Energy Stability of Hydromagnetic Flow

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The governing equations of motion for a viscous fluid with constant density ρ and finite conductivity σ flowing in a magnetic field are (see [3])

$$\frac{d\underset{\sim}{U}}{dt} = \frac{1}{\rho\mu}\,\underset{\sim}{B}\cdot\nabla\underset{\sim}{B} - \frac{1}{\rho}\nabla\left(p+\frac{1}{2\mu}\,|\underset{\sim}{B}|^2\right)+\nu\nabla^2\underset{\sim}{U}\ , \tag{1a}$$

$$\frac{d\underset{\sim}{B}}{dt} = \underset{\sim}{B}\cdot\nabla\underset{\sim}{U}+\frac{1}{\sigma\mu}\nabla^2\underset{\sim}{B}\ , \tag{1b}$$

and

$$\nabla\cdot\underset{\sim}{U} = \nabla\cdot\underset{\sim}{B} = 0\ . \tag{1c}$$

where B is the magnetic flux density, μ is the magnetic permeability and, as before, ν, U and p are viscosity, velocity and pressure. From (1b) one finds that $\partial\nabla\cdot\underset{\sim}{B}/\partial t = \frac{1}{\sigma\mu}\nabla^2\nabla\cdot\underset{\sim}{B}$. The condition div $\underset{\sim}{B} = 0$ is automatically guaranteed for solutions of (1b) which have div $\underset{\sim}{B} = 0$ at time 0 and on the boundary S at all times.

Here $\mathcal{V}$ is a bounded domain enclosed by a rigid surface on which U and B are assigned. As in [2], to analyze the stability of the basic motion $(\underset{\sim}{U},\underset{\sim}{B},p)$ we consider an altered motion $(\underset{\sim}{U}^*,\underset{\sim}{B}^*,p^*)$ which satisfies the same equations (1) and the same boundary conditions, but differs from the basic state initially. The differences

($\underset{\sim}{U}^*-\underset{\sim}{U} = \underset{\sim}{u}$, $\underset{\sim}{B}^*-\underset{\sim}{B} = \underset{\sim}{b}$ and $p^*-p = \delta p$) are called disturbances and they satisfy the equations

$$\frac{d\underset{\sim}{u}}{dt} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{U} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = \frac{1}{\rho\mu}(\underset{\sim}{b}\cdot\nabla\underset{\sim}{B} + \underset{\sim}{b}\cdot\nabla\underset{\sim}{b} + \underset{\sim}{B}\cdot\nabla\underset{\sim}{b})$$

$$- \frac{1}{\rho}\nabla\ [\delta p + \frac{1}{2\mu}(|\underset{\sim}{B}^*|^2 - |\underset{\sim}{B}^2|)] + \nu\nabla^2\underset{\sim}{u}, \tag{2a}$$

$$\frac{db}{dt} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{B} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{b} = \underset{\sim}{B}\cdot\nabla\underset{\sim}{u} + \underset{\sim}{b}\cdot\nabla\underset{\sim}{u} + \underset{\sim}{b}\cdot\nabla\underset{\sim}{U} + \frac{1}{\sigma\mu}\nabla^2\underset{\sim}{b}, \tag{2b}$$

$$\nabla\cdot\underset{\sim}{u} = \nabla\cdot\underset{\sim}{b} = 0 \tag{2c}$$

and

$$\underset{\sim}{u} = \underset{\sim}{b} = 0|_S\ . \tag{2d}$$

Following CARMI & LALAS (1970) we next determine sufficient conditions under which the altered flow will tend asymptotically to the basic flow at $t \to \infty$. Toward this end we form energy identities

$$\frac{1}{2}\frac{d}{dt}\langle|\underset{\sim}{u}|^2\rangle = -\langle\underset{\sim}{u}\cdot\underset{\sim}{D}\cdot\underset{\sim}{u} + \nu\nabla\underset{\sim}{u}:\nabla\underset{\sim}{u}\rangle + \frac{1}{\rho\mu}\langle\underset{\sim}{B}\cdot\nabla\underset{\sim}{b}\cdot\underset{\sim}{u}$$

$$+ \underset{\sim}{b}\cdot\nabla\underset{\sim}{b}\cdot\underset{\sim}{u} + \underset{\sim}{b}\cdot\nabla\underset{\sim}{B}\cdot\underset{\sim}{u}\rangle, \tag{3a}$$

and

$$\frac{1}{2}\frac{d}{dt}\langle|\underset{\sim}{b}|^2\rangle = \langle\underset{\sim}{b}\cdot\underset{\sim}{D}\cdot\underset{\sim}{b}\rangle - \langle\frac{1}{\sigma\mu}\nabla\underset{\sim}{b}:\nabla\underset{\sim}{b}\rangle$$

