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Bifurcating Time Periodic Solutions and their Stability

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Table of Contents

1. Introduction

Equilibrium configurations of mechanical systems are often characterized by stability parameters, such as the Reynolds number R in fluid mechanics. When R is small, the equilibrium configuration is stable; but when R is raised to a certain critical value R_c, the old equilibrium solution can lose stability and bifurcation of a new equilibrium or periodic solutions can take place. The bifurcating solution, which grows in size from zero as $|R-R_c|$ is increased from zero, may bifurcate upward and exist for $R>R_c$ (supercritical), or it may bifurcate downward and exist for $R<R_c$ (subcritical).

In the case of a simple eigenvalue, it is known that steady bifurcating solutions which bifurcate supercritically are stable and those which bifurcate subcritically are unstable [1, 2]. This paper establishes the same result relative to bifurcating time periodic solutions.

Subcritical instability is of the "snap through" type; at subcritical Reynolds numbers, a sufficiently large disturbance of the steady motion may pass through the unstable small norm solution and come to rest on a solution with a large norm (see Fig. 1). The most striking manifestation of the snap through instability is in turbulent pipe flow. Here, under subcritical conditions, the system "snaps through" the unstable "Tollmein-Schlichting" wave to a stable "turbulent solution" with a large norm.

For two-dimensional parallel flows, the snap through instability is a rephrasing of the "threshold amplitude" idea of MEKSYN & STUART [3] and STUART [4] which emphasizes that large disturbances are to be expected to lie in the domain of attraction of a large norm "turbulent solution".

In this paper, the existence of periodic solutions and the instability of subcritical periodic solutions are proved, not only for special cases but for all flows in a bounded domain in which the linearized stability problem has a pair of complex conjugate simple eigenvalues at $R=R_c$.

The snap through picture is neither consistent with the Landau-Hopf conjecture of turbulence as a "loss of stability through repeated branching" [5, 6] nor is it especially encouraging to those who would study subcritical nonlinear stability by perturbation theory. The physical description of turbulence as repeated branching is evidently better suited to supercritical problems. We give a fuller description of these remarks in the concluding discussion (12).

The mathematical problems to be solved here are the following: A fluid mechanical system admits a steady state when a stability parameter, say R, is small. This steady state loses its stability as R is increased past a critical value R_c as follows: A pair of simple complex conjugate eigenvalues associated with the linearized stability problem for the steady motion crosses the imaginary axis as R crosses R_c. Then, (*a*) there exists a unique bifurcating time periodic motion which may be constructed as a Taylor series in a parameter ε (the Poincaré-Lindstedt parameter) and (*b*) this time periodic motion is stable when ε is small and the bifurcation is supercritical and is unstable when ε is small and the bifurcation is subcritical. The precise results and conditions under which they are valid are stated in Sections 8, 9, and 10. Stability and instability criteria are determined in the linearized sense, that is, by a computation of the Floquet exponents of the linearized stability problem for the periodic motions. The techniques applied here are not restricted to the Navier Stokes equations and carry over quite directly to general nonlinear parabolic systems, so that all the results obtained here hold in more general circumstances. All of the results obtained here were obtained by HOPF [6] for systems of ordinary differential equations (see the discussion below).

The main mathematical contributions of this study we feel are (1) a perturbation construction which leads to a simple calculation of the critical Floquet exponent, (2) the proof of the instability of subcritical bifurcations of time periodic motions and of the stability of supercritical bifurcations when ε is small, and (3) the proof of existence and analyticity of the bifurcating periodic motion and its stability by an implicit function method (replacing the method of dominating majorants used in [7, 8]). The Poincaré-Lindstedt perturbation method for constructing time periodic solutions of the Navier Stokes equations is developed explicitly.

The work here is an outgrowth of earlier work of HOPF [9], SATTINGER [2, 7], and JOSEPH [1] and bears a definite relation to the work of STUART [4], WATSON [10], and REYNOLDS & POTTER [11]. Unlike the work of the four authors last mentioned, we do not tie the analysis to a special physical problem but instead analyze a class of problems with a common spectral property: loss of stability by simple eigenvalues crossing the imaginary axis. When the eigenvalue is real, we get bifurcating stationary solutions; when the eigenvalue is complex, we get a time periodic bifurcating solution.

The place of the works just mentioned is best deferred to a later stage. The work of HOPF, however, is particularly germane to the topic of the paper and deserves an early summary.

Hopf's paper appeared in the *Sach. Akad. der Wiss.* of 1942 and, unfortunately, is not generally available. HOPF established, for ordinary differential equations, all the basic results established here for the Navier Stokes equations. His results and techniques do not appear to be widely known. Especially the results on the relationship of stability to the direction of bifurcation and the uniqueness of the bifurcating solution (up to phase shifts) do not seem to be well-known (our opinion).

In trying to establish Hopf's results for partial differential equations, one is forced to analyze his arguments with great care. His use of the implicit function theorem rests on the fact that solutions of ordinary differential equations are analytic in the time variable. Possibly this result could be established (locally in time would do) for the Navier Stokes equations and Hopf's arguments carried over verbatim. We have chosen to approach the bifurcation problem somewhat differently, and we believe this approach is simpler in the long run. Our approach combines the Poincaré-Lindstedt perturbation method with the ideas behind the Lyapunov-Schmidt method to give a simple reduction of the bifurcation problem to an implicit function theorem.

One further aspect of Hopf's paper deserves attention. His argument in the calculation of the critical Floquet exponent is somewhat involved and seems to be not entirely clear at one point. Although that argument is easily rectifiable (see Section 11), we believe the perturbation scheme of the present paper to be simpler and more direct.

2. Loss of Stability; the Eigenvalue Problem

We consider the Navier Stokes equations on a bounded domain:

$$\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}-\frac{1}{R}\Delta\boldsymbol{V}+\nabla\Pi+\tilde{\boldsymbol{F}}(\boldsymbol{x})=0, \tag{2.1 a}$$

$$\operatorname{div}\boldsymbol{V}=0, \qquad \boldsymbol{V}=\boldsymbol{f}(\boldsymbol{x})|_{\partial\Omega}. \tag{2.1 b, c}$$

Here $\boldsymbol{V}$ is the velocity, Π is the pressure, R is the Reynolds number, $\tilde{\boldsymbol{F}}$ is a steady body force prescribed on Ω, $\boldsymbol{f}(\boldsymbol{x})$ is the steady velocity at the boundary. In geometries which allow spatial periodicity, we may consider Ω to be a period cell and alter the problem in the usual way.

The system (2.1) has a steady solution $\boldsymbol{V}=\tilde{\boldsymbol{u}}(\boldsymbol{x}, R)$ for all values R (LERAY, see LADYSHENSKAYA [12]) provided only that (2.1 b, c) are consistent: the normal component of $\boldsymbol{f}$ integrated over the whole of the closed boundary vanishes. For small values of R, the steady solution is unconditionally stable [13]; for larger values of R, it may be stable to small disturbances and unstable to larger disturbances; at a yet larger value of $R=R_c$ the steady solution may lose stability and can be replaced by another type of stable solution, say a secondary steady motion, or a time-periodic motion.

We wish to delineate the circumstances under which the steady solution $\tilde{\boldsymbol{u}}$ loses its stability to a time-periodic motion

$$V=\tilde{\boldsymbol{u}}(\boldsymbol{x}, R)+\hat{\boldsymbol{v}}(\boldsymbol{x}, t, R) \tag{2.2}$$

where $\hat{\boldsymbol{v}}$ is periodic in t with period $2\pi/\omega(R)$. The stability of the steady solution $\tilde{\boldsymbol{u}}(R)$ is determined by the spectrum of the linear eigenvalue problem [14]

$$-\gamma\zeta+\mathscr{L}\zeta+\nabla\Pi=0, \quad \operatorname{div}\zeta=0, \quad \zeta=0|_{\partial\Omega} \tag{2.3}$$

where $\mathscr{L}(R)$ is defined by

$$\mathscr{L}\boldsymbol{u}=-\frac{1}{R}\Delta\boldsymbol{v}+(\tilde{\boldsymbol{u}}\cdot\nabla)\boldsymbol{v}+\boldsymbol{v}\cdot\nabla\tilde{\boldsymbol{u}}.$$

This problem arises from (2.2) and (2.1) by the method of the exponential time factor

$$\hat{\boldsymbol{v}}=e^{-\gamma t}\zeta.$$

Here $\gamma=\xi+i\eta$ is a complex number, $\xi>0$ means stability to small disturbances, and $\xi<0$ means instability. In addition to (2.3) we have the conjugate of (2.3)

$$-\bar{\gamma}\bar{\zeta}+\mathscr{L}\bar{\zeta}+\nabla\bar{\Pi}=0, \quad \operatorname{div}\bar{\zeta}=0, \quad \zeta=0|_{\partial\Omega}. \tag{2.4}$$

The statement "$\mathscr{L}$ has a pair of simple eigenvalues" means that the homogeneous problem

$$\frac{\partial\boldsymbol{u}}{\partial t}+\mathscr{L}\boldsymbol{u}+\nabla\Pi=0, \quad \operatorname{div}\boldsymbol{u}=0, \quad \boldsymbol{u}=0|_{\partial\Omega} \tag{2.5}$$

has only two periodic solutions

$$z_1=e^{-\gamma t}\zeta \quad \text{and} \quad z_2=\bar{z}_1 \tag{2.6}$$

with corresponding representations for the pressure, when $R=R_c$ and $\xi=0$.

As R varies, the eigenvalues also vary. We assume that as R crosses R_c, a complex conjugate pair of simple eigenvalues crosses the imaginary axis leading to the loss of stability of the steady motion $\tilde{\boldsymbol{u}}$. We are going to study the oscillatory motions bifurcating when R is close to R_c. As R tends to R_c, the oscillations tend to zero in amplitude and their frequency $\omega(R)$ tends to $\eta(R_c)=\operatorname{Re}(\gamma(R_c))\equiv\omega_0$.

The periodic motion necessarily satisfies the following problem

$$\begin{gathered}\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\mathscr{L}\hat{\boldsymbol{v}}+(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}+\nabla\hat{\Pi}=0\\ \operatorname{div}\hat{\boldsymbol{v}}=0, \quad \hat{\boldsymbol{v}}=0|_{\partial\Omega} \quad \text{and} \quad \hat{\boldsymbol{v}}(\boldsymbol{x}, t, R)=\hat{\boldsymbol{v}}\left(\boldsymbol{x}, t+\frac{2\pi}{\omega}, R\right).\end{gathered} \tag{2.7}$$

To solve (2.7), we introduce a parameter ε and write

$$\begin{gathered}t=s/\omega, \quad \hat{\boldsymbol{v}}=\varepsilon\hat{\boldsymbol{u}}(\boldsymbol{x}, s, \varepsilon), \quad \hat{\Pi}=\varepsilon\hat{p}(\boldsymbol{x}, s, \varepsilon),\\ \omega=\omega(\varepsilon) \quad \text{and} \quad R=1/\lambda(\varepsilon)\end{gathered} \tag{2.8}$$

where as $\varepsilon\to0$, $R\to R_c$, $\lambda\to\lambda_0$, $\hat{\boldsymbol{u}}\to\boldsymbol{u}_0(\boldsymbol{x}, s)$, $\hat{p}\to\hat{p}_0(\boldsymbol{x}, s)$ and $\omega\to\omega_0$. Substituting (2.8) into (2.7) we get

$$\omega\frac{\partial\hat{\boldsymbol{u}}}{\partial s}+\mathscr{L}\hat{\boldsymbol{u}}+\varepsilon\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}+\nabla\hat{p}=0. \tag{2.9}$$

Letting $\varepsilon\to0$ we see that $\boldsymbol{u}_0$ must satisfy

$$\omega_0\frac{\partial\boldsymbol{u}_0}{\partial s}+\mathscr{L}_0\boldsymbol{u}_0+\nabla p_0=0 \tag{2.10}$$

where $\mathscr{L}_0=\mathscr{L}(\lambda_0)$, $\lambda_0=1/R_c$ where $\boldsymbol{u}_0$ is solenoidal and $\boldsymbol{u}_0=0|_{\partial\Omega}$. One real solution of this equation is

$$\boldsymbol{w}_1(s)=2\operatorname{Re}(e^{-\gamma s}\zeta). \tag{2.11}$$

The general (real) solution is of the form $\boldsymbol{u}_0=A\boldsymbol{w}_1(s+\delta)$ with a similar representation for p_0. There are two degrees of arbitrariness here; we shall take $\delta=0$ and $A=1$. We shall see later that this involves no loss of generality; that is, we do not miss any periodic solutions by this procedure.

