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Global stability of spiral flow. Part 2

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The stability of spiral flow between rotating and sliding cylinders is considered. In the limit of narrow gap, a 'modified' energy theory is constructed. This theory exploits the consequences of assuming the existence of a preferred spiral direction along which disturbances do not vary. The flow is also analyzed from the viewpoint of linearized theory. Both problems depend strongly on the sign of Rayleigh's discriminant, $-2\Omega\zeta$. Here Ω is the component of angular velocity, and ζ is the component of total vorticity of the basic flow in the direction perpendicular to the spiral ribbons on which the disturbance is constant. When the discriminant is negative, there is evidently no instability to infinitesimal disturbances, and the spiral disturbance whose energy increases at the smallest R is a roll whose axis is perpendicular to the stream. This restores and generalizes Orr's non-linear results for disturbances having a preferred spiral direction. When the discriminant is positive, the critical disturbances of linear theory and the modified energy theory are spiral vortices. The differences between the energy and linear limits can be made smaller in the restricted class of disturbances with coincidence achieved for axisymmetric disturbances in the rotating cylinder problem in the limit of narrow gap. For the sliding–rotating case, the critical disturbance of the linear theory appears as a periodic wave in a co-ordinate system fixed on the outer cylinder. This wave has a dimensionless frequency equal to $-\frac{1}{2}a\sin(\chi-\psi)$, where a is the wave-number, χ is the angle between the pipe axis and the direction of motion of the inner cylinder relative to the outer one, and ψ is the disturbance spiral angle.

Instability limits, frequencies and wave-numbers are computed numerically when the cylinder gap is not narrow. These are in even closer agreement with Ludwieg's experimental results than the approximate results which were given in part 1.

1. Introduction

This study continues the work reported in Joseph & Munson (1970), henceforth called part 1, in which the stability of Couette flow between rotating and sliding cylinders was studied. There we analyzed the basic spiral character of the instability, and showed how a suitably adjusted rotation could be chosen to bring the stability and instability limits together or nearly together. An approximate linear theory was given, and the results of this and the energy theory compared with the experiments of Ludwieg. In this paper, we continue the study of the stability of the spiral flow from the viewpoint of global theory, in which the linear stability theory gives sufficient conditions for instability, and energy theory gives sufficient conditions for stability.

The novel point here is the working out of a 'modified' energy theory, in which one exploits the consequences of assuming the existence of a preferred spiral direction along which disturbances do not vary. A basis for comparison of the results of such a non-linear theory with linear theory follows from the fact that the linear theory necessarily generates a preferred direction (wave-numbers along the axis and azimuth).

A modified energy theory is developed for Couette flow between rotating cylinders in Joseph & Hung (1971). There, the consequences of assuming axial symmetry of the (non-linear) disturbance from the start are explored. It is shown in Joseph & Hung that, in the class of axially symmetric solutions, the energy criterion for non-linear stability is in nearly perfect agreement with Taylor's linear stability boundary when the two cylinders rotate in the same sense. The criterion of Joseph & Hung is global (it applies to any initial condition) only for the rotating plane Couette flow limit (RPCF); otherwise, the criterion holds only when the initial amplitude is smaller than some critical value which is explicitly estimated in Joseph & Hung.

It is preferable, for several reasons connected with the clarity of the exposition and the finality of the results, to consider in detail (§§ 3, 4) the 'narrow gap' limit (i.e. RPCF). The effect of sliding cylinders along the axis in the round geometry is taken up in RPCF by the fact that the direction of shearing and rotation are not required to be perpendicular.

The modified energy analysis gives: (i) the form of the spiral disturbance that will make the disturbance 'energy' increase at the smallest R (Reynolds number); (ii) the smallest R; and (iii) the form of the 'energy' (the Lyaponouv functional) that increases initially at the smallest R. In explanation of the difference between energy analysis and modified energy analysis we note that the two analyses refer to different energies. The modified energy analysis holds only for spiral disturbances; though energy analysis leads finally to spiral disturbances, it is not restricted to these at the beginning.

In the analysis of rotating plane Couette flow, Rayleigh's discriminant $-2\Omega\zeta$ enters strongly (Ω is the angular velocity of the fluid and ζ is its total vorticity). The importance of this quantity in spiral flow was first noted by McIntyre & Pedley (in Pedley 1969). Here the relevant discriminant is the product $(-2\Omega_{y'}\zeta_{y'} = F)$ of the component of -2Ω and ζ normal to planes on which the disturbance is constant. Numerical analysis of the full linear theory problem gives solutions only when $F > 0$ (destabilizing), and never when $F \leqslant 0$ (stabilizing).

The modified energy theory for RPCF also depends strongly on the sign of F. We prove, in § 4, that when $F \leqslant 0$ the spiral disturbance that makes a disturbance 'energy' increase initially at the smallest R is a transverse vortex (of the Orr–

Sommerfeld type). This result restores with rigour and with greater generality (but only for a restricted set of (spiral) initial conditions) Orr's original energy result and the critical limit $R = 177{\cdot}22$. It still remains true, of course, that a longitudinal vortex disturbance can be found, which will make the actual disturbance energy increase initially when $R > 2\sqrt{1708}$ (cf. part 1).

Finally, in §5 we report results of exact numerical analysis of the linear stability problem for spiral Couette flow. This replaces the mean-radius approximation used for the linear theory in part 1. In some of the quantitative details of the instability, the exact result differs appreciably from the approximation. The agreement between the exact linear theory and the experiments of Ludwieg (1964) is good (§6).

2. Spiral vortex disturbances of spiral Couette flow

Spiral Couette flow is induced by the shearing motion of one cylinder (of radius R_1) relative to a second, larger cylinder (of radius R_2) on the fluid that fills the annulus between the cylinders. The velocities of the boundary are steady, and give rise to a steady solution of the Navier–Stokes equations,

$$\mathbf{U} = \left(A\hat{r} + \frac{B}{\hat{r}}\right)\mathbf{e}_\phi + U_c\,\frac{\ln(\hat{r}/R_2)}{\ln\eta}\,\mathbf{e}_x, \tag{2.1}$$

where $(x, \phi, \hat{r})$ are cylindrical co-ordinates, $\eta = R_1/R_2$, U_c is the sliding speed of the inner cylinder relative to the outer,

$$A = \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2}, \quad B = \frac{-R_2^2R_1^2(\Omega_2-\Omega_1)}{R_2^2 - R_1^2},$$

and Ω_1 and Ω_2 are the angular velocities of the inner and outer cylinders.

The equations governing the evolution of a disturbance $\mathbf{u} = (u_x, u_\phi, u_{\hat{r}})$ of the basic spiral flow $\mathbf{U} = (U_x, U_\phi, 0)$ are, for convenience, written in a co-ordinate system fixed on the outer cylinder and rotating at a constant angular velocity. Thus, in the annulus,

$$\left.\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega}\times\mathbf{u} + (\mathbf{U}+\mathbf{u}).\nabla\mathbf{u} + \mathbf{u}.\nabla\mathbf{U} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\mathbf{u},\\
&\operatorname{div}\mathbf{u} = 0, \quad \mathbf{u} = 0 \quad \text{at} \quad R_1 \quad \text{and} \quad R_2,\\
&\mathbf{u} \text{ is periodic in } \phi,\\
&\mathbf{u} \text{ is periodic, almost periodic or belongs to a}\\
&\qquad\qquad\qquad\text{Fourier transform class in } x,\\
\text{and}\quad &\mathbf{u} = \mathbf{u}_0 \not\equiv 0 \quad \text{at} \quad t = 0.
\end{aligned}\right\} \tag{2.2a–f}$$

The idea of this paper is to exploit several consequences of the assumption that disturbances on cylinders of radius r around the axis are constant along certain spiral lines. This idea is most readily carried out in RPCF. It may help the reader to turn to §4 for a discussion of the energy consequences of the assumption of a spiral disturbance. We choose, however, to start the analysis with a discussion of the linear theory. It is in the linear theory that the role of the Rayleigh discriminant is most critical; the sign of this discriminant seems decisive for the existence of instability in a range of parameters close to those investigated in Ludwieg's (1964) experiment.

