrees of freedom; in the limit, as Re tends to infinity, the number of degrees of freedom becomes likewise infinitely large."

Unfortunately, this very plausible description of transition to turbulence encounters difficulties when one tries to make it precise.* Though our construction is natural for Landau's conjectures (and works when $\tilde{\omega} = 0$), it cannot be generally carried out when $\tilde{\omega}/\omega_o$ is rational, and it leads to small divisors when $\tilde{\omega}/\omega_o$ is irrational (see Lemma 2). The first difficulty is perhaps superficial since in the rationally dependent case, the basic decomposition into two times does not follow from the solution at zeroth order. In the rationally independent case, the possibility remains that our series is asymptotic and that an asymptotic series

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*It is argued in (JS) that the instability of subcritical bifurcating periodic is inconsistent with a bifurcation description of transition to turbulence. In the subcritical case the disturbances escape the domain of attraction of the basic flow and must snap through the small norm unstable, bifurcating, periodic solution to a solution with a larger norm. It follows that the Landau-Hopf conjecture can hold only in the case of supercritical bifurcations. We have seen that the conjecture may fail also in the supercritical case. This failure can stem from considerations associated with the solvability Lemma 2 and with small divisors. A similar argument against the Landau-Hopf conjecture has recently been given by D. Ruelle and F. Takens (1971).

for $\omega(\epsilon)$ has a physical relevance. In reservation, we note that any result which depends in a strong way on rational independence must be questioned on physical grounds, and in the rational case the series cannot be constructed unless exceptional special conditions are satisfied (Lemma 2).

§5. THE PERIODIC BIFURCATING SOLUTIONS AND THEIR STABILITY

In the previous section we assumed that the basic periodic solution with frequency ω loses stability by a complex pair of eigenvalues γ of (1.4) crossing the $\text{im}\,\gamma = \eta(R_c) = \omega_o$ axis from the right. This was the case (b) (complex eigenvalue) of the introduction, and the solution which bifurcates off is quasi-periodic. Now we consider the easier case (a) in which the basic periodic solution loses its stability when a simple real eigenvalue $\gamma = \xi$ crosses through the origin of the complex γ plane as R is increased through R_c .

By our assumption, $\gamma = \xi$ $(\eta = 0)$, the spectrum of the operator J_o is real-valued and

$$\underset{\sim}{Z}_1 = \bar{\underset{\sim}{Z}}_1 = \underset{\sim}{\zeta}\ , \quad \underset{\sim}{Z}_2 = \bar{\underset{\sim}{Z}}_2 = \underset{\sim}{\zeta}^* \quad .$$

The perturbation problem to be considered is again (1.2) supplemented by the normalizing condition

$$1 = [\underset{\sim}{u}] = [\underset{\sim}{u}, \underset{\sim}{\zeta}^*] \quad . \tag{5.1}$$

The proposed solutions

$$\begin{bmatrix} \underset{\sim}{u}(x,t,\epsilon) \\ p(x,t,\epsilon) \\ \lambda(\epsilon) \end{bmatrix} = \sum_{n=0} \epsilon^n \begin{bmatrix} \underset{\sim}{u}_n(x,t) \\ p_n(x,t) \\ \lambda_n \end{bmatrix} \tag{5.2}$$

of (1.2) and (5.1) are to be periodic in t with period $2\pi/\tilde{\omega}$. Substitution of (5.2) into (1.2) and (5.1) leads to

$$J_o \underset{\sim}{u}_o + \nabla p_o = 0\ , \quad [\underset{\sim}{u}_o] = 1 \tag{5.3}$$

and

$$J_o \underset{\sim}{u}_m + L_m \underset{\sim}{u}_o + \tilde{\underset{\sim}{F}}_m + \nabla p_m = 0 \ , \quad [\underset{\sim}{u}_m] = 0 \quad . \tag{5.4}$$

Here $\underset{\sim}{u}_o$ and $\underset{\sim}{u}_m$ are solenoidal and vanish on the boundary $\partial\Omega(t)$ of Ω and $\underset{\sim}{u}_o$, $\underset{\sim}{u}_m$, p_o, p_n are $2\pi/\tilde{\omega}$ periodic in t. The vector field $\tilde{\underset{\sim}{F}}_m$ is the same as $\underset{\sim}{F}_m$ defined by (3.13) when $\omega_\ell = 0$.