Further explorations of the periodic solutions and their stability require a somewhat more detailed analysis of (2.10) and the associated spectral problem. We turn next to this analysis.

3. The Adjoint Problem and the Solvability Conditions

To study our problem further, it is convenient to define L_2 and P_2 scalar products in the following way: Consider any pair of complex valued vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$; then

$$(\boldsymbol{a}, \boldsymbol{b})\equiv\int_\Omega\boldsymbol{a}\cdot\bar{\boldsymbol{b}}\,d\boldsymbol{x}\equiv(\boldsymbol{a}, \boldsymbol{b})_{L_2}$$

where the overbar denotes complex conjugate. We also define

$$(\nabla\boldsymbol{a}, \nabla\boldsymbol{b})\equiv\int_\Omega\nabla\boldsymbol{a}:\nabla\bar{\boldsymbol{b}}\,d\boldsymbol{x},$$

as an integral of a doubly contracted dyadic. The P_2 scalar product is defined as

$$(\boldsymbol{a}, \boldsymbol{b})_{P_2}\equiv\frac{\omega}{2\pi}\int_0^{2\pi/\omega}(\boldsymbol{a}, \boldsymbol{b})\,dt$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are now time dependent $2\pi/\omega$ periodic vector fields.

The adjoint operator $\mathscr{L}^*$ is defined as the operator for which

$$(\boldsymbol{a}, \mathscr{L}\boldsymbol{b})=(\mathscr{L}^*\boldsymbol{a}, \boldsymbol{b})$$

for any solenoidal vectors $\boldsymbol{a}$, $\boldsymbol{b}$ which vanish on $\partial\Omega$. The j-th component ($j=1, 2, 3$) of $\mathscr{L}^*\boldsymbol{a}$ is

$$(\mathscr{L}^*\boldsymbol{a})_j=-\lambda\Delta a_j-\tilde{\boldsymbol{u}}\cdot\nabla a_j-\tilde{\boldsymbol{u}}\cdot\partial_j\boldsymbol{a}.$$

The adjoint eigenvalue problem is

$$-\bar{\gamma}\zeta^*+\mathscr{L}^*\zeta^*+\nabla\Pi^*=0, \quad \operatorname{div}\zeta^*=0, \quad \zeta^*=0|_{\partial\Omega}. \tag{3.1}$$

If γ is an eigenvalue of $\mathscr{L}$, then $\bar{\gamma}$ is an eigenvalue of $\mathscr{L}^*$; hence, $\mathscr{L}_0^*$ has eigenvalues $\pm i\omega_0$. The homogeneous adjoint problem

$$-\frac{\partial \boldsymbol{u}}{\partial t}+\mathscr{L}^*\boldsymbol{u}+\nabla \Pi^*=0, \qquad \operatorname{div}\boldsymbol{u}^*=0, \qquad \boldsymbol{u}^*=0|_{\partial\Omega}$$

has the two solutions

$$z_1^*=e^{-\bar{\gamma}t}\zeta^* \quad \text{and} \quad z_2^*=\overline{z_1^*} \tag{3.2}$$

with corresponding representations for the pressure.

The orthogonality relations

$$(\zeta,\overline{\zeta^*})=(\bar{\zeta},\zeta^*)=0 \tag{3.3}$$

follow from multiplying (2.3) by ζ^* and (3.1) by ζ, integrating, and noting that $(\mathscr{L}\zeta,\zeta^*)=(\mathscr{L}^*\bar{\zeta}^*,\bar{\zeta})$. Since γ is a simple eigenvalue, $(\zeta,\zeta^*)=(\bar{\zeta},\bar{\zeta}^*)\neq 0$, and we may normalize

$$(\zeta,\zeta^*)=(\bar{\zeta},\bar{\zeta}^*)=1. \tag{3.4}$$

The border between stability and instability is defined by the critical Reynolds number $R=R_c$. At $R=R_c$

$$\begin{aligned}
&\lambda=\lambda_0, \qquad \mathscr{L}=\mathscr{L}_0, \qquad \zeta=\zeta_0,\\
&\xi(R_c)=0, \qquad \eta(R_c)=\omega_0,\\
&z_1=e^{-i\omega_0 t}\zeta_0=e^{-is}\zeta_0, \qquad z_2=\overline{z_1},\\
&z_1^*=e^{-i\omega_0 t}\zeta_0^*=e^{-is}\zeta_0^*, \qquad z_2^*=\overline{z_1^*},\\
&\boldsymbol{u}_0=2\operatorname{Re}(e^{-is}\zeta_0).
\end{aligned} \tag{3.5}$$

We note that, when the steady solution $\tilde{\boldsymbol{u}}(\boldsymbol{x},\lambda)$, $\lambda=1/R$, loses its stability as R is increased past R_c in a strict sense, then at R_c, $\frac{d\xi}{dR}(R_c)=\operatorname{Re}(\gamma'(R_c))<0$, and hence

$$\frac{d\xi}{d\lambda}\equiv\xi'>0 \quad \text{at} \quad \xi=0, \quad \lambda=\lambda_0. \tag{3.6}$$

In the sequel, we will need the perturbation formula

$$\xi'=(\mathscr{L}_\lambda\zeta,\zeta^*) \tag{3.7}$$

where

$$\mathscr{L}_\lambda\zeta=\frac{d\tilde{\boldsymbol{u}}}{d\lambda}\cdot\nabla\zeta+\zeta\cdot\nabla\frac{d\tilde{\boldsymbol{u}}}{d\lambda}-\Delta\zeta.$$

To prove (3.7), we first differentiate (2.3) with respect to ξ. We note that

$$\frac{d\mathscr{L}(\lambda)}{d\xi}=\frac{d\lambda}{d\xi}\frac{d\mathscr{L}}{d\lambda}\equiv\frac{d\lambda}{d\xi}\mathscr{L}_\lambda$$

and find that

$$\frac{d\lambda}{d\xi}\mathscr{L}_\lambda\zeta+\mathscr{L}\zeta_1+\nabla\Pi_1=\zeta+\gamma\zeta_1,$$

$$\operatorname{div}\zeta_1=0 \quad \text{and} \quad \zeta_1=0|_{\partial\Omega}$$

where

$$\zeta_1=\left.\frac{d\zeta}{d\xi}\right|_{\lambda=\lambda_0} \quad \text{etc.}$$

Clearly

$$\frac{d\lambda}{d\xi}(\mathscr{L}_\lambda\zeta,\zeta^*)+(\mathscr{L}\zeta_1,\zeta^*)=(\zeta,\zeta^*)+\gamma(\zeta_1,\zeta^*).$$

To reduce this formula to (3.7), we recall that $(\zeta,\zeta^*)=1$ and note that

$$(\mathscr{L}\zeta_1,\zeta^*)=\gamma(\zeta_1,\zeta^*).$$

The perturbation series which are to be constructed in Sections 4 and 5 and the convergence proofs of Sections 8 and 9 rely on certain orthogonality relations. To define these relations, we first introduce the bracket notation

$$\left.\begin{matrix}[\boldsymbol{u}]_1\\ [\boldsymbol{u}]_2\end{matrix}\right\}=\left\{\begin{matrix}(\boldsymbol{u},z_1^*)_{P_2}\\ (\boldsymbol{u},z_2^*)_{P_2}.\end{matrix}\right. \tag{3.8}$$

It is easily seen that

$$[z_i]_j=\delta_{ij} \quad (i,j=1,2).$$

We also have the relation

$$\omega_0\left(\frac{\partial\boldsymbol{a}}{\partial s},z_j^*\right)_{P_2}+(\mathscr{L}_0\boldsymbol{a},z_j^*)_{P_2}=0 \tag{3.9}$$

for any 2π periodic vector $\boldsymbol{a}(\boldsymbol{x},s)$.

Now consider the problem

$$\omega_0\frac{\partial\boldsymbol{a}}{\partial s}+\mathscr{L}_0\boldsymbol{a}+\nabla P=\boldsymbol{b}. \tag{3.10}$$

From (3.10) we see immediately that

$$[\boldsymbol{b}]_1=[\boldsymbol{b}]_2=0 \tag{3.11}$$

is a necessary condition for the solvability of (3.10). In Lemma 7.1, we shall prove that this condition is also sufficient. This means that if (3.11) holds, then (3.10) has 2π periodic solutions $\boldsymbol{a}(s)$.

The 2π periodic solutions of (3.10) are not unique. One may add to such solutions any linear combination of solutions to the homogeneous equation corresponding to (3.10). The simplest *unique* solution of (3.10) will have no solutions of the homogeneous problem. To obtain such a simple solution of (3.10), it suffices to require that

$$[\boldsymbol{a}]_1=[\boldsymbol{a}]_2=0. \tag{3.12}$$

If $\boldsymbol{a}$ is a real valued 2π periodic vector, it is easily seen that $\overline{[\boldsymbol{a}]_1}=[\boldsymbol{a}]_2$. Hence, for real $\boldsymbol{a}$, $[\boldsymbol{a}]_1=0$ implies $[\boldsymbol{a}]_2=0$.

It is useful at this stage to introduce the Poincaré-Lindstedt parameter ε as

$$[\hat{\boldsymbol{v}}]_1=\varepsilon. \tag{3.13}$$

This scaling for $\hat{v}$ is always possible when $[\hat{v}]$ is non-zero and real. We shall construct a solution for which these two properties hold. First we note that, without loss of generality, we can take $[\hat{v}]$ as real. Suppose $[\hat{v}]$ were complex; define a shift operator

$$T_\delta \hat{v}(x,s) = \hat{v}(x, s+\delta). \tag{3.14}$$

Integration by parts shows that

$$[T_\delta \hat{v}]_1 = e^{-i\delta}[\hat{v}]_1. \tag{3.15}$$

Now we may always find δ such that

$$[\hat{v}(x, s+\delta)]_1 = \text{real}.$$

We work the problem for this δ.

We shall seek solutions in the form

$$\hat{v} = \varepsilon u_0 + \phi \equiv \varepsilon \hat{u} \tag{3.16}$$

where u_0 is given by (2.11). Note that u_0 satisfies

$$[u_0]_1 = 1 \quad \text{and} \quad \left[\frac{\partial u_0}{\partial s}\right]_1 = -i. \tag{3.17}$$

Hence,

$$[\hat{v}]_1 = \varepsilon + [\phi]_1.$$

In our construction we require that

$$[\phi]_1 = 0. \tag{3.18}$$

Hence,

$$[\hat{v}]_1 = \varepsilon \quad \text{and} \quad [\hat{u}]_1 = 1. \tag{3.19}$$

We also note that

$$\left[\frac{\partial \hat{u}}{\partial s}\right]_1 = -i[\hat{u}]_1 = -i = -\left[\frac{\partial \hat{u}}{\partial s}\right]_2. \tag{3.20}$$

4. Construction of the Time Periodic Solution

The solution of (2.7) will now be formally constructed as a Taylor series in ε. To keep the construction simple and clear, we will assume that $u(x, R)$ is independent of R as in the case of viscometric flow, e.g., Couette and Poiseuille flows. The R dependence of the basic motion complicates the algebra a bit but does not change the essential structure of the problem. In Section 8 where we prove convergence, we shall include the case where $\tilde{u}$ depends on R.