3. Linear stability analysis of RPCF

For definiteness, we shall choose $\mathbf{\Omega} = -\Omega_2 \mathbf{e}_x$, where $\Omega_2 > 0$ is the angular velocity of the outer cylinder. Then the co-ordinates rotate with the outer

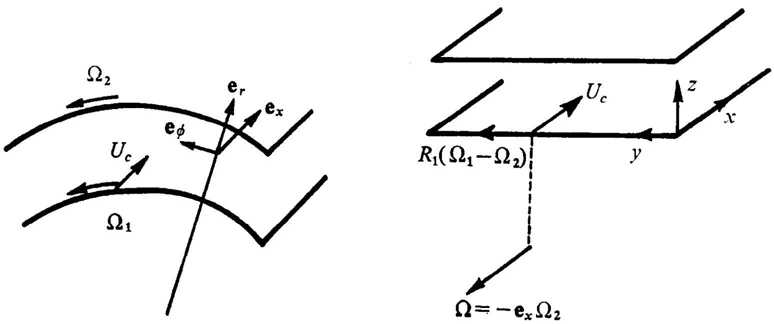

FIGURE 1. Rotating sliding cylinders in a co-ordinate system rotating with the outer cylinder in the narrow gap (rotating plane Couette flow) limit.

cylinder. In the rotating system, the outer cylinder is stationary, and the inner cylinder moves with a circumferential speed $R_1(\Omega_1-\Omega_2)$, and slides forward at the rate U_c (see figure 1). In the limit $\eta \to 1$ one may find that

$$U_\phi = \frac{\eta R_1(\Omega_2-\Omega_1)[(\hat{r}/R_2)^2-1]}{(1-\eta^2)\hat{r}/R_2} \to \frac{R_1(\Omega_1-\Omega_2)}{R_2-R_1}[(R_2-R_1)-\hat{z}] \qquad (3.1a)$$

and
$$U_x = U_c \ln(\hat{r}/R_2)/\ln\eta \to \frac{U_c}{R_2-R_1}[(R_2-R_1)-\hat{z}], \qquad (3.1b)$$

where we have set $\hat{r} = R_1+\hat{z}\,(0 \leqslant \hat{z} \leqslant R_2-R_1)$.

Since we are considering the limit $\eta \to 1$, we must allow for the possibility that, with a fixed rate of shear,
$$R_2-R_1 \to 0.$$
Then we should want

$$\frac{R_1(\Omega_1-\Omega_2)}{R_2-R_1} = \Omega_V, \quad \frac{U_c}{R_2-R_1} = \Omega_U \qquad (3.2)$$

to be bounded. The boundedness of Ω_V, when $\eta \to 1$, implies that $\Omega_2-\Omega_1 \to 0$. Hence, apart from the uninteresting case in which $\Omega_1 \to 0$, $\Omega_2 \to 0$, we are here restricted to the problem in which Ω_2 and Ω_1 have the same sense.

In dimensionless variables $z = \hat{z}/(R_2-R_1)$, we have

$$V = \frac{U_\phi}{U'} = (1-z)\sin\chi, \quad U = \frac{U_x}{U'} = (1-z)\cos\chi, \qquad (3.3a,b)$$

where $$U' = [R_1^2(\Omega_1-\Omega_2)^2+U_c^2]^{\frac{1}{2}} = (R_2-R_1)(\Omega_U^2+\Omega_V^2)^{\frac{1}{2}},$$

$$\sin\chi = \Omega_V(R_2-R_1)/U' = \Omega_V/(\Omega_U^2+\Omega_V^2)^{\frac{1}{2}}, \quad \cos\chi = \Omega_U/(\Omega_U^2+\Omega_V^2)^{\frac{1}{2}},$$

and $$\tilde{\Omega}_2 = \Omega_2/(\Omega_U^2+\Omega_V^2)^{\frac{1}{2}}.$$

Disturbances of rotating plane Couette flow necessarily satisfy

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}+(\mathbf{u}+\mathbf{U})\,.\,\nabla u - w\cos\chi = -\partial_x p+\frac{1}{R}\nabla^2 u,\\
&\frac{\partial v}{\partial t}+(\mathbf{u}+\mathbf{U})\,.\,\nabla v+(2\tilde{\Omega}_2-\sin\chi)\,w = -\partial_y p+\frac{1}{R}\nabla^2 v,\\
&\frac{\partial w}{\partial t}+(\mathbf{u}+\mathbf{U})\,.\,\nabla w - 2\tilde{\Omega}_2 v = -\partial_z p+\frac{1}{R}\nabla^2 w,\\
\text{and}\quad &\partial_x u+\partial_y v+\partial_z w = 0,
\end{aligned}\right\}\tag{3.4a–d}$$

where $$\mathbf{U} = \mathbf{e}_x U+\mathbf{e}_y V \quad\text{and}\quad R = U'(R_2-R_1)/\nu.$$

The disturbance $\mathbf{u}$ and p are assumed to be almost periodic (in x, y) and $\mathbf{u} = 0$ at $z = 0, 1$.

The equations which follow upon linearizing (3.4) may be combined into

$$\left(\frac{d}{dt}-\frac{1}{R}\nabla^2\right)\nabla^2 w - 2\tilde{\Omega}_2\phi = 0, \tag{3.5a}$$

$$\left(\frac{d}{dt}-\frac{1}{R}\nabla^2\right)\phi+\cos\chi\,\partial^2_{xy}w+(2\tilde{\Omega}_2-\sin\chi)\,\partial^2_{xx}w = 0, \tag{3.5b}$$

where $$\phi = (\partial^2_{xx}+\partial^2_{yy})\,v+\partial^2_{zy}w,$$

and $$\frac{d}{dt} = \frac{\partial}{\partial t}+(1-z)\sin\chi\,\partial_y+(1-z)\cos\chi\,\partial_x.$$

At the boundary, $$\phi = w = \partial_z w = 0. \tag{3.5c}$$

Assuming disturbances in the form of Fourier series with terms proportional to $\exp\{\sigma t+i\alpha x+i\beta y\}$, we find, for the Fourier coefficients $\hat{w}(z)$ and $\hat{\phi}(z)$,

$$\left.\begin{aligned}
&\left(\mathrm{Re}\,(\sigma)+i\mathscr{S}-\frac{1}{R}L\right)L\hat{w}-2\tilde{\Omega}_2\hat{\phi} = 0,\\
&\left(\mathrm{Re}\,(\sigma)+i\mathscr{S}-\frac{1}{R}L\right)\hat{\phi}-\alpha(\beta\cos\chi+\alpha(2\tilde{\Omega}_2-\sin\chi))\,\hat{w} = 0,\\
&\hat{\phi} = \hat{w} = D\hat{w} = 0|_{z=0,1},
\end{aligned}\right.\tag{3.6a–c}$$

where $L = D^2-a^2$, $D = d/dz$, $a^2 = \alpha^2+\beta^2$, $\mathscr{S} = c-az\sin(\chi-\psi)$,

$$c = \mathrm{Im}\,(\sigma)+a\sin(\chi-\psi),$$

and $$\sin(\chi-\psi) = \frac{\beta}{a}\sin\chi+\frac{\alpha}{a}\cos\chi, \tag{3.7}$$

where $$\alpha = -a\sin\psi, \quad \beta = a\cos\psi. \tag{3.8}$$

Here ψ is the angle between x and x', where x is the parallel to $\mathbf{\Omega} = -\tilde{\Omega}_2\mathbf{e}_x$, and x' is the direction along which disturbances are constant (figure 2).

Equations (3.6a, b) combine into

$$\left(\mathrm{Re}\,(\sigma)+i\mathscr{S}-\frac{1}{R}L\right)^2 L\hat{w}+a^2F\hat{w} = 0, \tag{3.9}$$

which is to be solved along with

$$\hat{w} = D\hat{w} = \left(\mathrm{Re}\,(\sigma)+i\mathscr{S}-\frac{1}{R}L\right) L\hat{w} = 0|_{z=0,1}. \tag{3.10}$$

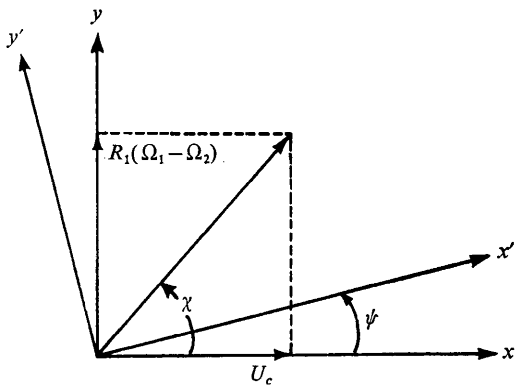

FIGURE 2. The basic flow spiral angle χ and the disturbance angle ψ. For the critical energy disturbance, $\psi = \psi_{\mathscr{E}}$. For the critical linear disturbance, $\psi = \psi_L$.