At zeroth order

$$\lambda_o = \frac{1}{R_c} \ , \quad \underset{\sim}{u}_o(\underset{\sim}{x},t) = \underset{\sim}{\zeta} \quad . \tag{5.5}$$

To solve (5.4) it is necessary to have

$$[L_m \underset{\sim}{u}_o] + [\underset{\sim}{F}_m] = 0 \quad . \tag{5.6}$$

If the operator $\tilde{L}(\lambda_o,t)$ were independent of time, the condition (5.6) would also be sufficient for bounded invertibility of the operator J_o. Assuming this, we could apply here the implicit function method used in (JS) to prove convergence. The value of λ_m is fixed by (5.6) (see (3.14a)). Hence, equations (5.4) are uniquely solvable.

When $m = 1$ we find, using (3.14a) and (3.13), that when $\epsilon = 0$,

$$\frac{d\lambda}{d\epsilon} \Big/ \frac{d\lambda}{d\xi} + [\underset{\sim}{\zeta}^*, \ \underset{\sim}{\zeta} \cdot \nabla\underset{\sim}{\zeta}] = 0 \quad . \tag{5.7}$$

Several interesting consequences follow from (5.7). First we note that if the bracket in (5.7) is non-zero,

$$\frac{d\lambda}{d\epsilon} = - \frac{1}{R^2} \frac{dR(0)}{d\epsilon} \neq 0 \quad .$$

In this case

$$R(\epsilon) = R_c + \frac{dR(0)}{d\epsilon} \epsilon + \frac{1}{2} \frac{d^2R(0)}{d\epsilon^2} \epsilon^2 + \ldots \tag{5.8}$$

has different values for small positive and negative values of ϵ. Returning to (5.1) we see that

$$\epsilon = [\underset{\sim}{V} - \tilde{\underset{\sim}{u}}, \ \underset{\sim}{\zeta}^*]$$

gives the sign of the projection of $\underset{\sim}{V}(\epsilon)$ - $\underset{\sim}{u}$ onto the eigensubspace of J_o and can be interpreted physically as the sign of the averaged motion (see (J)). Now (5.8) shows that if a motion of one sign branches off supercritically motion of the other sign branches subcritically. There are, therefore, two physically distinct solutions.

When $\lambda_1 = 0$ the sign of the bifurcation is determined by the sign of

$$\frac{d^2R(0)}{d\epsilon^2} = -R_c^2 \frac{d^2\lambda}{d\epsilon^2} ,$$

provided that this derivative does not vanish.

We turn now to the study of the stability of the periodic bifurcating solution. Setting

$$\underset{\sim}{V} = \underset{\sim}{\tilde{u}} + \epsilon \underset{\sim}{u} + \underset{\sim}{\nu} \tag{5.9}$$

into (1.1) and using the equations satisfied by the periodic basic solution u and the bifurcating periodic solution $\underset{\sim}{u}$, we find, after linearizing, that

$$J_o \underset{\sim}{\nu} + \epsilon \underset{\sim}{u} \cdot \nabla \underset{\sim}{\nu} + \epsilon \underset{\sim}{\nu} \cdot \nabla \underset{\sim}{u} + \nabla p = 0 \tag{5.10a}$$

and

$$\text{div } \underset{\sim}{\nu} = 0 , \quad \underset{\sim}{\nu} = 0|_{\partial\Omega(t)} , \quad [\underset{\sim}{\nu}] = 1 \quad . \tag{5.10b,c,d}$$

Again invoking Floquet theory

$$\underset{\sim}{\nu} = e^{-\sigma t}\underset{\sim}{\psi}(\underset{\sim}{x},t,\epsilon) \quad ,$$

where $\underset{\sim}{\psi}(\underset{\sim}{x},t,\epsilon)$ is $2\pi/\tilde{\omega}$ periodic in t, we get

$$-\sigma\underset{\sim}{\psi} + J_o\underset{\sim}{\psi} + \epsilon u \cdot \nabla\underset{\sim}{\psi} + \epsilon\underset{\sim}{\psi} \cdot \nabla\underset{\sim}{u} + \nabla p = 0 \quad . \tag{5.11}$$