We now reformulate (2.7) using (2.8) and (3.13) as

$$\omega \frac{\partial \hat{u}}{\partial s} + \mathscr{L}_0 \hat{u} + \mu \Delta \hat{u} + \varepsilon(\hat{u} \cdot \nabla)\hat{u} + \nabla \hat{p} = 0, \qquad \operatorname{div} \hat{u} = 0, \tag{4.1a, b}$$

$$\hat{u} = 0|_{\partial\Omega}, \qquad u(x,s,\varepsilon) = \hat{u}(x, s+2\pi, \varepsilon) \quad \text{and} \quad [\hat{u}(\varepsilon)]_1 = 1. \tag{4.1c, d, e}$$

Here $\mu = \lambda_0 - \lambda$ and

$$\mathscr{L}_0 \hat{u} = \mathscr{L}(\lambda_0)\hat{u} = -\lambda_0 \Delta \hat{u} + \tilde{u}_0 \cdot \nabla \hat{u} + \hat{u} \cdot \nabla \tilde{u}_0.$$

We seek a solution in series

$$\begin{Bmatrix} \hat{u}(x,t,\varepsilon) \\ \hat{p}(x,t,\varepsilon) \\ \omega(\varepsilon) \\ \mu(\varepsilon) \end{Bmatrix} = \sum_{l=0} \varepsilon^l \begin{Bmatrix} u_l(x,t) \\ \hat{p}_l(x,t) \\ \omega_l \\ \mu_l \end{Bmatrix} \tag{4.2}$$

where

$$\mu_0 = 0, \qquad \mu_l = -\lambda_l$$

and

$$u_l = \frac{1}{l!} \frac{\partial^l \hat{u}(x,t,0)}{\partial \varepsilon^l}, \quad \text{etc.}$$

are Taylor coefficients.

The boundary value problem satisfied by the Taylor coefficients are obtained by substituting (4.2) in (4.1) and (2.7). One finds that

$$\omega_0 \frac{\partial u_0}{\partial s} + \mathscr{L}_0 u_0 + \nabla p_0 = 0, \qquad u_0(s) = u_0(s+2\pi), \tag{4.3a, b}$$

$$\operatorname{div} u_0 = 0, \qquad u_0 = 0|_{\partial\Omega} \quad \text{and} \quad [u_0]_1 = 1, \tag{4.3c, d, e}$$

and $(m>0)$

$$\omega_0 \frac{\partial u_m}{\partial s} + \mathscr{L}_0 u_m + \omega_m \frac{\partial u_0}{\partial s} - \lambda_m \Delta u_0 + F_m + \nabla p_m = 0, \tag{4.4a}$$

$$u_m(s) = u_m(s+2\pi), \qquad \operatorname{div} u_m = 0, \qquad u_m = 0|_{\partial\Omega}, \qquad [u_m]_1 = 0. \tag{4.4b, c, d, e}$$

Here

$$F_m = \sum_{l+j=m-1} (u_l \cdot \nabla) u_j + \tilde{\sum_{l+j=m}} \left(\omega_l \frac{\partial u_j}{\partial s} - \lambda_l \Delta u_j\right), \tag{4.5}$$

and the summation with the tilda overbar denotes a sum over all non-negative integers such that $l+j=m$ minus all the "highest order" terms (order m). Thus

$$\tilde{\sum_{l+j=m}} a_l b_j \equiv \sum_{l+j=m} a_l b_j - (a_m b_0 + a_0 b_m). \tag{4.6}$$

It is important to note that only the lower order terms (order $m-1$ and less) appear in the expression (4.5) for F_m.

For subsequent reference, we note here that

$$[\Delta u_0]_1 = -(\nabla u_0, \nabla z_1^*)_{P_2} = -(\nabla \zeta_0, \nabla \zeta_0^*). \tag{4.7}$$

Consider (3.7) when, as is assumed here, $d\tilde{u}/d\lambda = 0$. We have

$$\xi' = -(\Delta \zeta_0, \zeta_0^*) = -[\Delta u_0]_1 = -[\Delta u_0]_2. \tag{4.8}$$

We now demonstrate that the equations (4.4) can be solved sequentially. To solve these problems we must choose the constants ω_m and λ_m to make the inhomogeneous terms of the differential equations orthogonal to both solutions (z_1^* and z_2^*) of the adjoint problem (see 3.11). Taking the P_2 scalar product (4.4a) with z_1^* and using (3.20) and (4.8) we find that

$$-i\,\omega_m+\xi'\,\lambda_m+[\boldsymbol{F}_m]_1=0. \tag{4.9}$$

At each stage we construct real solutions of (4.4) so that $\boldsymbol{u}_1, \dots \boldsymbol{u}_{m-1}, \lambda_1, \dots \lambda_{m-1}, \omega_1, \dots \omega_{m-1}$; hence $\boldsymbol{F}_m$ are real. Taking real and imaginary parts of (4.9) we get

$$-\,\omega_m=+\mathrm{Im}[\boldsymbol{F}_m]_1, \tag{4.10}$$

$$\xi'\,\lambda_m=-\mathrm{Re}[\boldsymbol{F}_m]_1. \tag{4.11}$$

Since $\boldsymbol{F}_m$ is independent of $\boldsymbol{u}_m$, ω_m, and λ_m, we see that if derivatives of order $l<m$ are known, then ω_m and λ_m can be computed from (4.10, 11). With these values of ω_m and λ_m (4.4a, b, c, d) may be solved and the solution is unique up to a linear combination of solutions of the homogeneous problem, which we take to be zero.

For $m=1$ we have

$$\boldsymbol{F}_1=\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0,$$

and using (3.5) we easily see that

$$[\boldsymbol{F}_1]_1=0.$$

Therefore,

$$\omega_1=\lambda_1=0. \tag{4.12}$$

For later use we note here that $\boldsymbol{u}_1$ solves the following problem

$$\begin{gathered}\omega_0\frac{\partial\boldsymbol{u}_1}{\partial s}+\mathscr{L}_0\,\boldsymbol{u}_1+\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0+\nabla p_1=0,\\ \boldsymbol{u}_1(s)=\boldsymbol{u}_1(s+2\pi),\quad \operatorname{div}\boldsymbol{u}_1=0,\quad \boldsymbol{u}_1=0|_{\partial\Omega}\ \text{ and }\ [\boldsymbol{u}_1]_1=0.\end{gathered} \tag{4.13}$$

Since $\boldsymbol{u}_0$ is a real odd polynomial linear in e^{is} and e^{-is}, $\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0$ is even, and u_1 is even, since we choose u_1 so that $[u_1]_1=0$.

The first effects of the nonlinearity on $\lambda(\varepsilon)$ and $\omega(\varepsilon)$ occur at second order. When $m=2$, we evaluate

$$\boldsymbol{F}_2=\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_1+\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_0,\quad \omega_2=\mathrm{Im}[\boldsymbol{F}_2]_1,\quad -\lambda_2=\mathrm{Re}[\boldsymbol{F}_2]_1/\xi'. \tag{4.14a, b, c}$$

Recalling that

$$\lambda(\varepsilon)=\frac{1}{R(\varepsilon)}=\frac{1}{R_c}+\lambda_2\,\varepsilon^2+O(\varepsilon^3), \tag{4.15}$$

we see that the bifurcation is supercritical $(R(\varepsilon)>R_c)$ when $\lambda_2<0$ and is subcritical $(R(\varepsilon)<R_c)$ when $\lambda_2>0$.

As a final result we prove that

$$\omega_{2l+1}=\lambda_{2l+1}=0. \tag{4.16}$$

It follows from this that $\lambda(\varepsilon)$ and $\omega(\varepsilon)$ are even functions and the bifurcations are definitely subcritical or supercritical (see Fig. 1a). We prove (4.16) by induction. First note that by the nature of the perturbation construction, $\boldsymbol{u}_m$ is a trigonometric polynomial in e^{is} and e^{-is}. We call $\boldsymbol{u}_m$ an even (odd) polynomial if the integers l in the polynomials are even (odd). The hypothesis of the induction is that

$$\begin{gathered}\boldsymbol{u}_{2l}\ \text{is an odd polynomial,}\\ \boldsymbol{u}_{2l+1}\ \text{is an even polynomial,}\end{gathered} \tag{4.17a}$$

and

$$\lambda_{2l+1}=\omega_{2l+1}=0 \tag{4.17b}$$

when $l<m$. (4.17a, b) are true for $m=0$: $\lambda_1=\omega_1=0$ and $u_0=e^{-is}\zeta+e^{is}\bar{\zeta}$. To prove that (4.17) holds for $l=m$, we note that given (4.17a, b), $\boldsymbol{F}_{2m}$ is odd. Returning now to (4.4a), we see that the inhomogeneous terms in the equation governing $\boldsymbol{u}_{2m}$ are odd polynomials; it is then easily seen that $\boldsymbol{u}_{2m}$ is odd. Now we may compute λ_{2m+1} and ω_{2m+1} from (4.10), (4.11). Since $\boldsymbol{u}_{2m}$ is odd, $\boldsymbol{F}_{2m+1}$ is even; so $[\boldsymbol{F}_{2m+1}]_1=0$, proving that $\lambda_{2m+1}=\omega_{2m+1}=0$. But then the inhomogeneous terms in (4.4a) for $\boldsymbol{u}_{2m+1}$ are even; so $\boldsymbol{u}_{2m+1}$ is even (recall we choose $[\boldsymbol{u}_{2m+1}]=0$).

Equation (4.16) shows that the time periodic solution bifurcates subcritically or supercritically; a two-sided bifurcation, as in steady convection (see Fig. 1), is not possible here (see "remarks" following Corollary 9.3).

The sign of the coefficient λ_2 is unknown theoretically even for particular flows (in fact, it is introduced here for the first time). However, for spatially periodic disturbances of plane Poiseuille flow, heavy numerical calculations of REYNOLDS & POTTER [11] show that a coefficient (E_2, see Section 6) related to λ_2 has the sign appropriate to subcritical bifurcation. All the rectilinear flows which are bounded by straight walls are observed experimentally to break down under subcritical conditions. For example, plane Poiseuille flow breaks down at $R\sim\frac{1}{5}R_c$, and plane Couette flow is unstable for $R>O(10^3)$ but $R_c=\infty$.

Our next task is to test the stability of the periodic solution to small disturbances.

5. Perturbation of the Floquet Exponents

In trying to determine the stability of a periodic motion, one is led to consider the Floquet exponents of the motion [15]. There are various formulations of the definition of exponents, all equivalent, and the one we shall adopt is the following: Let $\hat{\boldsymbol{v}}$ be a periodic solution of (2.1a, b, c) and form the *linearized* equations for disturbances $\boldsymbol{u}$. We get

$$\begin{gathered}\omega\frac{\partial\boldsymbol{u}}{\partial s}+\mathscr{L}(\lambda)\,\boldsymbol{u}+\hat{\boldsymbol{v}}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\hat{\boldsymbol{v}}+\nabla p=0,\\ \operatorname{div}\boldsymbol{u}=0,\quad \boldsymbol{u}=0|_{\partial\Omega}.\end{gathered} \tag{5.1}$$

The coefficients $\hat{\boldsymbol{v}}$ are now periodic with period $2\pi/\omega(\lambda)$: $\hat{\boldsymbol{v}}=\hat{\boldsymbol{v}}(\boldsymbol{x},s,\varepsilon)$ is 2π periodic in s. We look for solutions of (5.1) of the form

$$\boldsymbol{u}=e^{-\sigma s}\boldsymbol{\Gamma}(\boldsymbol{x},s,\varepsilon),\qquad p'=e^{-\sigma s}p(\boldsymbol{x},s,\varepsilon) \tag{5.2}$$

where $\sigma=\sigma(\lambda)$ is a (possibly) complex number, called the Floquet exponent, and $\boldsymbol{\Gamma}$ and p are 2π periodic in s.