Here $$F(\psi, \chi, \tilde{\Omega}_2) = 2\tilde{\Omega}_2 \sin\psi(\cos(\chi-\psi)-2\tilde{\Omega}_2\sin\psi) \tag{3.11}$$

and $$-\infty \leqslant F \leqslant \tfrac{1}{4}\cos^2(\chi-\psi).$$

The linear stability limit R_L is the smallest value of R over all neutral eigensolutions ($\mathrm{Re}\,(\sigma) = 0$) and over the wave-number radius a and angle ψ.

A special solution of (3.9) can be found when $\mathscr{S} = 0$. It will not ordinarily be possible to put $\mathscr{S}(z) \equiv 0$, but this is possible when $\chi = \psi$. Then the spiral angle χ, which is equal to the energy spiral angle $\psi_{\mathscr{E}}$ (see figure 2) is also the angle ψ along which disturbances do not vary. In this case, the problem $L^3\hat{w}+R^2a^2F\hat{w} = 0$ and (3.10) define the Bénard problem,

$$R^2F(\chi,\chi,\tilde{\Omega}_2) = g(a^2), \quad \min_{a^2} g(a^2) = g((3{\cdot}12)^2) = 1708. \tag{3.12}$$

Equations (3.12) were first given by Kiessling (1963). They imply that

$$R^2 = 1708/F(\chi,\chi,\tilde{\Omega}_2).$$

Of course, the special solution (3.12) could hold only so long as $\tilde{\Omega}_2$ ($-\infty < \tilde{\Omega}_2 < \infty$) is compatible with the inequality

$$F(\chi,\chi,\tilde{\Omega}_2) = 2\tilde{\Omega}_2\sin\chi(1-2\tilde{\Omega}_2\sin\chi) > 0.$$

The values of χ and $\tilde{\Omega}_2$ which make $F = \frac{1}{4}$ have a special importance. *Suppose that* $\tilde{\Omega}_2 = \frac{1}{4}\sin\chi$. *Then* $F = \frac{1}{4}$ *and the criterion*

$$R < ((R_2-R_1)/\nu)(\Omega_U^2+\Omega_V^2)^{\frac{1}{2}} < 2\sqrt{1708}$$

is both necessary and sufficient for all periodic disturbances to decay monotonically. The proof of this theorem follows by comparing the linear solution with the unmodified energy problem (Busse 1970; see part 1). It follows that the criterion $R < 2\sqrt{1708}$ is both necessary and sufficient for stability, and that the most persistent infinitesimal disturbance is just the one whose energy increases initially at the smallest R. In the general case the energy spiral angle χ and the disturbance spiral angle ψ do not coincide. Then $\mathscr{S}$ cannot vanish everywhere.

	$\tilde{\Omega}_2$	$a_{\mathscr{E}}$	$\psi_{\mathscr{E}}$	λ	$R_{\mathscr{E}}$	$F_{\mathscr{E}}$	a_L	ψ_L	Im (σ)	R_L	F_L
$\eta = 1{\cdot}0$	0·30	3·12	34·80°	0·6057	91·95	0·1936	2·925	20·915°	0·27693	111·19	0·1644
$\chi = 10°$	0·50	3·12	25·85°	0·8290	85·68	0·2293	3·009	19·55°	0·24961	92·97	0·2180
	0·70	3·12	19·77°	0·9250	83·74	0·2424	3·061	17·10°	0·18917	86·62	0·2390
	0·90	3·12	15·75°	0·9649	83·01	0·2474	3·10	14·50°	0·12161	84·08	0·2462
	1·44	3·12	10·00°	1·000	82·66	0·2500	3·12	10·00°	0	82·66	0·2500
	1·80	3·12	8·25°	1·0696	82·73	0·2495	3·114	8·10°	−0·05162	82·84	0·2497
	2·20	3·12	6·75°	1·0747	82·83	0·2489	3·101	6·63°	−0·091144	83·24	0·2491
	2·40	3·12	6·15°	1·0635	82·86	0·2486	3·097	6·20°	−0·10264	83·44	0·2485
$\eta = 1{\cdot}0$	0·20	3·12	51·01°	0·4993	92·84	0·1936	3·07	35·70°	0·152456	100·11	0·1778
$\chi = 30°$	0·30	3·12	43·02°	0·7246	85·48	0·2325	3·093	34·75°	0·12806	88·36	0·2239
	0·40	3·12	35·90°	0·8925	83·15	0·2466	3·108	32·60°	0·070494	83·79	0·2448
	0·50	3·12	30·00°	1·000	82·66	0·2500	3·12	30·00°	0	82·66	0·2500
	0·60	3·12	25·40°	1·0678	82·92	0·2481	3·11	27·15°	−0·077317	83·46	0·2471
	0·70	3·12	21·73°	1·0998	83·43	0·2443	3·086	24·38°	−0·151106	85·43	0·2412
	0·80	3·12	18·90°	1·1193	84·06	0·2400	3·042	21·77°	−0·21773	88·13	0·2352
	0·90	3·12	16·65°	1·1280	84·65	0·2358	3·00	19·48°	−0·27392	91·23	0·2299
	1·00	3·12	14·81°	1·1265	85·20	0·2320	2·935	17·51°	−0·317375	94·55	0·2254
$\eta = 1{\cdot}0$	0·10	3·12	75·87°	0·2526	105·83	0·1489	3·106	62·13°	0·05772	108·76	0·1454
$\chi = 60°$	0·15	3·12	71·61°	0·4097	92·48	0·1978	3·105	62·21°	0·059868	93·91	0·1948
	0·20	3·12	67·89°	0·5978	86·06	0·2297	3·110	61·50°	0·040705	86·67	0·2278
	0·28868	3·12	60·00°	1·000	82·66	0·2500	3·12	60·00°	0	82·66	0·2500
	0·40	3·12	47·35°	1·5192	86·14	0·2279	3·10	56·10°	−0·105424	88·49	0·2216
	0·50	3·12	35·54°	1·7669	92·52	0·1912	2·98	50·86°	−0·23668	108·33	0·1642
	0·60	3·12	26·40°	1·7823	98·95	0·1597	2·404	42·82°	−0·35504	176·31	0·1137

TABLE 1. Values of the critical parameters of the linear and energy theory when $\eta = 1$. (For $\chi = 90°$, see figure 3)

For the case $\mathscr{S} \equiv 0$, McIntyre and Pedley (Pedley 1969) have shown that a necessary and sufficient condition for the existence of an inviscid ($R \to \infty$) solution of (3.6*a*) is that $F > 0$. Furthermore, they show that $-F$ is the product of the overall angular velocity and the total vorticity, i.e. F is Rayleigh's discriminant. In the general case $\chi \neq \psi$, F is still Rayleigh's discriminant, i.e.

$$F(\psi, \chi, \tilde{\Omega}_2) = -2\Omega_{y'}\zeta_{y'}, \tag{3.13}$$

where $\Omega_{y'} = \mathbf{e}_{y'}.\mathbf{\Omega}$ and $\zeta_{y'} = \mathbf{e}_{y'}.(2\mathbf{\Omega} + \text{curl}\,\mathbf{U})$ are components of the overall angular velocity ($\mathbf{\Omega}$) and total vorticity ($2\mathbf{\Omega} + \text{curl}\,\mathbf{U}$) in the direction y' normal to the direction x' in which disturbances do not vary. To verify (3.13) note that $\mathbf{\Omega} = -\tilde{\Omega}_2\mathbf{e}_x$, $\mathbf{U} = (\mathbf{e}_x \cos\chi + \mathbf{e}_y \sin\chi)(1-z)$ and use the geometry of figure 2.