We seek solutions of (5.10,11), $\underset{\sim}{\psi}(\underset{\sim}{x},t,\epsilon)$, $p(\underset{\sim}{x},t,\epsilon)$, $\sigma(\epsilon)$ as power series in ϵ. When $\epsilon = 0$, since zero is a simple eigenvalue of J_o (cf. discussion associated with (1.6), we get

$$\sigma(0) = \sigma_o = 0 , \quad \underset{\sim}{\psi}(\underset{\sim}{x},t,0) = \underset{\sim}{\psi}_o = \underset{\sim}{\zeta} \quad . \tag{5.12}$$

At first order

$$J_o \underset{\sim}{\psi}_1 + \lambda_1 L_\lambda \underset{\sim}{\zeta} - \sigma_1 \underset{\sim}{\zeta} + 2\underset{\sim}{\zeta} \cdot \nabla \underset{\sim}{\zeta} + \nabla p_1 = 0 \quad .$$

Applying the solvability condition, we find that

$$\lambda_1 \Big/ \frac{d\lambda}{d\xi} - \sigma_1 + 2[\underset{\sim}{\zeta}^*, \ \underset{\sim}{\zeta} \cdot \nabla \underset{\sim}{\zeta}] = 0 \quad .$$

Using (5.7) we get

$$\sigma_1 = -\lambda_1 \Big/ \frac{d\lambda}{d\xi} = - \frac{dR(0)}{d\epsilon} \Big/ \frac{dR(0)}{d\xi} \quad . \tag{5.13}$$

Since $dR(0)/d\xi < 0$, we have two cases:

$$\frac{dR}{d\epsilon} > 0 , \quad \sigma_1 > 0 \quad ,$$

$$\frac{dR}{d\epsilon} < 0 , \quad \sigma_1 < 0 \quad .$$

We may conclude that in the case of a simple eigenvalue, when $\lambda_1 \neq 0$ supercritical bifurcating solutions are stable and subcritical bifurcating solutions are unstable.

When $\lambda_1 = 0$ we find that $\sigma_1 = 0$. Then

$$J_o \underset{\sim}{\psi}_1 + 2\underset{\sim}{\zeta} \cdot \nabla \underset{\sim}{\zeta} + \nabla p_1 = 0 \quad ,$$

and comparison with (5.4) when $m = 1$ shows that

$$\underset{\sim}{\psi}_1 = 2\underset{\sim}{u}_1 \quad . \tag{5.14}$$

At second order using (5.12,14), we get

$$-\sigma_2 \underset{\sim}{\zeta} + J \underset{\sim}{\psi}_2 + \lambda_2 L_\lambda \underset{\sim}{\zeta} + 3\underset{\sim}{u}_o \cdot \nabla \underset{\sim}{u}_1 + 3\underset{\sim}{u}_1 \cdot \nabla \underset{\sim}{u}_o + \nabla p_2 = 0 \quad .$$

Noting that

$$\underset{\sim}{u}_1 \cdot \nabla \underset{\sim}{u}_o + \underset{\sim}{u}_o \cdot \nabla \underset{\sim}{u}_1 = \underset{\sim}{F}_2 \qquad \text{and}$$

applying the solvability condition, we get

$$-\sigma_2 + \lambda_2 \Big/ \frac{d\lambda}{d\xi} + 3\ [\underset{\sim}{F}_2] = 0 \quad .$$

We note that when $\lambda_1 = 0$, equation (5.6) shows that

$$\lambda_2 \Big/ \frac{d\lambda}{d\xi} + [\underset{\sim}{F}_2] = 0 \quad .$$

Hence

$$\sigma_2 = -2\lambda_2 \Big/ \frac{d\lambda}{d\xi} = -2\frac{d^2 R(0)}{d\epsilon^2} \Big/ \frac{dR(0)}{d\xi} \quad .$$

We may again conclude that subcritical bifurcating solutions are unstable and supercritical bifurcating solutions are stable.

It is important that our demonstration of the instability of small subcritical bifurcating solutions implies that the branch of the solution on which $R(\epsilon)$ is decreasing is unstable. The instability of this decreasing branch might have been anticipated on physical grounds. For example, in the problem of convection the heat transported decreases when the temperature difference increases on this decreasing subcritical branch. Such solutions are not observed.