Substitution of (5.2) into (5.1) leads to

$$-\sigma\,\omega(\varepsilon)\,\boldsymbol{\Gamma}+\omega\frac{\partial\boldsymbol{\Gamma}}{\partial s}+\mathscr{L}\boldsymbol{\Gamma}+\hat{\boldsymbol{v}}\cdot\nabla\boldsymbol{\Gamma}+\boldsymbol{\Gamma}\cdot\nabla\hat{\boldsymbol{v}}+\nabla p=0. \tag{5.3}$$

When $\varepsilon=0$, $\omega(0)=\omega_0$, $\lambda=1/R_c$, $\mathscr{L}=\mathscr{L}_0$ and $\hat{\boldsymbol{v}}=0$ in our bifurcation problem. So we get

$$-\sigma\boldsymbol{\Gamma}+\omega_0\frac{\partial\boldsymbol{\Gamma}}{\partial s}+\mathscr{L}_0\boldsymbol{\Gamma}+\nabla p=0. \tag{5.4}$$

Two solutions clearly are

$$\sigma=0 \quad\text{and}\quad \boldsymbol{\Gamma}=e^{-is}\boldsymbol{\zeta}, \quad\text{or}\quad \boldsymbol{\Gamma}=e^{is}\bar{\boldsymbol{\zeta}}. \tag{5.5}$$

Thus for $\varepsilon=0$, there is a double Floquet exponent at the origin. We get all the Floquet exponents by looking for solutions of (5.4) of the form $\boldsymbol{\Gamma}(s,\boldsymbol{x})=e^{iks}\boldsymbol{\Phi}(\boldsymbol{x})$ where k is an integer. This leads to the eigenvalue problem

$$-\sigma\boldsymbol{\Phi}+(i\,k\,\omega_0)\,\boldsymbol{\Phi}+\mathscr{L}_0\boldsymbol{\Phi}+\nabla p=0,$$

$$\operatorname{div}\boldsymbol{\Phi}=0, \qquad \boldsymbol{\Phi}|_{\partial\Omega}=0.$$

Thus, $(-\sigma+ik\omega_0)$ is an eigenvalue of $\mathscr{L}_0$. It follows that at criticality the Floquet exponents are of the form

$$\sigma=-\gamma+i\,k\,\omega_0$$

where γ is an eigenvalue of $\mathscr{L}_0$. Since all eigenvalues of $\mathscr{L}_0$ except two have negative real parts, all other Floquet exponents have negative real parts.

Now consider the case $\varepsilon=0$. Assuming that perturbation theory holds for the Floquet exponents, we have only to worry about the critical ones, namely those at the origin. One solution of (5.1)

$$\boldsymbol{u}=\frac{\partial\hat{\boldsymbol{u}}}{\partial s}, \qquad \sigma=0$$

always exists even with $\varepsilon\neq0$. To check this, differentiate (4.1a) with respect to s

$$\omega\frac{\partial^2\hat{\boldsymbol{u}}}{\partial s^2}+\mathscr{L}(\lambda)\frac{\partial\hat{\boldsymbol{u}}}{\partial s}+\hat{\boldsymbol{v}}\cdot\nabla\frac{\partial\hat{\boldsymbol{u}}}{\partial s}+\frac{\partial\hat{\boldsymbol{u}}}{\partial s}\cdot\nabla\hat{\boldsymbol{v}}+\nabla\frac{\partial\hat{p}}{\partial s}=0. \tag{5.6}$$

Therefore, of the two eigenvalues $\sigma=0$ at the origin when $\varepsilon=0$, one remains at the origin even when $\varepsilon\neq0$.

The problem now is to calculate the second Floquet exponent. (In ordinary differential equations, this is the one that determines the stability; [15].) We look for an eigenfunction of (5.3) in the form

$$\boldsymbol{\Gamma}(\boldsymbol{x},s,\varepsilon)=a(\varepsilon)\frac{\partial\hat{\boldsymbol{u}}}{\partial s}+\boldsymbol{\gamma}(\boldsymbol{x},s,\varepsilon) \tag{5.7}$$

where

$$\boldsymbol{\gamma}(\boldsymbol{x},s,\varepsilon)=\boldsymbol{u}_0(\boldsymbol{x},s)+\varepsilon\boldsymbol{\gamma}_1(\boldsymbol{x},s)$$

and $a(\varepsilon)$ is a coefficient to be determined. Substituting (5.7) into (5.3) and using (5.6), we get for a, σ, and $\boldsymbol{\gamma}$ the equation

$$\omega\frac{\partial\boldsymbol{\gamma}}{\partial s}-\sigma\,\omega\,\boldsymbol{\gamma}+\mathscr{L}\boldsymbol{\gamma}+\varepsilon(\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{\gamma}+\boldsymbol{\gamma}\cdot\nabla\hat{\boldsymbol{u}})-\sigma\,\omega\,a\frac{\partial\hat{\boldsymbol{u}}}{\partial s}+\nabla p=0. \tag{5.8}$$

We again seek a solution in series

$$\begin{pmatrix}\boldsymbol{\gamma}(\boldsymbol{x},s,\varepsilon)\\ p(\boldsymbol{x},s,\varepsilon)\\ \sigma(\varepsilon)\\ a(\varepsilon)\end{pmatrix}=\sum_{l=0}^{\infty}\varepsilon^l\begin{pmatrix}\boldsymbol{\gamma}_l(\boldsymbol{x},s)\\ p_l(\boldsymbol{x},s)\\ \sigma_l\\ a_l\end{pmatrix}.$$

We shall require that, as in the construction of the periodic solutions, that

$$[\boldsymbol{\gamma}_0]_1=1, \qquad [\boldsymbol{\gamma}_l]_1=0, \qquad l>0.$$

As in Section 4 we get

$$-\sigma_0\,\omega_0\left(a_0\frac{\partial\boldsymbol{u}_0}{\partial s}+\boldsymbol{\gamma}_0\right)+\omega_0\frac{\partial\boldsymbol{\gamma}_0}{\partial s}+\mathscr{L}_0\boldsymbol{\gamma}_0+\nabla p_0=0,$$

$$\operatorname{div}\boldsymbol{\gamma}_0=0, \qquad \boldsymbol{\gamma}_0(\boldsymbol{x},s)=\boldsymbol{\gamma}_0(\boldsymbol{x},s+2\pi), \qquad \boldsymbol{\gamma}_0=0|_{\partial\Omega}.$$

Using (3.11) and (4.7)

$$-\sigma_0\,\omega_0(i\,a_0+1)=0. \tag{5.9}$$

Hence, $\sigma_0=0$ and $\boldsymbol{\gamma}_0=\boldsymbol{u}_0$.

At first order we must solve the problem

$$-\sigma_1\,\omega_0\left(a_0\frac{\partial\boldsymbol{u}_0}{\partial s}+\boldsymbol{u}_0\right)+\omega_0\frac{\partial\boldsymbol{\gamma}_1}{\partial s}+\mathscr{L}_0\boldsymbol{\gamma}_1+2\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0+\nabla p_1=0 \tag{5.10}$$

and

$$\operatorname{div}\boldsymbol{\gamma}_1=0, \quad\text{periodicity}, \quad \boldsymbol{\gamma}_1=0|_{\partial\Omega} \quad\text{and}\quad [\boldsymbol{\gamma}_1]_1=0.$$

Applying the orthogonality conditions to (5.10), we get $\sigma_1=0$, since $[\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0]_1=0$. Now comparing (5.10) for $\boldsymbol{\gamma}_1$ with (4.13) for $\boldsymbol{u}_1$, we see that $\boldsymbol{\gamma}_1=2\boldsymbol{u}_1$ since $[\boldsymbol{u}_1]_1=[\boldsymbol{\gamma}_1]_1=0$. This relation enables us to establish the stability of supercritical bifurcation and the instability of subcritical bifurcation.

At second order

$$\left(\omega_0\frac{\partial\boldsymbol{\gamma}_2}{\partial s}+\mathscr{L}_0\boldsymbol{\gamma}_2\right)-\sigma_2\,\omega_0\left(a_0\frac{\partial\boldsymbol{u}_0}{\partial s}+\boldsymbol{u}_0\right)+\omega_2\frac{\partial\boldsymbol{u}_0}{\partial s}-\lambda_2\Delta\boldsymbol{u}_0+\boldsymbol{T}_2+\nabla p_2=0, \tag{5.11}$$

$$\operatorname{div}\boldsymbol{\gamma}_2=0, \quad\text{periodicity}, \quad \boldsymbol{\gamma}_2=0|_{\partial\Omega} \quad\text{and}\quad [\boldsymbol{\gamma}_2]_1=0$$

where, since $\boldsymbol{\gamma}_1=2\boldsymbol{u}_1$, we find that

$$\boldsymbol{T}_2=3(\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_0+\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_1)=3\boldsymbol{F}_2$$

where $\boldsymbol{F}_2$ is given by (4.14).

The solvability conditions applied to (5.11) using (4.8) give

$$-\sigma_2\,\omega_0(a_0\,i+1)+i\,\omega_2+\lambda_2\xi'+3[\boldsymbol{F}_2]_1=0.$$

Taking real and imaginary parts we get, using (4.10) and (4.11),

$$\sigma_2 \omega_0 + 2\lambda_2 \xi' = 0,$$
$$\sigma_2 \omega_0 a_0 + 2\omega_2 = 0.$$

Hence,

$$\sigma_2 = -\frac{2\lambda_2 \xi'}{\omega_0} \tag{5.12}$$

and, assuming $\lambda_2 \neq 0$,

$$a_0 = -\frac{2\omega_2}{\sigma_2 \omega_0}.$$

Remark. In Section 10, an implicit function argument will be used to prove the convergence of the series for γ, σ, and a. It will be an immediate consequence of that development that σ is real.

Equation (5.12) contains an important result. Recall that solutions of the linearized equation decay when $\sigma > 0$ and that $\xi' > 0$ when the steady solution is unstable with $R > Rc$.

Theorem 5.1. *Subcritical periodic motions* ($\lambda_2 > 0$) *are unstable* ($\sigma_2 < 0$) *and supercritical periodic motions* ($\lambda_2 > 0$) *are stable* ($\sigma_2 > 0$) *in the linearized theory.*

The results here have been obtained under the assumption that the steady motion is independent of λ, $d\tilde{u}/d\lambda = 0$. However, the same stability result (5.17) can be obtained in the general case as well (see Section 9). Analogous results hold when $\xi' < 0$.

In the ordinary differential equation case (5.17) was calculated by HOPF [9] (see the discussion of his result in our introduction). In the case of two-dimensional parallel flow, the physical result implied by (5.17) was obtained first by STUART [4]. His result follows from integration of an amplitude equation (6.11), truncated after A^2.

6. The Method of False Problems

When $R - R_c \neq 0$, then $\xi \neq 0$ and small disturbances of the steady flow $\tilde{u}(x)$ grow ($\xi < 0$) or decay ($\xi > 0$). Of course, it is possible to have a time periodic solution $\hat{u}$ with $R = R_c$, but the perturbation which proceeds with R fixed could be expected to involve time changes other than those associated with periodicity. This is basically the reason why perturbation methods such as STUART'S [4] and WATSON'S [10] involve an amplitude equation which will allow for the growth or decay of disturbances.

One can proceed, however, at fixed R, directly to the time periodic solution by a "method of false problems". Such a method was first suggested by REYNOLDS & POTTER [11] in connection with their study of the stability of Poiseuille flow to a class of disturbances which do not vary in one (oblique) direction. They consider the perturbation problem for their special version of (2.7) when R is fixed. Then, only the parameter $\omega(\varepsilon)$ and not $\lambda(\varepsilon) = 1/R(\varepsilon)$ is available to insure that the perturbation equations satisfy the necessary solvability conditions. To have a second parameter, REYNOLDS & POTTER allow $\omega(\varepsilon)$ (in their notation, $\omega(A)$) to be complex: this is the false problem. One gets *real* periodic solutions

if the perturbation series converges in some sense for those values $\varepsilon = \tilde{\varepsilon}$ for which $\omega(\tilde{\varepsilon})$ is real, that is, for the roots of the equation

$$\sum_{l=0}^{\infty} \varepsilon^l \operatorname{Im}(\omega_l) = 0.$$

The roots $\tilde{\varepsilon}$ of this equation are called "equilibrium amplitudes".

SATTINGER [7] also uses a method of false problems which differs from the one just described but which is very close to the Stuart-Watson method. SATTINGER proves the convergence of his perturbation series. A simplified version of the Sattinger perturbation will now be given.