Numerical analysis of (3.9), (3.10) gives solutions for whenever $F > 0$, and not otherwise (see table 1). Problem (3.9), (3.10) contains plane Couette flow ($F = 0$)

as a special case. A rigorous demonstration that there are no solutions of (3.9), (3.10) with $\mathrm{Re}(\sigma) \geqslant 0$ and $F \leqslant 0$ has yet to be constructed. The determination of the wave speed $\mathrm{Im}(\sigma)$ of the most persistent small disturbance has a particularly simple solution. The answer is that

$$\mathrm{Im}(\sigma) = -\tfrac{1}{2}a\sin(\chi-\psi). \tag{3.14}$$

The argument leading to (3.14) starts with the observation that every solution of (3.9), (3.10), or the equivalent problem (3.6*a–c*), has

$$\langle \mathscr{S}[|2\tilde{\Omega}_2\hat{\phi}|^2 + a^2F(|D\hat{w}|^2 + a^2|\hat{w}|^2)]\rangle = 0.\dagger \tag{3.15}$$

To prove (3.15) set $f = -2\tilde{\Omega}_2\hat{\phi}$ and introduce F of (3.11) into (3.6*b*). This leads us to

$$[\mathrm{Re}(\sigma) + i\mathscr{S}]L\hat{w} = \frac{1}{R}L^2\hat{w} - f, \tag{3.16a}$$

and
$$[\mathrm{Re}(\sigma) + i\mathscr{S}]f = \frac{1}{R}Lf + a^2F\hat{w}. \tag{3.16b}$$

Form $\langle \hat{w}^*(3.16a)\rangle - \langle f^*(3.16b)\rangle/a^2F$ to produce

$$\left\langle (\mathrm{Re}(\sigma) + i\mathscr{S})\left(\hat{w}^*L\hat{w} - \frac{|f|^2}{a^2F}\right)\right\rangle = \frac{1}{R}\left\langle |L\hat{w}|^2 + \frac{|Df|^2 + a^2|f|^2}{a^2F}\right\rangle - \langle fw^* + f^*w\rangle. \tag{3.17}$$

where * designates the complex conjugate. Subtraction of the complex conjugate of (3.17) from (3.17) gives

$$0 = \left\langle \mathscr{S}\left(\hat{w}^*L\hat{w} + \hat{w}L\hat{w}^* - \frac{2|f|^2}{a^2F}\right)\right\rangle = -2\langle \mathscr{S}(|D\hat{w}|^2 + a^2|\hat{w}|^2 + |f|^2/a^2F)\rangle,$$

where the term
$$\langle D\mathscr{S}(\hat{w}^*D\hat{w} + \hat{w}D\hat{w}^*)\rangle = 0,$$

which arises from integration by parts, vanishes because $D\mathscr{S}$ is a constant.

Equation (3.15) shows that $\mathscr{S}(z)$ must change sign on (0, 1). In fact $\mathscr{S}(\frac{1}{2}) = 0$. To see this, choose $c = \mathrm{Im}(\sigma) + a\sin(\chi-\psi) = \frac{1}{2}a\sin(\chi-\psi)$, so that $\mathscr{S}(\frac{1}{2}) = 0$. Then write (3.15) and (3.16) in the variable $z' = z-\frac{1}{2}$, $-\frac{1}{2} \leqslant z' \leqslant \frac{1}{2}$. The form of (3.15), (3.16) is unchanged by the variable change, but

$$\mathscr{S}(z') = -z'a\sin(\chi-\psi). \tag{3.18}$$

Next, decompose f and $\hat{w}$ into even and odd parts; insert these representations into (3.16), and identify the even and odd parts of the resulting equations; e.g. from (3.16*b*), we find

$$\mathrm{Re}(\sigma)\hat{f}_0 - iz'\sin(\chi-\psi)\,a\hat{f}_e = \frac{1}{R}L^2\hat{f}_0 + a^2F\hat{w}_0.$$

This, and the other three equations, show that we may take

$$\hat{w} = w_e(z') + iw_0(z'), \quad \hat{f} = f_e(z') + if_0(z'), \tag{3.19}$$

where w_e, f_e, w_0 and f_0 are real functions. Now, using (3.18) and (3.19), the condition (3.15) is satisfied identically.

† $\langle g\rangle \equiv \int_0^1\left\{\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} g(y,z)\,dy\right\}dz.$

4. The energy problem for rotating plane Couette flow when spiral vortex disturbances are assumed from the start

A distinguished spiral direction ($\mathbf{e}_{x'}$) is found as a part of the solution of the linear stability problem. This direction is determined by the wave-number that gives the smallest eigenvalue R for a neutral solution ($\mathrm{Re}\,(\sigma) = 0$) of the spectral problem. The spiral vortex disturbance is also observed in experiments. It is, therefore, reasonable to examine the consequence of assuming the preferred direction from the start. The aim here is an energy analysis, which takes advantage of the presumed spiral form for the disturbance.

It will be convenient to decompose the motion along and normal to the direction x' in which u, v, w and p do not vary, e.g. $u = u(y', z, t)$:

$$\mathbf{U}(z) = (1-z)\,[\cos(\chi-\psi)\,\mathbf{e}_{x'} + \sin(\chi-\psi)\,\mathbf{e}_{y'}], \tag{4.1a}$$

and

$$-\mathbf{\Omega} = \tilde{\Omega}_2 \cos\psi\,\mathbf{e}_{x'} - \tilde{\mathbf{\Omega}}_2\,\mathbf{e}_{y'} \sin\psi = \tilde{\Omega}_2\,\mathbf{e}_x. \tag{4.1b}$$

The governing equations for the x' independent disturbances are

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} + (\mathbf{u}+\mathbf{U})\,.\,\nabla u + (2\tilde{\Omega}_2 \sin\psi - \cos(\chi-\psi))\,w = \frac{1}{R}\nabla_2^2 u,\\
&\frac{\partial v}{\partial t} + (\mathbf{u}+\mathbf{U})\,.\,\nabla v + (2\tilde{\Omega}_2 \cos\psi - \sin(\chi-\psi))\,w = -\frac{\partial p}{\partial y'} + \frac{1}{R}\nabla_2^2 v,\\
&\frac{\partial w}{\partial t} + (\mathbf{u}+\mathbf{U})\,.\,\nabla w - 2\tilde{\Omega}_2 (v\cos\psi + u\sin\psi) = -\frac{\partial p}{\partial z} + \frac{1}{R}\nabla_2^2 w,
\end{aligned}\right\} \tag{4.2a–c}$$

where

$$\nabla_2^2 = \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z^2} \quad \text{and} \quad \frac{\partial v}{\partial y'} + \frac{\partial w}{\partial z} = 0. \tag{4.2d}$$

The boundary conditions are

$$\mathbf{u} = 0|_{z=0,1} \tag{4.2e}$$

and $\mathbf{u}$ is almost periodic in y'. The disturbance velocity component u cannot be driven by a disturbance pressure gradient because the assumption that $\partial \mathbf{u}/\partial x' = 0$ implies that $\partial^2 p/\partial x'^2 = 0$. Then $\partial p/\partial x' = K(y', z)$, and, since p is almost periodic in x', it is bounded as $x'^2 \rightarrow \infty$, and it follows that $K \equiv 0$.

There are several consequences of the independence of p upon x'. One consequence is the existence of *two* energy identities: one governing the energy of the longitudinal disturbances,

$$\frac{1}{2}\frac{d}{dt}\langle u^2\rangle + (2\tilde{\Omega}_2 \sin\psi - \cos(\chi-\psi))\langle wu\rangle = -\frac{1}{R}\langle|\nabla_2 u|^2\rangle, \tag{4.3a}$$

and one governing the evolution of the energy of the transverse components,

$$\frac{1}{2}\frac{d}{dt}\langle w^2+v^2\rangle - \sin(\chi-\psi)\langle wv\rangle - 2\tilde{\Omega}_2 \sin\psi\langle wu\rangle = -\frac{1}{R}\langle|\nabla_2 w|^2 + |\nabla_2 v|^2\rangle. \tag{4.3b}$$

Equations (4.3) are the subject of analysis of this section. We form the sum $(4.3b)+\lambda(4.3a)$, with $\lambda > 0$, and let $\phi = \sqrt{\lambda}\,u$, to obtain

$$\begin{aligned}
&\tfrac{1}{2}\, d\langle w^2+v^2+\phi^2\rangle/dt - \sin(\chi-\psi)\langle wv\rangle - \sqrt{\lambda}\cos(\chi-\psi)\langle w\phi\rangle\\
&\qquad - 2\tilde{\Omega}_2 \sin\psi\left(\frac{1}{\sqrt{\lambda}} - \sqrt{\lambda}\right)\langle w\phi\rangle = -\frac{1}{R}\langle|\nabla_2 w|^2 + |\nabla_2 v|^2 + |\nabla_2\phi|^2\rangle.
\end{aligned} \tag{4.4}$$

This is the equation governing the evolution of the disturbance energy when spiral vortices are assumed from the start; it can be written as

$$\frac{d\mathscr{E}}{dt} = -\mathscr{H}_\lambda - \frac{\mathscr{D}}{R} = -\mathscr{D}\left(\frac{1}{R} - \frac{(-\mathscr{H}_\lambda)}{\mathscr{D}}\right) \leqslant -\mathscr{D}\left(\frac{1}{R} - \frac{1}{R_\lambda}\right), \tag{4.5}$$

where

$$\mathscr{E} = \tfrac{1}{2}\langle w^2 + v^2 + \phi^2\rangle,$$

$$\mathscr{D} = \langle |\nabla_2 w|^2 + |\nabla_2 v|^2 + |\nabla_2 \phi|^2\rangle,$$

$$-\mathscr{H}_\lambda = -2\tilde{\Omega}_2 \sin\psi\left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right)\langle w\phi\rangle + \sqrt{\lambda}\cos(\chi - \psi)\langle w\phi\rangle + \sin(\chi - \psi)\langle wv\rangle,$$

and

$$\frac{1}{R_\lambda} = \max_{\mathbf{H}_2} \frac{-\mathscr{H}_\lambda}{\mathscr{D}}, \tag{4.6}$$

where $\mathbf{H}_2$ is the set of x' independent kinematically admissible vectors, i.e. vectors $\mathbf{u}$ satisfying (4.2d, e), which are almost periodic functions of y'.