It is well-known, from energy estimates, that any single valued branch $R(\epsilon)$ must have a positive minimum. Therefore, it is anticipated that the subcritical branch will decrease with ϵ to a positive minimum and then turn up. This is just what happens in generalized convection problems (Joseph, 1971). There one can demonstrate that the subcritical branch regains its stability as it passes through its minimum.

$$\lambda_2 \;/\; \frac{d\lambda}{d\xi} + [\underset{\sim}{F}_2] = 0 \quad .$$

Hence

$$\sigma_2 = -2\lambda_2 \;/\; \frac{d\lambda}{d\xi} = -2\frac{d^2R(0)}{d\epsilon^2} \;/\; \frac{dR(0)}{d\xi} \quad .$$

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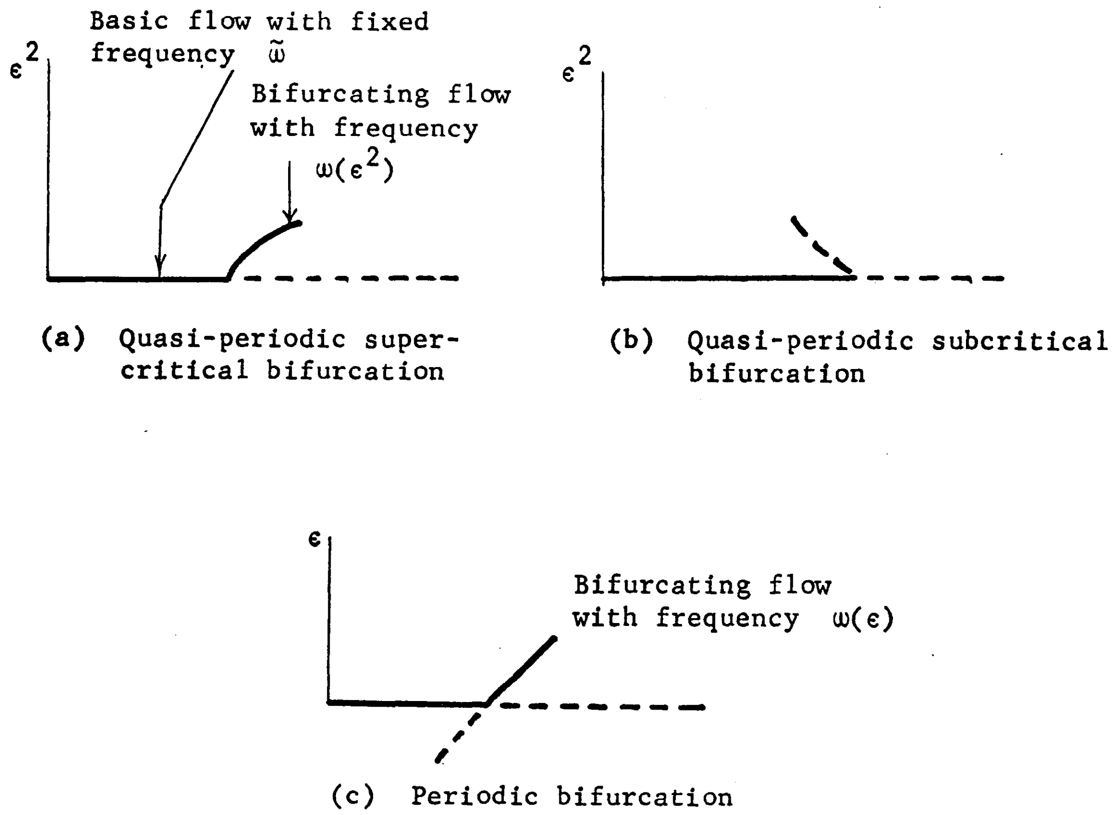
Fig. 1: Bifurcation of a time periodic basic flow with a fixed frequency $\tilde{\omega}$



CAPTION FOR FIG. 1

Fig. 1. Bifurcation diagrams for the solutions which bifurcate from a basic periodic solution of fixed frequency when the Floquet exponent is a simple eigenvalue of the linearized stability problem for the basic flow. The cases (a) and (b) represent the situation when the critical Floquet exponent is complex. In these cases the bifurcating solution is quasi-periodic; in addition to the fixed frequency ω a natural frequency $\omega(\epsilon)$ arises through instability. The bifurcation is symmetric about $\epsilon = 0$ for small ϵ, and only one-sided bifurcation can occur. Case (c) gives the typical situation which arises when the Floquet exponent is zero at criticality. In this case the bifurcating solution and basic solution have the same frequency, and the bifurcation can be two-sided.

Dashed lines show solutions which are unstable to small disturbances; heavy lines show solutions which are stable to small disturbances.

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