Replacing problem (2.7), one considers the false problem

$$\omega \frac{\partial \hat{u}}{\partial s} + \mathscr{L}\hat{u} + (E(\varepsilon, R) - \xi(R))\hat{u} + \varepsilon(\hat{u} \cdot \nabla)\hat{u} + \nabla \hat{p} = 0 \tag{6.1}$$

with the old side conditions. In the perturbation, we hold R fixed and regard the frequency $\omega(\varepsilon, R)$ and $E(\varepsilon, R)$ as the two perturbation parameters (in [5], $\xi(R)$ was called ε_R and $E(\varepsilon, R)$ was called $\varepsilon(\mu, R)$). One seeks the series at fixed R

$$E(\varepsilon, R) = E_1(R)\varepsilon + E_2(R)\varepsilon^2 + \cdots \tag{6.2}$$

and a similar series for $\hat{u}$, $\hat{p}$, and ω. At zeroth order, $R - R_c$ fixed, we have $E(0, R) = 0$, $\omega(0, R) = \eta(R)$ where $\xi(R)$ and $\eta(R)$ are the real and imaginary part the eigenvalue $\gamma(R)$ of the $\mathscr{L}$ defined in (2.3).

We get a periodic solution of the true problem for $\varepsilon = \tilde{\varepsilon}$ such that

$$E(\tilde{\varepsilon}, R) = \xi(R). \tag{6.3}$$

The perturbation to be used for (6.1) is close to that described in Section 4. However, since $R - R_c$ need not vanish and $\xi(R) = 0$, we must span the eigenspace of $\mathscr{L}$ with the vectors ζ and $\bar{\zeta}$ satisfying (2.3) and its conjugate.

The differences between the perturbation method of Section 4 and the one appropriate to (6.1) do, however, require explanation. Consider (6.1) when $\varepsilon = 0$:

$$\eta \frac{\partial \hat{u}_0}{\partial s} + \mathscr{L}\hat{u}_0 - \xi \hat{u}_0 + \nabla p_0 = 0. \tag{6.4}$$

There are two independent complex-valued solutions of (6.4),

$$e^{-is}\zeta \quad \text{and} \quad e^{is}\bar{\zeta}. \tag{6.5}$$

For the first solution we have

$$-\gamma\zeta + \mathscr{L}\zeta + \nabla p = 0, \tag{6.6}$$

and for the second we have the complex conjugate of (6.6). Hence, (6.5) gives the two solutions of (6.4) implied by the simplicity of eigenvalues γ and $\bar{\gamma}$ of the operator $\mathscr{L}$. Naturally, the real solutions $\hat{u}_0$ of (6.4) are linear combinations of those given by (6.5). We choose

$$\hat{u}_0 = e^{-is}\zeta + e^{is}\zeta \tag{6.7}$$

and compute

$$\left\{\frac{d}{ds}+i\right\}(\hat{u}_0,\zeta^*)=0. \tag{6.8}$$

An important difference between the perturbation of Sections 4 and 5 and the one to be used with (6.1) is associated with the solvability condition for the equation

$$\eta\frac{\partial \boldsymbol{a}}{\partial s}+\mathscr{L}\boldsymbol{a}-\xi\boldsymbol{a}+\nabla p=\boldsymbol{b} \tag{6.9}$$

where $\boldsymbol{b}$ is a complex-valued 2π periodic vector field. Here, a necessary condition for the solvability of (6.9) is that

$$\{\boldsymbol{b}\}_1=\{\boldsymbol{b}\}_2=0 \tag{6.10}$$

where

$$\{\boldsymbol{b}\}_1\equiv\frac{1}{2\pi}\int_0^{2\pi}(\hat{u}_0,\bar{\zeta}_*)(\boldsymbol{b},\zeta^*)\,ds$$

and

$$\{\boldsymbol{b}\}_2\equiv\frac{1}{2\pi}\int_0^{2\pi}(\hat{u}_0,\zeta_*)(\boldsymbol{b},\bar{\zeta}^*)\,ds.$$

With these differences in mind, the rest of the perturbation can now be constructed in an extensive analogy to the perturbation given in Section 4. It is also easy to carry out the Floquet stability analysis of true periodic solutions by treating a false problem.

The false problem (6.1) is closely related to the Stuart-Watson method. In fact, we may set

$$E(\varepsilon,R)-\xi(R)=\frac{1}{A}\frac{dA}{dt}=-\xi(R)+a^{(2)}A^2+O(A^4) \tag{6.11}$$

where A is the Stuart-Watson amplitude as used in [11] and $a^{(2)}$ is the critical parameter of the Stuart-Watson theory and is defined by equation (2.9) of [11]. We get a periodic solution when A is stationary; then, $A=\tilde{\varepsilon}$ and $E_2(R)=a^{(2)}$. The numerical calculation of [11] yields the values $E_2(R_c)$ for plane Poiseville flow: $E_2(R_c)=19.7$. Given the numerical calculation, there will be a periodic solution for $\xi>0$; these solutions would decay for amplitudes $|\varepsilon|<|\tilde{\varepsilon}|$ and are *subcritical.*

The method of false problems can be used to give rigorous justification for the Stuart-Watson series used by REYNOLDS & POTTER in the case when $dA/dt=0$.

The series solutions constructed until now are formal and not rigorous. We turn next to the convergence proofs which give these series rigorous justification.

7. Preliminary Estimates for the Navier-Stokes Equations

It will be convenient to write the Navier-Stokes equations in a more abstract form. In this way, it will become apparent that the arguments which follow hold more generally.

We introduce the Hilbert space of vector fields $u=(u_1,u_2,u_3)$

$$L_2(\Omega)=\{u:\int_\Omega u_i\bar{u}_i\,dx=|u|^2<+\infty\},$$

and we let H_σ be the subspace consisting of weakly divergent, real vector fields* (see [12]):

$$\int u_i\partial_i p\,dx=0$$

for all p in C^1. We denote the orthogonal projection onto H_σ by P_σ. We also introduce the Banach spaces $C^{k+2\alpha,l+\alpha}$ of vector fields with Hölder norms $|\ |_{k+2\alpha,l+\alpha}$ on $\Omega_T=\bar{\Omega}\times[0,T]$. (See [7], p. 69.) We assume here that $\partial\Omega\in C^{2+2\alpha}$ for $0<\alpha<\frac{1}{2}$. We remark that P_σ is continuous in the Hölder norms ([12], p. 97).

Let A be the linear operator $Au=P_\sigma[-\Delta u]$. Since P_σ is a continuous linear operator from $C^{2\alpha}$ to itself, $|Au|_{2\alpha}\leqq C|u|_{2+2\alpha}$ for some constant C. Applying P_σ to the equations (2.7) for the perturbed flow, we can write them in the more concise form

$$\frac{\partial v}{\partial t}+Lv+N(v,v)=0 \tag{7.1}$$

where

$$N(v,v)=P_\sigma v\cdot\nabla v,$$

$$L(\lambda)v=P_\sigma[-\lambda\Delta v+\tilde{u}\cdot\nabla v+v\cdot\nabla\tilde{u}].$$

As is well-known, equations (7.1) and (2.7) are entirely equivalent [12].

We first prove

Lemma 7.1. *The equation $L_0u=f$ is invertible $(L_0=L(\lambda_0))$ and $|u|_{2+2\alpha}\leqq C|f|_{2\alpha}$, $0<\alpha<\frac{1}{2}$, where the constant C is independent of f.*

Proof. We need the *a priori* estimate $|u|_{2+2\alpha}\leqq C|Au|_{2\alpha}$ ([12], p. 79). We write

$$L_0u=P_\sigma[-\Delta u+\tilde{u}\cdot\nabla u+u\cdot\nabla\tilde{u}]=f$$

and operate by A^{-1} to get

$$u+A^{-1}M(\tilde{u},u)=A^{-1}f \tag{7.2}$$

where $M(\tilde{u},u)=N(\tilde{u},u)+N(u,\tilde{u})$. Now the operation $A^{-1}M$ is continuous from $C^{1+2\alpha}$ to $C^{2+2\alpha}$, since $|M(\tilde{u},u)|_{1+2\alpha}\leqq C|u|_{2+2\alpha}$. So $A^{-1}M$ is a compact mapping from $C^{2+2\alpha}$ to itself, and the Fredholm alternative applies: either equation (7.2) is boundedly invertible or the homogeneous equation has a non-trivial solution. A non-trivial solution ψ of (7.2) leads to $\psi+A^{-1}M(\tilde{u},\psi)=0$, or $L_0\psi=0$, but we assume that zero is not an eigenvalue of L_0 at criticality. So (7.2) is boundedly invertible, and for any f there is a solution u satisfying $|u|_{2+2\alpha}\leqq C|A^{-1}f|_{2+2\alpha}\leqq C'|f|_{2\alpha}$.

Lemma 7.2. *Let $\tilde{u}_0$ be a solution in $C^{2+2\alpha}$ of*

$$\begin{aligned}-\lambda_0\Delta\tilde{u}_0+\tilde{u}_0\cdot\nabla\tilde{u}_0&=-\nabla p+\tilde{F},\\ \operatorname{div}\tilde{u}_0=0,\quad \tilde{u}_0|_{\partial\Omega}&=f.\end{aligned} \tag{7.3}$$

* From now on we drop the boldface notation.

Then there is an analytic solution $\tilde{u}(\lambda)=\tilde{u}_0+(\lambda-\lambda_0)u_1+\cdots$ *of the equation*

$$-\lambda\Delta\tilde{u}+\tilde{u}\cdot\nabla\tilde{u}=-\nabla p+\tilde{F},\qquad \operatorname{div}\tilde{u}=0,\quad \tilde{u}|_{\partial\Omega}=f. \tag{7.4}$$

Proof. Write $\mu=\lambda-\lambda_0$ and $\tilde{u}=\tilde{u}_0+v$. We find for v the equivalent equation

$$L_0 v+N(v,v)+\mu A v=\mu A u_0.$$

Operating through by $K=L_0^{-1}$, we have

$$v+KN(v,v)+\mu KAv=\mu KAu_0. \tag{7.5}$$

By Lemma 1 we have $KAu_0\in C^{2+2\alpha}$, and KA is a bounded mapping from $C^{2+2\alpha}$ to itself. Furthermore, it is easily seen that $KN(v,v)$ is a Frechet differentiable completely continuous mapping from $C^{2+2\alpha}$ to itself. Equation (7.5) can be written as $F(\mu,v)=0$, where $F(\mu,v)=v+KN(v,v)+\mu KAv-\mu KAu_0$. For $\mu=0$, we have

$$F(0,0)=0,$$

$$F_v(0,0)=I.$$

Furthermore, the mapping $KN(v,v)$ is bilinear, hence, analytic in the sense of VAINBERG & TRENOGIN [16], p. 19. Therefore, by the analytic version of the implicit function theorem in a Banach space ([16], Theorem 3.2), (7.5) has a solution v which may be expanded in a convergent power series in μ. This completes the proof.

As a consequence of Lemma 7.2, we can write

$$\tilde{u}(\mu)=\tilde{u}_0+\mu u_1+\mu^2\tilde{u}_2+\cdots=\tilde{u}_0+v$$

and equations (7.1) may be written in the form

$$\frac{\partial v}{\partial t}+L_0 v+\mu L_1(\mu)v+N(v,v)=0$$

where

$$L_1(\mu)v=Av+M(\tilde{u}_1,v)+\mu M(\tilde{u}_2,v)+\cdots.$$

We assume L_0 has simple eigenvalues $\pm i$ and consider the equation

$$\frac{\partial u}{\partial s}+L_0 u=f. \tag{7.6}$$

The homogeneous equation (7.6) has the 2π periodic solutions $z_1=e^{-is}\zeta$, $z_2=\bar{z}_1$ as in Section 2. We introduce the Hilbert space $P_{2\pi}$ of 2π periodic vector fields in H_σ with inner product

$$(u,v)_{P_2}=\frac{1}{2\pi}\int_0^{2\pi}(u,v)\,ds$$

where (u,v) is the inner product on H_σ.

Define $P_{2\pi}^{\perp}$ to be the subspace of $P_{2\pi}$ of periodic vector fields u for which $[u]_1=[u]_2=0$ (see 3.8).