The energy inequality (4.5) integrates to

$$\mathscr{E}(t) \leqslant \mathscr{E}(0)\exp\left\{-2\hat{\Lambda}\left(\frac{1}{R} - \frac{1}{R_\lambda}\right)t\right\}, \tag{4.7}$$

where $\mathscr{D} > 2\hat{\Lambda}\mathscr{E}$ for all $\mathbf{u} \in \mathbf{H}_2$ and provided that $R < R_\lambda$. The criterion here is independent of the size of the initial disturbances and applies globally. It is clear that the vector $\mathbf{u} \in \mathbf{H}_2$ which solves (4.6) is also the form of the x' independent disturbance whose energy increases initially at the smallest R.

Again, $\lambda > 0$ is a free parameter, which is selected to maximize the interval $0 < R < R_\lambda$ on which global monotonic stability can be assured. Thus,

$$\tilde{R}(\chi, \psi, \tilde{\Omega}_2) = \max_{\lambda > 0} R_\lambda. \tag{4.8}$$

If $R < \tilde{R}$, rotating Couette flow is monotonically and globally stable to x' independent disturbances, making an angle ψ with x ($\mathbf{e}_x = -\mathbf{\Omega}/|\mathbf{\Omega}|$).

Of course, one cannot know at the start whether nature will select a single direction x' along which disturbances are constant. Moreover, even if such a direction is selected, it will not be possible to specify its angle ψ with x at the start. However, one can seek the angle $\psi = \psi_{\mathscr{g}}$ for which

$$R_{\mathscr{g}}(\chi, \tilde{\Omega}_2) = \tilde{R}(\chi, \psi_{\mathscr{g}}(\chi, \tilde{\Omega}_2), \tilde{\Omega}_2) = \min_{\psi} \tilde{R}, \tag{4.9}$$

where $0 \leqslant \psi \leqslant 2\pi$. *If $R < \tilde{R}_{\mathscr{g}}$, rotating Couette flow is monotonically and globally stable to x' independent disturbances, making any angle ψ with x.*

The values of $\tilde{R}$ and $\tilde{R}_{\mathscr{g}}$ depend strongly on the sign of the Rayleigh discriminant. The first case to be considered is the case $F \leqslant 0$. This corresponds to the situation in which the angular momentum increases outward in the rotating cylinder problem. This case includes plane Couette flow ($F = 0$). The following theorem holds. *Let $F \leqslant 0$. The x' independent disturbance whose energy increases initially at the smallest value of R ($> R_{\mathscr{g}}$) is a transverse vortex perpendicular to direction of the shearing motion. Moreover, $R_{\mathscr{g}} = 177{\cdot}22$ is the value calculated by*

Orr (1907). For, when $F \leqslant 0$, one can find a value $\lambda = \tilde{\lambda} = -4\tilde{\Omega}_2^2 \sin^2\psi/F \geqslant 0$ which will make the coefficient of $\langle w\phi\rangle$ in (4.6) vanish. Any other choice of λ gives a larger value R_λ^{-1}. Suppose $(\tilde{w}, \tilde{\phi})$ gives (4.6) its smallest value when $\lambda = \tilde{\lambda}$. Then the same maximum is attained for $(\tilde{w}, -\tilde{\phi})$. One of these two pairs clearly makes $R_\lambda^{-1} > R_{\tilde{\lambda}}^{-1}$ when $\lambda \neq \tilde{\lambda}$. Hence $\lambda = \tilde{\lambda}$ solves (4.8), i.e.

$$\frac{1}{\tilde{R}} = \max_{\mathbf{H}_2} \frac{\sin(\chi-\psi)\langle wv\rangle}{\langle |\nabla w|^2 + |\nabla v|^2 + |\nabla\phi|^2\rangle}. \tag{4.10}$$

The maximizing vector clearly has $\phi \equiv 0$. The maximum value of

$$\langle wv\rangle / \langle |\nabla w|^2 + |\nabla v|^2\rangle$$

in $\mathbf{H}_2$ is 1/177·22 (Orr 1907). Hence, $\tilde{R} = 177{\cdot}22/\sin(\chi-\psi)$ and

$$R_{\mathscr{E}} = \min_{\psi} \tilde{R} = 177{\cdot}22$$

is attained when

$$\chi - \psi_{\mathscr{E}} = \tfrac{1}{2}\pi.$$

This is an Orr–Sommerfeld type of two-dimensional disturbance which does not vary on lines perpendicular to the plane of the motion.

When $F > 0$, we cannot select a positive value of λ which will make the coefficient of $\langle w\phi\rangle$ in (4.6) vanish. Then the optimizing value for (4.8), $\lambda = \tilde{\lambda}$, is sought as the root of

$$0 = \frac{\partial}{\partial\lambda} R_\lambda^{-1} = \frac{1}{\mathscr{D}} \frac{\partial}{\partial\lambda}(-\mathscr{H}_\lambda).$$

Since $\mathbf{u}$ is a maximizing vector,

$$\frac{d}{d\lambda}(-\mathscr{H}_\lambda) = \frac{1}{2\sqrt{\lambda}} \frac{\partial(-\mathscr{H}_\lambda)}{\partial\sqrt{\lambda}} = \frac{1}{2\sqrt{\lambda}}\left\{-2\tilde{\Omega}_2 \sin\psi\left(1+\frac{1}{\lambda}\right) + \cos(\chi-\psi)\right\}\langle w\phi\rangle.$$

Hence,

$$\tilde{\lambda} = \frac{\tilde{\Omega}_2 \sin\psi}{\frac{1}{2}\cos(\chi-\psi) - \tilde{\Omega}_2 \sin\psi} = \frac{4\tilde{\Omega}_2^2 \sin^2\psi}{F}, \tag{4.11}$$

$$-\mathscr{H}_{\tilde{\lambda}} \equiv -\mathscr{H} = 2\sqrt{F}\langle w\phi\rangle + \sin(\chi-\psi)\langle wv\rangle, \tag{4.12}$$

and

$$\frac{1}{\tilde{R}} = \max_{\mathbf{H}_2}(-\mathscr{H}/\mathscr{D}). \tag{4.13}$$

It is convenient to find $\tilde{R}$ as an eigenvalue of Euler's equations for (4.13):

$$\sqrt{F}\phi + \tfrac{1}{2}\sin(\chi-\psi)v + \frac{1}{R}\nabla_2^2 w = \partial_z p,$$

$$\sin\tfrac{1}{2}(\chi-\psi)w + \frac{1}{R}\nabla_2^2 v = \partial_{y'} p,$$

and

$$\sqrt{F}w + \frac{1}{R}\nabla_2^2\phi = 0.$$

Using the continuity equation (4.2*d*), and after normal-mode reduction to ordinary differential equations, one finds that

$$L^3\hat{w} - 2i(\tfrac{1}{2}R)a\sin(\chi-\psi)LD\hat{w} + 4(\tfrac{1}{2}R)^2 Fa^2\hat{w} = 0, \tag{4.14a}$$

with

$$\hat{w} = D\hat{w} = L^2\hat{w} = 0|_{z=0,1}. \tag{4.14b}$$

The required stability limit is found as an eigenvalue of (4.14):

$$\tilde{R} = \min_a R(\tilde{\Omega}_2, \chi, \psi, a). \tag{4.15}$$

The criterion $R < \tilde{R}$ guarantees stability for all disturbances making an angle ψ with x. The criterion

$$R < \min_\psi \tilde{R} \equiv R_g \tag{4.16}$$

suffices for stability to disturbances making any angle with x. The values $\tilde{R}_g$ and the minimizing angles ψ_g are displayed in table 1 and in figure 3.

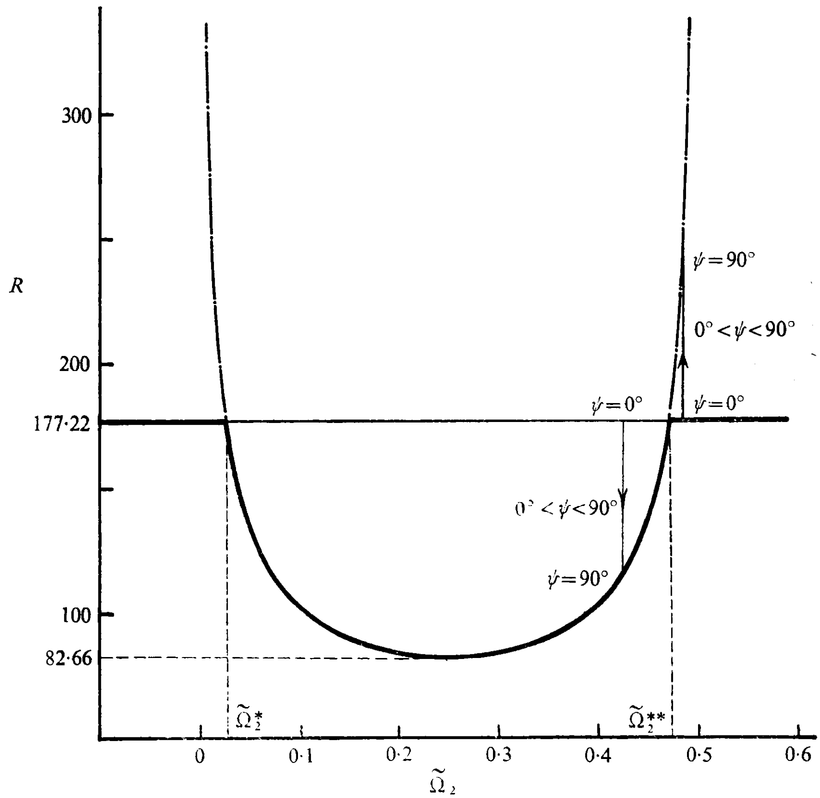

FIGURE 3. Energy stability boundary for spiral disturbances of Couette flow in the rotating plane Couette flow limit ($\chi = 90°$). The heavy dark line is the stability boundary. The values $\tilde{R}(\psi)$, $0° \leqslant \psi \leqslant 90°$, lie between $\tilde{R}(0°)$ and $\tilde{R}(90°)$. When $\tilde{\Omega}_2 = \tilde{\Omega}_2^*$ or $\tilde{\Omega}_2 = \tilde{\Omega}_2^{**}$, we have $\tilde{R}(\psi) = 177{\cdot}2$ for all ψ in the first quadrant. For these values of $\tilde{\Omega}_2$, the critical energy spiral disturbance is not a Taylor vortex ($\psi = 90°$). It is a 'roller bearing' vortex ($\psi = 0°$). The 'roller bearing' initial condition can be achieved in the laboratory (see Coles 1965, plate O).

The stability criterion $R < \tilde{R}$ has an interesting consequence when the velocity of sliding is zero ($\chi = 90°$), and the disturbances are axisymmetric ($\psi = 90°$). In this case, referring first to the linear equations (3.9) and (3.10), we find that $\mathscr{S} = \mathrm{Im}(\sigma)$, and (3.17) then implies $\mathrm{Im}(\sigma) = 0$. This reduces (3.9) and (3.10) to the Bénard problem with minimum eigenvalue $R_L^2 F = 1708$, $a = 3{\cdot}12$. On the other hand, with $\chi = \psi$, the energy equations (4.14a, b) imply $\tilde{R}^2 F = 1708$,

$a = 3{\cdot}12$. *Hence,* $\tilde{R} = R_L$ and the criterion $R < \tilde{R}$ *is both necessary and sufficient for global stability to axisymmetric disturbances.*†

What is the energy limit $R_{\mathscr{E}}(\tilde{\Omega}_2) = \min\limits_{\psi} \tilde{R}(\psi, \tilde{\Omega}_2) = \tilde{R}(\psi_{\mathscr{E}}(\tilde{\Omega}_2), \tilde{\Omega}_2)$ for the narrow-gap rotating cylinder problem ($\chi = 90°$) among spiral disturbances which are not necessarily axisymmetric? (Axisymmetric disturbances are Taylor vortices with $\psi = 90°$.) The answer, given by numerical analysis, is

$$\tilde{\Omega}_2^* = 0{\cdot}028, \quad \tilde{\Omega}_2^{**} = 0{\cdot}472,$$

$$\text{(i)}\ \tilde{\Omega}_2^* < \tilde{\Omega}_2 < \tilde{\Omega}_2^{**}, \quad \psi_{\mathscr{E}}(\tilde{\Omega}_2) = 90°, \quad 82{\cdot}66 \leqslant R_{\mathscr{E}}(\tilde{\Omega}_2) < 177{\cdot}2,$$

$$\text{(ii)}\ \tilde{\Omega}_2 < \tilde{\Omega}_2^*, \quad \tilde{\Omega}_2 > \tilde{\Omega}_2^{**}, \quad \psi_{\mathscr{E}}(\tilde{\Omega}_2) = 0°, \quad R_{\mathscr{E}} = 177{\cdot}2,$$

$$\text{(iii)}\ \tilde{\Omega}_2 = \tilde{\Omega}_2^*, \quad \tilde{\Omega}_2 = \tilde{\Omega}_2^{**}, \quad 0° \leqslant \psi_{\mathscr{E}} \leqslant 90°, \quad \tilde{R}_{\mathscr{E}} = 177{\cdot}2.$$

For values of $\tilde{\Omega}_2$ outside the interval $[\tilde{\Omega}_2^*, \tilde{\Omega}_2^{**}]$, the spiral vortex whose energy increases initially at the smallest R is not a Taylor vortex; it is a transverse vortex of Orr's type. At the end of the interval $[\tilde{\Omega}_2^*, \tilde{\Omega}_2^{**}]$, every spiral angle $0° \leqslant \psi_{\mathscr{E}} \leqslant 90°$ gives the same eigenvalue $\tilde{R}_{\mathscr{E}} = 177{\cdot}2$.

5. Linear stability analysis of spiral Couette flow

When $1-\eta$ is not small, the stability problem becomes complicated by the geometry. In part 1, we simplified the problem by a mean-radius narrow-gap approximation. It was shown in part 1 that this approximation is reasonably accurate for the energy equations; but it turns out that the linear theory results based on the approximation need correction. The difference between the energy and linear theory, with regard to the approximation, can be traced to the fact that only derivatives of the basic spiral flow appear in the energy equations, and, for spiral Couette flow, these are very nearly constant. In the linear theory, the effect of $\mathbf{U}.\nabla\mathbf{u}$ is not well approximated by a constant.‡

We have calculated the linear stability limits numerically, without approximation. The working equations for the calculation, derived below, are written in dimensionless variables with length scale $R_2 - R_1$ and velocity scale $[U_c^2 + R_1^2(\Omega_1 - \Omega_2)^2]^{\frac{1}{2}}$. The Reynolds number is defined as in (3.4), $\mathbf{\Omega} = -\mathbf{e}_x \tilde{\Omega}_2$ and

$$\mathbf{U} = \mathbf{e}_x \frac{\cos\chi}{\ln\eta} \ln[r(1-\eta)] + \mathbf{e}_\phi \frac{\eta \sin\chi}{(1+\eta)(1-\eta)^2}\left[\frac{1}{r} - (1-\eta)^2 r\right], \tag{5.1}$$

† The coincidence between energy and linear limits here applies to all $\tilde{\Omega}_2$ for which $F > 0$. In contrast, if axial symmetry is not assumed from the start the coincidence of the energy and linear limits occurs only when $\tilde{\Omega}_2 = \frac{1}{4}$ (Busse 1969).