Lemma 7.3. *The operator*

$$\frac{\partial}{\partial s}+L_0$$

has a bounded inverse from $P_{2\pi}^{\perp}$ *to itself.*

Proof. In (7.6) we assume $f\in P_{2\pi}^{\perp}$. Writing

$$f=\sum_n f_n(x)\,e^{-ins}$$

we see that $[f]_1=[f]_2=0$ implies $(f_1(x),\zeta^*)=(f_{-1},\bar{\zeta}^*)=0$. (Note that if f is real, then $f_k=\overline{f_{-k}}$.) If we put $u=\sum_n u_n(x)e^{-ins}$, then equation (7.6) leads to

$$(-in+L_0)u_n=f_n.$$

These equations are boundedly invertible except in the case $n=\pm 1$. However, the conditions $(f_1,\zeta^*)=(f_{-1},\bar{\zeta}^*)=0$ imply, by the Fredholm alternative, that $(-i+L_0)u_1=f_1$ and $(i+L_0)u_{-1}=f_{-1}$ have solutions in L_2. These are not unique but may be made so by requiring that $(u_1,\zeta^*)=(u_{-1},\bar{\zeta}^*)=0$. Then there is a constant C such that $|u_1|\leqq C|f_1|$, $|u_{-1}|\leqq C|f_{-1}|$. For $n^2\neq 1$ we have $u_n=(-in+L_0)^{-1}f_n$. We have

$$|u|_{P_2}^2=\frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_n u_n e^{-ins},\sum_m u_m e^{-ims}\Big)ds$$

$$=\sum_n |u_n|^2\leqq C\sum_n |f_n|^2=C|f|_{P_2}^2.$$

The uniform estimate $|u_n|\leqq C|f_n|$ follows from the fact that the imaginary axis is a ray of minimal growth for L_0 [14]. We even get the stronger estimate

$$|u_n|\leqq\frac{C}{1+n}|f_n| \tag{7.7}$$

which will be useful later.

Lemma 7.4. *Let v satisfy the linear initial boundary value problem*

$$\frac{\partial v}{\partial t}+L_0 v=f,\qquad v(x,0)=\psi(x). \tag{7.8}$$

Assume

$$\psi\in C_{2+2\alpha}(\bar{D})\quad\textit{and}\quad f\in C_{2\alpha,\alpha}(\bar{D}\times[0,T]).$$

Then

$$v\in C_{2+2\alpha,1+\alpha}(\bar{D}\times[0,T])\quad\textit{and}\quad |v|_{2+2\alpha,1+\alpha}\leqq C(|f|_{2\alpha,\alpha}+|\psi|_{2+2\alpha}).$$

Proof. We can decompose $v=u+w$ where w satisfies the inhomogeneous equation with zero initial data while u satisfies the homogeneous equation and $u(0)=\psi$. The estimate for w is given by Lemma 3.2 of [7]. The estimate for u can be obtained by arguments similar to those used in the proof of that lemma and by using the *a priori* estimates contained in [12], Theorem 5, p. 100.

Lemma 7.5. *The operator* $\hat{K}=\left(\frac{\partial}{\partial s}+L_0\right)^{-1}$ *given by Lemma 7.3 is a continuous mapping from* $P_{2\pi}^{\perp}\cap C_{2\alpha,\alpha}$ *to* $P_{2\pi}^{\perp}\cap C_{2+2\alpha,1+\alpha}$.

Proof. Let f be 2π periodic and let $|f|_{2\alpha,\alpha}<+\infty$. The periodic solution u constructed in Lemma 7.3 has the representation

$$u(t)=e^{-tL_0}u(0)+\int_0^t e^{-(t-s)L_0}f(s)\,ds \tag{7.9}$$

where e^{-tL_0} denotes the semi group generated by L_0. Since u is periodic, we get

$$(I-e^{-2\pi L_0})u(0)=\int_0^{2\pi} e^{-(2\pi-s)L_0}f\,ds. \tag{7.10}$$

Denote the right side of (7.10) by g. Then $|g|_{2+2\alpha}\leqq C|f|_{2\alpha}$ by Lemma 7.4. The operator $e^{-2\pi L_0}$ is a compact operator from $C^{2+2\alpha}$ to itself; in fact, the injection i: $C^{2+2\alpha}\to C^{1+2\alpha}$ is compact, while $e^{-2\pi L_0}$ maps $C^{2\alpha}$ continuously to $C^{2+2\alpha}$.

This last statement can be obtained from the representation

$$e^{-tL_0}f=\frac{1}{2\pi i}\int_C e^{-\lambda t}(\lambda-L_0)^{-1}f\,d\lambda$$

for the semigroup e^{-tL_0}. The contour C encloses the spectrum of L_0 and tends to infinity in the complex plane on the rays $\lambda=|\lambda|\,e^{\pm i\theta}$, where $0<\theta<\pi/2$. One then applies the estimate $|(\lambda-L_0)^{-1}f|_{2+2\alpha}\leqq C|\lambda|\,|f|_{2\alpha}$ to the integrand in the integral above. For a derivation of this estimate, the reader is referred to [21], Chapter 5.

Therefore, the Fredholm alternative applies to (7.10). Now $e^{-2\pi L_0}\zeta=\zeta$ because $z_1=e^{-is}\zeta$ is a periodic solution of $\left(\frac{\partial}{\partial s}+L_0\right)z_1=0$; so $(I-e^{-2\pi L_0})$ is not invertible. But (7.10) expresses the fact that the right side lies in the range of $(I-e^{-2\pi L_0})$ – that is, that g satisfies the required orthogonality relations $(g,\zeta^*)=(g,\bar{\zeta}^*)=0$. Let P be the projection on the null space of $I-e^{-2\pi L_0}$ and let $Q=I-P$ be the projection onto the range. By the Fredholm theory there is a bounded operator S such that $S(I-e^{-2\pi L_0})Q=Q$. Applying SQ to (7.10), we get $Qu(0)=SQg=Sg$; hence, $|Qu(0)|_{2+2\alpha}\leqq C|f|_{2\alpha}$.

On the other hand,

$$Pu=(u,\zeta^*)\zeta+(u,\bar{\zeta}^*)\bar{\zeta}$$

so

$$|Pu(0)|_{2+2\alpha}\leqq 2|u(0)|\;|\zeta^*|\;|\zeta|_{2+2\alpha}\leqq C|u(0)|.$$

But we can write

$$u(0)=\sum_n u_n(x)=\sum_n(-in+L_0)^{-1}f_n.$$

From (7.7),

$$|u(0)|\leqq\sum_n\frac{C}{1+n}|f_n|,$$

so

$$|u(0)|^2\leqq C'\sum_n|f_n|^2=C'|f|^2_{P_2}$$
$$\leqq C''|f|^2_{2\alpha,\alpha}.$$

Therefore,

$$|Pu|_{2+2\alpha}\leqq C|f|_{2\alpha,\alpha},\quad\text{and}\quad |u|_{2+2\alpha}\leqq|Pu|_{2+2\alpha}+|Qu|_{2+2\alpha}\leqq C|f|_{2\alpha,\alpha}.$$

Let $P_{2\pi}^{2+2\alpha,1+\alpha}$ be the Banach space of 2π periodic vector fields with finite Hölder norm $|\;|_{2+2\alpha,1+\alpha}$.

Lemma 7.6. *Let* J *be the operator*

$$J=\left(\frac{\partial}{\partial s}+L_0\right)$$

on $P_{2\pi}^{2+2\alpha,1+\alpha}$. *There exist projections* $\hat{P}$ *and* $\hat{Q}=I-\hat{P}$ *onto the null and range spaces of* J, *and the operator* $\hat{K}$ *satisfies* $\hat{K}J\hat{Q}=\hat{Q}$. $\hat{K}\hat{Q}$ *is a bounded linear transformation from* $P_{2\pi}^{2\alpha,\alpha}$ *to* $P_{2\pi}^{2+2\alpha,1+\alpha}$.

Proof. We take $Pu=(u,z_1^*)_{P_2}z_1+(u,z_2^*)_{P_2}z_2$ and $\hat{Q}=I-\hat{P}$. It is easily seen that for any u, $[\hat{Q}u]_1=[\hat{Q}u]_2=0$ so that $\hat{Q}u$ lies in the range of J by Lemma 7.5. The operator $\hat{K}$ defined on $P_{2\pi}^{\perp}\cap C^{2\alpha,\alpha}$ to $P_{2\pi}^{\perp}\cap C^{2+2\alpha,1+\alpha}$ by Lemma 7.5 is thus defined everywhere on the range of $\hat{Q}$, so $\hat{K}\hat{Q}$ is defined everywhere on $P_{2\pi}^{2\alpha,\alpha}$. Since $\hat{P}$ is a smooth mapping from $P_{2\pi}^{2\alpha,\alpha}$ to itself, so is $\hat{Q}=I-\hat{P}$, and the final statement of Lemma 7.6 follows.

8. Bifurcation of Periodic Solutions; Implicit Function Theorem

In this section we construct periodic solutions of frequency ω of (7.1) for λ near λ_0 when a simple complex conjugate pair γ, $\bar{\gamma}$ of eigenvalues of $L(\lambda)$ cross the imaginary axis. We assume $\gamma(\lambda_0)=i$ and that the nondegeneracy condition

$$\operatorname{Re}\gamma'(\lambda_0)\neq 0 \tag{8.1}$$

holds.

As in Section 3, this is easily seen to be equivalent to

$$\operatorname{Re}[L_1u_0]_1\neq 0 \tag{8.2}$$

where $L_0\zeta=i\zeta$, and $u_0(s)=\operatorname{Re}e^{-is}\zeta$.

If we put $s=\omega t$, our equations become

$$\omega\frac{\partial v}{\partial s}+L(\mu)v+N(v)=0 \tag{8.3}$$

where $N(v)$ is a general analytic operator of degree at least two. Here $L(\mu)=L_0+\mu L_1+\cdots$.

We look for 2π periodic solutions of (8.3) satisfying

$$[v]_1=\varepsilon. \tag{8.4}$$

Putting $v=\varepsilon u$, $[u]_1=1$ and writing $u=\hat{P}u+\hat{Q}u$ where $\hat{P}$ and $\hat{Q}$ are the projections defined in Section 7, we see that $[\hat{Q}u]_1=0$ and $[\hat{P}u]_1=1$, so that $\hat{P}u=u_0$. Finally, write $\hat{Q}u=\phi$ and substitute $v=\varepsilon(u_0+\phi)$ into (8.3):

$$\left(\frac{\partial\phi}{\partial s}+L_0\phi\right)+(\omega-1)\frac{\partial\phi}{\partial s}+\mu L_1\phi +(\omega-1)\frac{\partial u_0}{\partial s}+\mu L_1 u_0+\varepsilon N(u_0+\phi)+O(\mu^2)=0. \tag{8.5}$$

Now apply the operator $\hat{K}\hat{Q}$:

$$\phi+\hat{K}\hat{Q}\left\{(\omega-1)\frac{\partial\phi}{\partial s}+\mu L_1\phi+(\omega-1)\frac{\partial u_0}{\partial s}+\mu L_1 u_0 +\varepsilon N(u_0+\phi)+O(\mu^2)\right\}=0.^{\star} \tag{8.6}$$

Now applying the orthogonality relation to the quantity in braces, we get

$$\mu[L_1\phi]_1-i(\omega-1)+\mu[L_1u_0]_1+\varepsilon[N(u_0+\phi)]_1+O(\mu^2)=0, \tag{8.7}$$

where we have used the relations

$$\left[\frac{\partial\phi}{\partial s}\right]_1=0,\qquad \left[\frac{\partial u_0}{\partial s}\right]_1=-i.$$

Taking real and imaginary parts of (8.6), we get

$$\mu\,\mathrm{Re}[L_1u_0]_1+\mu\,\mathrm{Re}[L_1\phi]_1+\varepsilon R[N(u_0+\phi)]_1+O(\mu^2)=0, \tag{8.8}$$

$$-(\omega-1)+\mu\,\mathrm{Im}[L_1u_0]_1+\mu\,\mathrm{Im}[L_1\phi]_1+\varepsilon\,\mathrm{Im}[N(u_0+\phi)]_1+O(\mu^2)=0. \tag{8.9}$$

Equations (8.6), (8.8), and (8.9) are equivalent to (8.5), as in the usual Lyapunov-Schmidt procedure. We view the left side of these equations as the components of a mapping $F(\phi,\omega,\mu,\varepsilon)$ from $B\times R\times R\times R$ into $B\times R\times R$, where B is the range space of the operator J. Thus, B is simply the null space in $P_{2\pi}^{2+2\alpha,1+\alpha}$ of the projection $\hat{P}$.

For $\varepsilon=0$, one set of solutions is $\omega=1$, $\mu=0$, $\phi=0$. Let us calculate the Frechet derivative of F at this point. The derivative of (8.6) with respect to ϕ is the operator

$$I+\hat{K}\hat{Q}\left\{(\omega-1)\frac{\partial}{\partial s}+\mu L_1+\varepsilon M(u_0,\cdot)+O(\mu^2)\right\}$$

where $M(u_0,\cdot)$ is the Frechet derivative of N at u_0. But at $\mu=0$, $\omega=1$, and $\varepsilon=0$, this reduces to I, the identity. Similarly, the Frechet derivative of (8.6) relative to ω is the vector $\hat{K}\hat{Q}\,\dfrac{\partial\phi}{\partial s}$, which becomes zero for $\phi=0$. Proceeding in this fashion, we may write down the Frechet derivative of F in the form of a matrix

$\star$ *Remark.* The term $O(\mu^2)$ comes from higher powers in the Taylor series for $L(\mu)=L_3+\mu L_1+\mu^2L_2+\cdots$. In the case of the Navier-Stokes equations L_k ($k\geqq 2$) is a first order partial differential operator. $\hat{K}\hat{Q}$ gains two derivatives, however, and $\hat{K}\hat{Q}(\mu^2L_2+\mu^3L_3+\cdots)$ is a bounded analytic operator.

operator

$$\begin{bmatrix}\dfrac{\partial F_1}{\partial\phi} & \dfrac{\partial F_1}{\partial\mu} & \dfrac{\partial F_1}{\partial\omega}\\[2mm] \dfrac{\partial F_2}{\partial\phi} & \dfrac{\partial F_2}{\partial\mu} & \dfrac{\partial F_2}{\partial\omega}\\[2mm] \dfrac{\partial F_3}{\partial\phi} & \dfrac{\partial F_3}{\partial\mu} & \dfrac{\partial F_3}{\partial\omega}\end{bmatrix}=\begin{bmatrix} I & \hat{K}\hat{Q}L_1u_0 & 0\\ 0 & \mathrm{Re}[L_1u_0]_1 & 0\\ 0 & \mathrm{Im}[L_1u_0]_1 & -1\end{bmatrix}. \tag{8.10}$$

The components F_1, F_2, F_3 are the left sides of (8.6), (8.8), and (8.9), respectively. It is easily seen that the operator in (8.10) is invertible under the assumption (8.2).

Now assuming that all operations in (8.6) are continuous – that is, that $\hat{K}\hat{Q}$ applied to the operations in braces is a continuous operator on $P_{2\pi}^{2+2\alpha,1+\alpha}$, we can apply the implicit function theorem of Vainberg & Trenogin ([16], Theorem 3.1 and 3.2). That these operations are continuous in the hydrodynamics case was proved in Section 7. Our conclusion, then, is that the quantities ϕ, μ, and ω have convergent Taylor series in ε such that $\phi(\varepsilon)$, $\mu(\phi)$, and $\omega(\varepsilon)$ provide a solution to the problem (8.3) for all ε sufficiently small. This completes the proof.

We make one final remark, which will be useful in the next section. A consequence of the implicit function theorem is that the curve $\phi(\varepsilon)$, $\mu(\varepsilon)$, and $\omega(\varepsilon)$ gives the *unique* family of periodic solutions of the problem (8.3) satisfying the normalization condition (8.4). In other words, in a sufficiently small neighborhood of $\phi=0$, $\mu=0$, and $\omega=1$, there are no other periodic solutions of (8.3) satisfying (8.4).

9. Uniqueness and Direction of Bifurcating Periodic Solutions

In Section 4 we saw that in the case of a quadratic nonlinearity the bifurcation is always one-sided (excepting the singular case of a one-parameter family of periodic solutions at criticality). In this section, we prove the same result for general nonlinear terms, as was first shown by Hopf. For simplicity, we assume the operator equation (8.3) represents a parabolic system, that $N(v)$ is a general power series in v (convergent in a suitable sense), and that estimates in the Hölder norms $|\ |_{2+2\alpha,1+\alpha}$ hold as in the case of the Navier-Stokes equations.

Theorem 9.1. *There exist positive numbers $a>0$, $b>0$ such that any periodic solution v of* (8.3) *for which $|v|<a$ and $|\mu|<b$ belongs to the one-parameter family of periodic solutions constructed in Section* 8 *for $\varepsilon>0$. That is, there is an $\varepsilon>0$ such that $v=\varepsilon(u_0(s)+\phi(s,\varepsilon))$, $\mu=\mu(\varepsilon)$ and $\omega=\omega(\varepsilon)$.*

Proof. If Theorem 9.1 is not true, then there exists a sequence of periodic solutions v_n with $|v_n|_{2+2\alpha,1+\alpha}\to 0$ and $\mu_n\to 0$:

$$\frac{\partial v_n}{\partial t}+L(\mu_n)v_n+N(v_n)=0. \tag{9.1}$$

Furthermore, by appropriate choice of phase for v_n, we may assume $[v_n]_1=\sigma_n\geqq 0$.

In (9.1) $N(v_n)$ is a general nonlinear operator with a power series in v_n. Therefore, $N(\rho v)=\rho^2 N(\rho,v)$ where $N(\rho,v)$ is a power series in ρ and v. Put $\rho_n=$

$|v_n|_{2+2\alpha,1+\alpha}$ and $u_n=\rho_n^{-1}v_n$. Then $|u_n|_{2+2\alpha,1+\alpha}=1$ and

$$\frac{\partial u_n}{\partial t}+L(\mu_n)u_n+\rho_n N(\rho_n,u_n)=0. \tag{9.2}$$

Some subsequence of u_n (which we may also call u_n) converges in the norm $|\ |_{2+2h,1+h}$ for $h<\alpha$. Letting $n\to\infty$ in (9.2), we find that $z=\lim_n u_n$ satisfies

$$\frac{\partial z}{\partial t}+L_0 z=0.$$

Thus, we see that $u_n(t)$ tends uniformly to a 2π periodic function. If ω_n denotes the frequency of u_n, then $\omega_n\to 1$. We now have

$$\omega_n\frac{\partial v_n}{\partial s}+L(\mu_n)v_n+N(v_n)=0,$$

$$[v_n]_1=\sigma_n,$$

with $\omega_n\to 1$, $\sigma_n\to 0$, $\mu_n\to 0$. Therefore, for sufficiently large n, by the remarks at the end of Section 8, $v_n=v(s,\sigma_n)$, where $v(s,\sigma)$ is the one-parameter family of periodic solutions given by the implicit function theorem.

Corollary 9.2. *The family of periodic solutions constructed in Section* 8 *satisfies the symmetry condition* $v(s,-\varepsilon)=v(s+\pi,\varepsilon)$ *for all* ε *for which the series converge.*

Proof. From (3.13) and (3.14) we find that $[v(s+\pi,\varepsilon)]_1=e^{-i\pi}[v(s,\varepsilon)]_1=-\varepsilon$. This holds for all ε, so $[v(s+\pi,-\varepsilon)]_1=\varepsilon$. Theorem 9.1 then shows that $v(s+\pi,-\varepsilon)=v(s,\rho)$ for some $\rho>0$, where $\rho=[v(s,\rho)]_1=[v(s+\pi,-\varepsilon)]_1=\varepsilon$.

In the special case of a quadratic nonlinearity, we have the additional symmetry conditions:

$$\mu(\varepsilon)=\mu(-\varepsilon), \qquad \omega(\varepsilon)=\omega(-\varepsilon).$$

In the case of an arbitrary (analytic) nonlinearity, we have

Corollary 9.3. *In the expansion* $\mu(\varepsilon)=\mu_k\varepsilon^k+\mu_{k+1}\varepsilon^{k+1}+\cdots$ *the first nonvanishing term is even. Thus, the bifurcation is either only supercritical, only subcritical, or occurs only at criticality* ($\mu\equiv 0$).

Proof. The periodic solutions satisfy

$$\omega(\varepsilon)\frac{\partial v}{\partial s}(s,\varepsilon)+L(\mu(\varepsilon))v+N(v)=0 \tag{9.3}$$

(for general nonlinearity). Changing ε to $-\varepsilon$ and using Corollary 9.2, we get

$$\omega(-\varepsilon)\frac{\partial v}{\partial s}(s+\pi,\varepsilon)+L(\mu(-\varepsilon))v(s+\pi,\varepsilon)+N(v(s+\pi,\varepsilon))=0.$$

Now operate with T_π:

$$\omega(-\varepsilon)\frac{\partial v}{\partial s}(s,\varepsilon)+L(\mu(-\varepsilon))v(s,\varepsilon)+N(v(s,\varepsilon))=0. \tag{9.4}$$

Now take the bracket $[\ ]_1$, of (9.3) and (9.4); using (3.19) and (3.20) we find:

$$-i\varepsilon\omega(\varepsilon)+i\varepsilon+\mu_k\varepsilon^k\varepsilon[L_1v_0]_1+O(\varepsilon^{k+2})+[N(v)]_1=0,$$

$$-i\varepsilon\omega(-\varepsilon)+i\varepsilon+\mu_k(-\varepsilon)^k\varepsilon[L_1v_0]_1+O(\varepsilon^{k+2})+[N(v)]_1=0.$$

Taking real parts of these equations and subtracting, we have

$$\mu_k\varepsilon(\varepsilon^k-(-\varepsilon)^k)\operatorname{Re}[L_1u_0]_1=O(\varepsilon^{k+2})$$

which implies that k must be even.

Remark. The results contained in Theorem 9.1 and Corollary 9.3 are in striking contrast to the case of bifurcation of stationary solutions, which occurs when a simple real eigenvalue crosses the origin. In the stationary case under the same hypotheses introduced here (critical eigenvalue simple and $\operatorname{Re}\gamma'(0)\neq 0$), the bifurcation may be two-sided – that is, one supercritical and one subcritical solution. This happens, for example, in the generalized convection problems [1]. Furthermore, even when the bifurcation is one-sided, there are two physically distinct solutions. In the case of bifurcation of periodic solutions, however, the content of the preceeding theorem and corollaries is that, apart from *phase differences*, there is only *one* periodic solution which bifurcates.

10. Perturbation Theory of Critical Floquet Exponents

We wish to establish the convergence of the formal power series constructed in Section 5 for the critical Floquet exponent and eigenfunction which satisfy (5.3). We write the problem in the simpler operator notation

$$\omega(\varepsilon)\frac{\partial\phi}{\partial s}-\sigma\omega\phi+L(\mu(\varepsilon))\phi+M(v,\phi)=0. \tag{10.1}$$

For simplicity we shall assume (10.1) came from an equation with quadratic nonlinearity $N(v,v)$, but the argument applies to general analytic nonlinearities. We assume throughout that $\mu_2\neq 0$.