‡ The mean radius results in figures 7 and 8 of part 1 are to be replaced with the numerical results of tables 2 and 3 and figure 4 of this part. The following are corrigenda to slips in part 1. Equation (4.26): $\{\frac{1}{2}, \frac{1}{2}(1-\eta)^{-1}[\ldots]^{-\frac{1}{2}}\} \to \{\frac{1}{2}, (1-\eta)^{-1}[\ldots]^{-\frac{1}{2}}\}$. (5.1): $\mathrm{Re}(\omega)[1,1,1] \to \mathrm{Re}(\omega)[w, v, u]$. (6.5): add $U\dfrac{\partial}{\partial x}\,\mathrm{curl}\,\mathbf{u}$ to the left-hand side. (8.1): $U_\phi \to U_x$. (8.4): $[(1-\eta)/(1+\eta)\ln\eta]^2 \to [1-\eta)/\ln\eta]^2$. (9.4) $\mathscr{S} \to \rho_L' \mathscr{S}$. Definitions of a_2, a_3, a_5, b_2 on pp. 553–554: a_2, $\lambda(1,\eta) \to \lambda(0,\eta)$; a_3, $\{\ln\eta \ldots \to \ln\eta\{\ldots$ and $|U_C/U_0| \to [U_C/U_0](1-\eta)$; b_2, $\eta\ln\{\ldots \to \eta\ln\eta\{\ldots$.

where $\eta/(1-\eta) \leqslant r \leqslant 1/(1-\eta)$. Define

$$\boldsymbol{\tau} = \mathbf{U}.\nabla\mathbf{u}+\mathbf{u}.\nabla\mathbf{U}+2\boldsymbol{\Omega}\times\mathbf{u} = \mathbf{e}_r\left(\mathbf{U}.\nabla w-\left(\frac{2V}{r}+2\tilde{\Omega}_2\right)v\right)$$

$$+\mathbf{e}_\phi\left(\mathbf{U}.\nabla v+\frac{1}{r}D(rV+\tilde{\Omega}_2r^2)\,w\right)+\mathbf{e}_x(\mathbf{U}.\nabla u+wDU),$$

and note that the spectral problem of the linear theory is defined by

$$\sigma\mathbf{u}+\boldsymbol{\tau} = -\nabla p+\frac{1}{R}\nabla^2\mathbf{u}, \tag{5.2}$$

where $\mathbf{u}$ satisfies (2.2*b*–*e*). Now observe that

$$\mathbf{r}.\nabla^2\mathbf{v} \equiv \nabla^2\mathbf{r}.\mathbf{v}+2\partial_x v_x-2\,\mathrm{div}\,\mathbf{v}, \tag{5.3}$$

where $\mathbf{r}$ is the cylindrical radius vector ($\mathbf{r}.\mathbf{e}_x = 0$), and $\mathbf{v}$ is any smooth vector field. Then form $\mathbf{r}.\mathrm{curl}$ (5.2) and $\mathbf{r}.\mathrm{curl}(\mathrm{curl}$ (5.2)), using $\mathrm{div}\,\mathbf{u} = 0$, $\mathrm{curl}^2\boldsymbol{\tau} \equiv -\nabla^2\boldsymbol{\tau}+\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\tau}$, etc., and (5.3), to find

$$\sigma(r\zeta_r)+\mathbf{r}.\mathrm{curl}\,\boldsymbol{\tau} = \frac{1}{R}(\nabla^2(r\zeta_r)+2\partial_x\zeta_x), \tag{5.4}$$

where (see figure 1)

$$r\zeta_r = \mathbf{r}.\mathrm{curl}\,\mathbf{u} = -\partial u/\partial\phi+r\,\partial v/\partial x, \quad \zeta_x = \frac{1}{r}[-\partial_r(rv)+\partial w/\partial\phi],$$

and
$$\sigma(\nabla^2 f+2\partial_x u)+\nabla^2(r\tau_r)+2\partial_x\tau_x-\frac{1}{r}\partial_r(r^2\,\mathrm{div}\,\boldsymbol{\tau}) = \frac{1}{R}(\nabla^4 f+4\partial_x\nabla^2 u), \tag{5.5}$$

where $f = wr$. It is convenient to differentiate (5.4) with respect to x and to eliminate u through the continuity equation. This leads to

$$\partial_x u = -\frac{1}{r}(\partial_\phi v+\partial_r f)$$

and
$$\partial_x(r\zeta_r) = r\left(\partial^2_{xx}v+\frac{1}{r^2}\partial^2_{\phi\phi}v+\frac{1}{r^2}\partial^2_{r\phi}f\right).$$

Eliminating u in this way leads to a coupled fourth-order and second-order equation for f and v. This set of equations is then reduced, by the usual Fourier methods, to

$$L^2 f-4L\left(\frac{Df}{r}\right)-4inL\left(\frac{v}{r}\right)$$

$$= R\left\{\frac{2n\mathscr{S}v}{r}-\frac{2i\mathscr{S}Df}{r}+i\mathscr{S}Lf+iD\mathscr{S}Df-\frac{in}{r^2}D(rV+\tilde{\Omega}_2r^2)\,Df\right.$$

$$\left.-i\alpha DU\left(Df-\frac{2f}{r}\right)+\frac{2(n^2+\alpha^2r^2)}{r^2}(V+\tilde{\Omega}_2 r)\,v\right\}, \tag{5.6a}$$

and

$$L\left[(n^2+\alpha^2r^2)\frac{v}{r}\right] = inL\left(\frac{Df}{r}\right)-\frac{2in\alpha^2}{r^2}f+\frac{2\alpha^2}{r}D(rv)+\frac{R}{n}\left\{i\mathscr{S}(n^2+\alpha^2r^2)\left(\frac{nv}{r}-\frac{iDf}{r}\right)\right.$$
$$\left.-\alpha^2r\mathscr{S}Df-\alpha n^2\frac{DU}{r}f+\alpha^2n\left[\frac{1}{r}D(rV+\tilde{\Omega}_2r^2)\right]f\right\}, \qquad (5.6b)$$

	$\tilde{\Omega}_2$	n	α	Im (σ)	R_L
$\eta = 0{\cdot}8$	0·40	13	−1·098	0·276288	96·397
$\chi = 10°$	0·70	13	−0·896	0·182815	85·431
	1·00	14	−0·750	0·095444	83·000
	1·2914	14	−0·605	0·026828	82·576
	1·40	14	−0·564	0·0073055	82·607
	1·80	14	−0·447	−0·048508	83·030
	2·20	14	−0·366	−0·086823	83·605
$\eta = 0{\cdot}8$	0·10	11	−1·892	0·193735	106·431
$\chi = 30°$	0·20	11	−1·876	0·189563	89·193
	0·30	11	−1·788	0·152884	82·663
	0·35	12	−1·814	0·113088	81·321
	0·424	12	−1·698	0·066105	80·605
	0·50	12	−1·559	0·009080	81·193
	0·60	12	−1·383	−0·063900	83·368
	0·70	12	−1·211	−0·135777	86·572
	0·80	12	−1·061	−0·198695	90·415
	0·90	12	−0·937	−0·25082	94·638
	1·10	12	−0·751	−0·329160	103·62
	1·30	11	−0·562	−0·354939	112·83
$\eta = 0{\cdot}8$	0·05	6	−2·804	0·114541	90·184
$\chi = 60°$	0·10	6	−2·794	0·113722	82·291
	0·15	6	−2·762	0·107739	78·328
	0·2125	7	−2·825	0·030643	76·907
	0·30	7	−2·683	0·0018589	79·878
	0·40	8	−2·566	−0·117776	93·323
	0·50	8	−2·045	−0·242451	135·95
	0·56	8	−1·694	−0·329572	251·97

TABLE 2. Values of the critical parameters of linear theory when $\eta = 0{\cdot}8$

where
$$\mathscr{S} = \mathrm{Im}\,(\sigma)+\frac{nV}{r}+\alpha U,$$

and
$$L = D^2+D/r-(n^2+\alpha^2r^2)/r^2.$$

Equations (5.6*a*, *b*) are to be solved relative to the boundary conditions at $r = 1/(1-\eta)$ and $r = \eta/(1-\eta)$,

$$f = Df = v = 0, \qquad (5.6c)$$

by a standard Runge–Kutta forward integration scheme in which both R and Im (σ) are varied using the method of chords. Here, either of the pair of values may be regarded as the eigenvalue parameter. Despite the fact that Im $(\sigma) \neq 0$ when $\chi \neq 90°$ the numerical integration is straightforward. The numerical results are summarized in table 2 ($\eta = 0{\cdot}8$) and table 3 ($\eta = 0{\cdot}5$). The instability limit R_L, the wave speed Im (σ) and the azimuthal periodicity n of the spiral disturbance are important experimental observables. The number of vortices seen in any given experiment should correspond to the member of zeros ($2n$) of $\cos n\phi$ of the eigenfunction belonging to R_L.

	$\tilde{\Omega}_2$	n	α	Im (σ)	R_L
$\eta = 0{\cdot}5$	0·20	4	−1·160	0·29303	113·073
$x = 10°$	0·50	4	−1·020	0·24254	88·413
	0·80	4	−0·779	0·14046	84·435
	1·0962	4	−0·606	0·06506	84·158
	1·30	4	−0·519	0·02682	84·492
	1·70	4	−0·402	−0·02485	85·390
	2·00	4	−0·342	−0·05142	86·065
	2·50	4	−0·273	−0·08198	86·996
	3·00	4	−0·226	−0·10279	87·700
$\eta = 0{\cdot}5$	0	3	−1·900	0·28267	103·246
$\chi = 30°$	0·10	3	−1·8725	0·278809	86·623
	0·20	4	−2·1175	0·22923	80·398
	0·3112	4	−1·942	0·16945	78·660
	0·40	4	−1·761	0·10412	79·864
	0·50	4	−1·550	−0·02531	83·150
	0·60	4	−1·349	−0·05216	87·943
	0·70	4	−1·1725	−0·121304	93·818
	0·80	4	−1·025	−0·17955	100·535
$\eta = 0{\cdot}5$	−0·17	2	−3·163	0·17473	102·471
$\chi = 60°$	−0·10	2	−3·028	0·14854	83·136
	−0·0	2	−2·941	0·14210	72·601
	0·0844	2	−2·864	0·13640	70·730
	0·12	2	−2·820	0·130254	71·311
	0·16	2	−2·764	0·12309	73·049
	0·24	2	−2·590	0·09340	80·989
	0·30	2	−2·360	0·04702	93·8165
	0·35	2	−2·068	−0·01678	114·558

TABLE 3. Values of the critical parameters of linear theory when $\eta = 0{\cdot}5$

6. Comparison of theory and experiment

Consider first the narrow-gap problem for which rotating plane Couette flow is representative. Here the linear theory gives instability when $F > 0$, and stability when $F \leqslant 0$. The equality may be thought to represent the situation in which either the angular momentum of the basic spiral flow is constant in planes parallel to the plane of the disturbance, or it can represent plane Couette flow in a non-rotating system $\tilde{\Omega}_2 = F = 0$. The condition $F < 0$ is satisfied by rotating cylinders in which the outer cylinder rotates much faster than the inner one. The linear theory of stability of plane Couette flow without rotation has never indicated anything other than absolute stability for this flow.†

† Hopf (1914), Southwell & Chitty (1930), Morawetz (1952), Wasow (1953), Grohne (1954), Gallagher & Mercer (1962), Riis (1962), Deardorf (1963), Dikii (1960).

The experimental results of Schulz-Grunow (1958), for the allied problem of rotating cylinders with the inner one at rest, do seem to indicate that, if the amplitude of the disturbances is suppressed, then Couette flow is stable even at very large Reynolds numbers. On the other hand, for the same flow in natural circumstances, Couette (1890), Mallock (1881) and Taylor (1923), among others, find a natural transition to turbulence. Reichardt (1956) claims to find that the plane Couette flow achieved in his experiments is stable when $U_c(R_2-R_1)/\nu < 1500$.

It seems that the mechanics of instability of non-rotating plane Couette flow in natural circumstances is associated with the non-linear term $\mathbf{u}\times\text{curl}\,\mathbf{u}$ neglected in linear analysis. In the case of Couette flow, instability at any R has yet to be established theoretically.

The following is true when the sign of the Rayleigh discriminant F is negative (stability according to Rayleigh's criterion). The experimental limits of instability are larger by a factor of 20 than the critical energy value $R = 2\sqrt{1708}$, below which all kinematically admissible disturbances decay. Among all the kinematically admissible disturbances, the disturbance which makes the energy $\langle u^2+v^2+w^2\rangle$ increase at the smallest $R \geqslant 2\sqrt{1708}$ is a longitudinal vortex.

Among all the kinematically admissible x' independent disturbances (spiral vortices with axis x'), the one whose weighted energy $\langle \lambda u^2+v^2+w^2\rangle$ increases at the smallest R ($> 177{\cdot}2$) is a transverse vortex $u \equiv 0$ whose axis is perpendicular to the basic flow. This critical energy value, $R = 177{\cdot}2$, is about ten times lower than the experimentally observed instability limit. The situation is greatly changed when $F > 0$: there is a dynamic source for converting the energy of rotation into disturbance energy. The threshold of *instability* is lowered to energy-like values, and, when the rotation parameters are optimally adjusted (see part 1), there is perfect, or nearly perfect, agreement between the energy and linear theories.

The comparison of the theoretical results with the experiment of Ludwieg (1964) is developed below. Ludwieg's apparatus is like a long sleeve bearing, which is rotated around its axis at a fixed angular velocity, and is geared to a shaft in the bearing in such a way that the shaft can be made to turn and slide relative to the rotating bearing. Since the clearance is small ($\eta = 0{\cdot}8$), the flow develops almost instantly, and is very nearly linear shear. The relations, between the parameters of Ludwieg's experiment and those in parts 1 and 2, are given in part 1. The critical parameters of Ludwieg's experiment, and of the various theories, are shown in figure 4; the graphs marked 'linear theory' are taken from the numerical integration for $\eta = 0{\cdot}8$ (table 2).

The nearly perfect agreement between the linear theory and Ludwieg's experiment shows that the instability observed here is not subcritical. Figure 4 shows good agreement between the threshold limit and the spiral angle. The wave speed ($\text{Im}\,(\sigma)$, see table 2) and the spiral vortex spacing (values n, see table 2) are not reported in Ludwieg's experiment. The curves marked 'energy (i)' give values associated with the disturbance whose energy increases initially at the smallest R (see part 1). The curves marked 'energy (ii)' are taken from the calculation of §4 for the rotating plane Couette flow. The spiral angle ψ is, in this case, the limiting $\eta \to 1$ value.

In explanation of the two energy analyses consider points A, B and C in

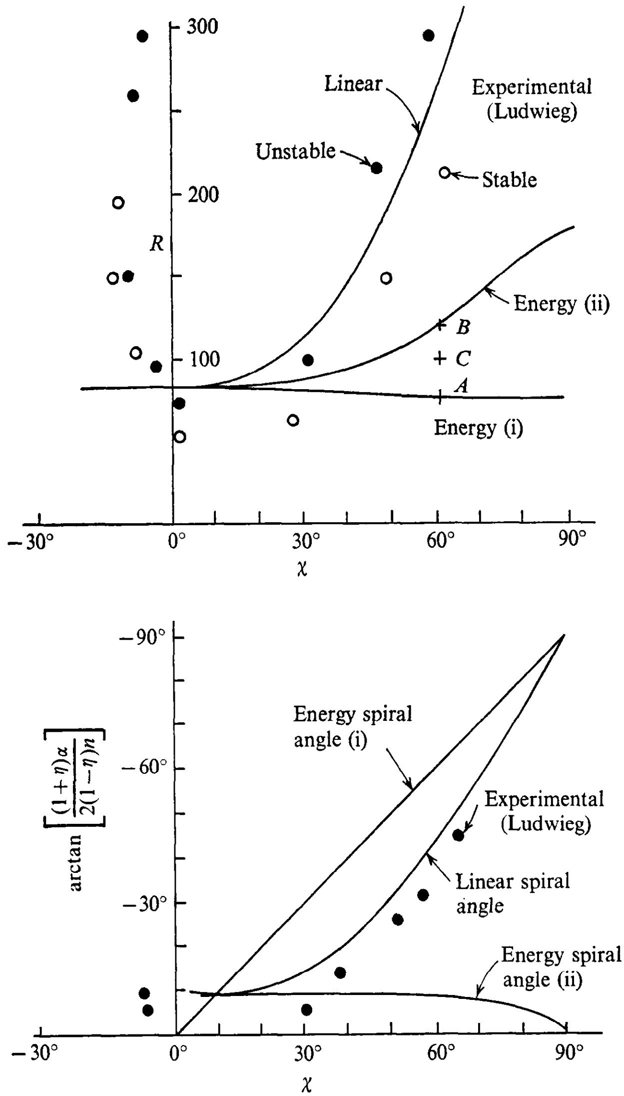

FIGURE 4. Comparison of theory and experiment (Ludwieg 1964) for $\eta = 0{\cdot}8$, $(R_2 - R_1)^2\Omega_2/\nu = 150$. Black dots are unstable, white ones stable.

figure 4. At A we take critical energy disturbance which is a spiral vortex along energy spiral (i). At B we consider the extremalizing solution of problem (4.9). This is also a spiral vortex whose axis lies along energy spiral (ii). At C the energy of disturbance A increases initially, and the weighted energy of disturbance B decreases. The same weighted energy, but of A rather than B, decreases at a yet faster rate than B. Hence, the difference between the rate of change of the energy

of A which is positive, and the weighted energy which is negative, is strongly positive.

To test the theoretical predictions of energy theory, it would be necessary to determine if the initial conditions whose energy will increase at values of $R > R_{\mathscr{E}}$ are sufficiently representative of physically realizable initial conditions. It would appear, from experiments, that, even if such energetic disturbances are realizable, they are globally stable and *eventually* decay.

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