One solution of (10.1) is, of course, $\sigma=0$ and $\phi=\frac{\partial u}{\partial s}$: (recall our convention $v=\varepsilon u$)

$$\omega(\varepsilon)\frac{\partial^2 u}{\partial s^2}+L(\mu(\varepsilon))\frac{\partial u}{\partial s}+M\left(v,\frac{\partial u}{\partial s}\right)=0. \tag{10.2}$$

(Recall that u has the expansion $u=u_0+\varepsilon u_1+\cdots$ where we assume that $[u_0]_1=1$.) The second order term u_1 satisfies

$$\frac{\partial u_1}{\partial s}+L_0u_1+N(u_1,u_1)=0, \qquad [u_1]_1=0.$$

The second eigenfunction is sought in the form

$$\phi=a(\varepsilon)\frac{\partial u}{\partial s}+\psi(\varepsilon). \tag{10.3}$$

Substituting (10.3) into (10.1) we obtain

$$\omega \frac{\partial \psi}{\partial s} - \sigma \omega \psi + L(\mu(\varepsilon))\psi + \varepsilon M(u, \psi) - \sigma \omega a \frac{\partial u}{\partial s} = 0. \qquad (10.4)$$

We construct a solution of (10.4) of the form

$$\psi = u_0 + \varepsilon \eta, \quad [\eta]_1 = 0,$$
$$\sigma(\varepsilon) = \varepsilon^2 \delta(\varepsilon),$$
$$a(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots$$

where all quantities are convergent power series. Recall

$$\omega(\varepsilon) = 1 + \omega_2 \varepsilon^2 + \cdots, \quad \mu(\varepsilon) = \mu_2 \varepsilon^2 + \cdots.$$

We assume $\mu_2 \neq 0$. Substitution gives, after some simplification,

$$\left[\frac{\partial \eta}{\partial s} + L_0 \eta + M(u_0, u_0)\right] + \varepsilon \left[\omega_2 \frac{\partial u_0}{\partial s} - \delta u_0 + \mu_2 L_1 u_0 + \omega_2 \frac{\partial \eta}{\partial s} + M(u_1, u_0) + M(u_0, \eta) + \delta a \frac{\partial u_0}{\partial s}\right] + O(\varepsilon^2) = 0. \qquad (10.5)$$

The orthogonality condition is

$$-\delta - i\omega_2 + \mu_2 [L_1 u_0]_1 + [M(\phi, u_0)]_1 + [M(u_0, \eta)]_1 + i \delta a + O(\varepsilon) = 0.$$

Taking real and imaginary parts of this equation, we obtain

$$-\delta + \mu_2 \operatorname{Re}[L_1 u_0]_1 + \operatorname{Re}[M(u_1 u_0)]_1 + \operatorname{Re}[M(u_0, \eta)]_1 + O(\varepsilon) = 0, \qquad (10.6)$$

and

$$-\delta a - \omega_2 + \mu_2 \operatorname{Im}[L_1 u_0]_1 + \operatorname{Im}[M(u_1, u_0)]_1 + \operatorname{Im}[M(u_0, \eta)]_1 + O(\varepsilon) = 0. \qquad (10.7)$$

For $\varepsilon = 0$ these equations have the solutions

$$\frac{\partial \eta}{\partial s} + L_0 \eta + M(u_0, u_0) = 0, \quad \eta = 2u_1,$$
$$-\delta + \mu_2 \operatorname{Re}[L_1 u_0]_1 + 3 \operatorname{Re}[M(u_1, u_0)]_1 = 0, \qquad (10.8)$$
$$\delta_0 = -2\mu_2 \operatorname{Re}[L_1 u_0]_1.^*$$

Similarly when $\varepsilon = 0$, (10.7) may be solved for a_0 provided $\delta_0 \neq 0$, that is, that $\mu_2 \neq 0$. This means we must assume Re $[M(u_1, u_0)]_1 \neq 0$.

We can now apply an implicit function theorem argument to solve (10.5)–(10.7) for $\varepsilon \neq 0$, just as in Section 8. The Frechet derivative of these equations

* From the development of the Poincaré-Lindstedt series for the periodic solution one obtains

$$\mu_2 \operatorname{Re}[L_1 u_0] + \operatorname{Re}[M(u_1, u_0)] = 0.$$

The last relation of (10.8) therefore follows from the second.

at $\varepsilon = 0$ is (apply $\hat{K}\hat{Q}$ to (10.5))

$$\begin{bmatrix} I & 0 & 0 \\ \operatorname{Re}[M(u_0, \cdot)]_1 & 1 & 0 \\ \operatorname{Im}[M(u_0, \cdot)]_1 & a_0 & \delta_0 \end{bmatrix}$$

which is invertible provided $\delta_0 \neq 0$.

Theorem 10.1. *An implicit function theorem can be applied to prove the convergence of the series for a and ψ in* (10.3) *and the power series for σ provided that* $\mu_2 \neq 0$. *In the case of a quadratic nonlinearity, this amounts to* Re $[L_1 u_0]_1 \neq 0$ *and* Re$[M(u_1, u_0)]_1 \neq 0$. *In the general case of arbitrary nonlinearities, the relationship* (10.8) *is still valid*:

$$\sigma_2 = -2\mu_2 \operatorname{Re}[L_1 u_0]_1. \qquad (10.9)$$

11. Remarks on Hopf's Paper

We wish to point out briefly how the Hopf's perturbation scheme for the critical Floquet exponents may be clarified. Hopf's argument comes down to showing that the eigenvalue problem (5.19, p. 19)

$$\beta_2 \rho = A_1 \rho + B_1 \sigma,$$
$$\beta_2 \sigma = A_2 \rho + B_2 \sigma,$$

has distinct eigenvalues. In Hopf's paper, $\beta(\varepsilon) = \beta_2 \varepsilon^2 + \cdots$ is *either* one of the critical Floquet exponents, while the Floquet eigenfunction has an expansion

$$\phi = \rho z + \sigma \dot{z} + \varepsilon \phi_1 + \cdots$$

(z and $\dot{z}$ are the two solutions of $\dot{z} + L_0 z = 0$).

The one solution, obtained from differentiating the periodic solution with respect to time ($\dot{\eta}$ – see 2.29), is $\rho = 0$, $\sigma = 1$, and $\beta_2 = 0$. This implies that $B_1 = B_2 = 0$. Now in (5.23) (p. 21) he states,

"Da (5.19) die Lösung $\beta_2 = \rho = 0$ hat und das zweite $\beta_2 = 0$ ist, ist

(5.23) $\quad B_1 = B_2 = 0, \quad A_1 \neq 0$."

However, it is not clear to us how it is established that the second eigenvalue is distinct from zero.

On the other hand, by (5.20)

$$A_1 = [\mathscr{L}_1(g) + \mathscr{L}_2(z)]_1$$

and comparing this term with the first equation in (4.16), one can prove that $A_1 \neq 0$ under the assumption $\mu_2 \neq 0$. This the second eigenvalue is certainly non-zero, as claimed.

12. Discussion

The results obtained in this paper have some relevance to the problem of transition from laminar to turbulent flow. We want to distinguish two different kinds of transition: (*a*) transition to turbulence via repeated branching and (*b*) transition to turbulence via a snap through instability.

The basis of our discussion is the formula (5.12) for the Floquet exponent, $\sigma=\sigma_2\varepsilon^2+\cdots$, where

$$\sigma_2=-2\xi'\lambda_2.$$

Here $\xi'=d\xi/d\lambda$ is positive in the usual situation (whenever the basic steady motion loses stability as R is increased past R_c). As we have seen in Theorem 5.1, supercritical periodic solutions are stable and subcritical periodic solutions are unstable.

The Floquet analysis of the stability of bifurcating solutions, which leads to equation (5.12), gives a mathematical structure to ideas set out by LANDAU in his essay on the problem of turbulence. Three ideas have a prominent place in Landau's essay: (1) The initial exponential growth of small disturbances when $R>R_c$ will actually be limited by nonlinear interactions of a type described by the truncated amplitude equation (6.11). This idea of Landau's has been further developed by STUART [4]. (2) The terminal state for a disturbance of steady flow with R slightly greater than R_c is a time periodic motion. (3) As R is increased further, the time periodic solution loses its stability to a quasi-periodic solution of the type $e^{\sigma t}f(x,t)$ where f has period ω, but $\sigma=\sigma_1+i\omega_1$ is complex and $\omega\neq\omega_1$. "The result is a quasi-periodic (almost periodic) motion characterized by two different periods."

"In the course of the further increase of the Reynolds number, new periods appear in succession, and the motion assumes an involved character typical of a developed turbulence .... So a turbulent motion is, to a certain extent, a quasi-periodical motion."

In the supercritical case, our analysis is a realization of Landau's idea (2). The analysis gives the periodic solution and shows that in the supercritical case, when $R-R_c$ is small, the periodic solution is stable to infinitesimal disturbances. (Of course, the problem of establishing the validity of the Floquet theory for partial differential equations remains.) Landau's conjecture about the bifurcation of quasi-periodic solutions from the periodic solution when it loses stability at some $R=R_1>R_c$ is a highly interesting one.

The transition to turbulence through repeated branching cannot, however, be the relevant description in the case of subcritical bifurcations. In this case, the time periodic solution which bifurcates from the steady solution is unstable ($\sigma_2<0$) from the start; it has no domain of attraction and an arbitrary initial disturbance of the steady solution either decays or is attracted to something else, perhaps a stable "turbulent solution".*

Experiments seem to indicate that in pipe and channel flows the domain of attraction of the "turbulent solution" is larger than the steady solution; the steady solution can be maintained for large $R<R_c$ only if considerable efforts are made to suppress the size of disturbances [17, p. 432). It would, therefore, appear that in the subcritical case there are two "stable solutions" (shown as heavy lines in Fig. 1a): the steady flow and the turbulent flow. Initial disturbances of appropriate size and form snap through the unstable periodic solution and come to rest on the turbulent solution.

* Our use here of the words "turbulent solution" is loose. Behind these words is the observation that when the external conditions are steady, certain average values of turbulent fields are reproducible in experiments [17]. Thus, a "turbulent solution" could be a set of actual solutions possessing the same statistical properties.

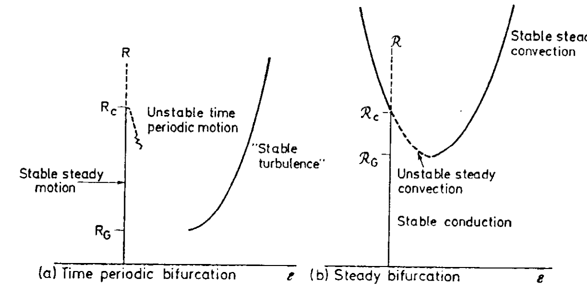

Fig. 1a and b. Bifurcation curves for simple eigenvalues: ---- unstable solutions, —— stable solutions. Here $|\varepsilon|$ could be taken as some measure of the amplitude of the motion (say an L_2 norm) averaged over all time.

(a) *The time periodic bifurcation.* The picture near R_c is established. The bifurcation is either up or down and not up and down as in (b). Upward bifurcations are stable and downward bifurcations are unstable. The situation which holds when $|\varepsilon|$ is not small is not known. Experiments show that there is a global stability limit R_G below which all disturbances decay and that the realized motions have the property that the amplitude $|\varepsilon|$ of the turbulence increases with R.

(b) *The steady bifurcation.* This picture applies, for example, to generalized convection in a bounded domain [1] when $\mathscr{R}_c$ is a simple eigenvalue. Here the principal eigenvalue for the linearized stability problem for steady heat conduction is real and the bifurcating convective solutions are steady. The picture shown can be completely established in the small by perturbation theory.

The existence of two stable subcritical "solutions" which one can infer from experiments in the case of pipe and channel flows can be completely established for some cases of bifurcation from a real, simple eigenvalue. Subcritical convection in a bounded domain is an example where such stable subcritical solutions occur [1]. In convection problems, the critical Rayleigh number $\mathscr{R}_c$ is a simple eigenvalue, and the bifurcation curve may have the form shown in Fig. 1b.

The existence of subcritical solutions and the "snap through" description of their instability would appear to be germane to many physical problems. The physical importance of the snap through phenomena, however, would ordinarily be related to the extent of subcriticality. For example, in steady convection, extreme conditions are often required to induce a modest degree of subcriticality [18]. On the other hand, in pipe flow, say Poiseuille flow down an annular pipe, the extent of subcriticality is always deep [19, Fig. 1]. Experimentally the bottom of the bifurcation curve is $R=R_G\cong 2000$. However, calculations show that R_c varies from a minimum of about 11,600 monotonically to infinity as the radius of the inner pipe is reduced. In the round pipe, $R_c=\infty$; hence, Poiseuille flow is stable in the linearized theory for all R and any other motions (the observed ones) are of the subcritical type.

Addendum. Since this article went to press the authors have become aware of two recent articles which treat the subject of bifurcation and stability of periodic solutions of the Navier Stokes equations by different methods. We should like to refer the reader to JUDOVIC, *Generation of selfoscillations within a viscous fluid*, Prikl. Mat. Mek. **35**, 638–655 (1971) (Russian), and a forthcoming article by G. IOOSS, *Existence et stabilité de la solution périodique secondaire intervenant dans les problèmes d'evolution du type Navier-Stokes*.

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