$$+ \langle\underset{\sim}{B}\cdot\nabla\underset{\sim}{u}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\nabla\underset{\sim}{u}\cdot\underset{\sim}{b} - \underset{\sim}{u}\cdot\nabla\underset{\sim}{B}\cdot\underset{\sim}{b}\rangle\ . \tag{3b}$$

where the angle brackets designate volume-averaged integrals and $\underset{\sim}{D}$ is the strain-rate tensor for $\underset{\sim}{U}$. In carrying out the integration, the integral $\langle \nabla \cdot \underset{\sim}{A}_1 \rangle = 0$ which is added on the right of (3a) and the integral $\langle \nabla \cdot A_2 \rangle = 0$ which added on the right of (3b) has been carried to the boundary by the divergence theorem. The vector fields

$$\underset{\sim}{A}_1 = \nu\nabla\frac{1}{2}|\underset{\sim}{u}|^2 - \underset{\sim}{u}[\frac{1}{2}|\underset{\sim}{u}|^2 + \frac{\delta p}{\rho} + \frac{1}{2\mu\rho}(|\underset{\sim}{B}^*|^2 - |\underset{\sim}{B}|^2)]$$

and

$$\underset{\sim}{A}_2 = \frac{1}{\sigma\mu}\nabla\frac{1}{2}|\underset{\sim}{b}|^2 - \underset{\sim}{u}\frac{1}{2}|\underset{\sim}{b}|^2$$

vanish on $\partial\mathcal{V}$.

The reader's attention is drawn to the fact that some of the cubic nonlinearities in the disturbance, which arise from the quadratic terms in $\underset{\sim}{u}$ and $\underset{\sim}{b}$ of (2), do not integrate to zero, that is, though

$$\underset{\sim}{u}\cdot(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = \underset{\sim}{b}\cdot(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{b} = 0\,,$$

the terms

$$\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}\cdot\underset{\sim}{u} \quad \text{and} \quad \underset{\sim}{b}\cdot\nabla\underset{\sim}{u}\cdot\underset{\sim}{b}$$

are not necessarily zero and effect the energy balances (3a) and (3b).

There is a linear combination of (3a) and (3b) in which the cubic nonlinearities subtract out: thus,

$$\frac{1}{2}\frac{d}{dt}\left\langle \rho\mu|\underset{\sim}{u}|^2 + |\underset{\sim}{b}|^2\right\rangle = -\rho\mu\left\langle \underset{\sim}{u}\cdot\underset{\sim}{D}\cdot\underset{\sim}{u}\right\rangle + \left\langle \underset{\sim}{b}\cdot\underset{\sim}{D}\cdot\underset{\sim}{b}\right\rangle$$

$$+ 2\left\langle \underset{\sim}{b}\cdot\underset{\sim}{\Omega}_B\cdot\underset{\sim}{u}\right\rangle - \nu\mu\rho\left\langle \nabla\underset{\sim}{u}:\nabla\underset{\sim}{u}\right\rangle - \frac{1}{\sigma\mu}\left\langle \nabla\underset{\sim}{b}:\nabla\underset{\sim}{b}\right\rangle \quad , \qquad (4)$$

where

$\underset{\sim}{\Omega}_B$ = antisym metric part of ∇B

This is the energy identity considered by CARMI & LALAS.

There are four fundamental measures of the basic flow; the strain rate tensor $\underset{\sim}{D}$, the vorticity tensor $\underset{\sim}{\Omega}_B$ the symmetric part of the dyadic gradient of magnetic flow $\underset{\sim}{D}_B$ and the antisymmetric part $\underset{\sim}{\Omega}_B$ of the same tensor; that is

$$\nabla U = \underset{\sim}{D} + \underset{\sim}{\Omega}_U \quad \text{and} \quad \nabla B = \underset{\sim}{D}_B + \underset{\sim}{\Omega}_B \ .$$

Of the four measures only $\underset{\sim}{D}$ and $\underset{\sim}{\Omega}_B$ appear in the identity (4).

The fact that (4) is homogeneous of degree two in the disturbance opens the possibility of finding a global result which is independent of the scale of the motion. This possibility is realized in the theorem of unconditional stability of CARMI * LALAS (1970).*

* This theorem is also proved in the paper of BHATTACHARYYA and JAIN (1972). These authors also consider hydro-thermoconvective flows.

In preparation for the statement of this theorem the energy identity (4) is made dimensionless by dividing $[\underset{\sim}{x}, t, \underset{\sim}{u}, \underset{\sim}{b}, \underset{\sim}{B}, \underset{\sim}{D}]$ by $[l, l^2/\nu, U_o, B_o P_m^{1/2}/A, B_o P_m^{1/2}/A, \hat{D}_m]$ where U_o and B_o are typical values of the velocity and magnetic field, $P_m = \mu\sigma\nu$ is the magnetic Prandtl number, $A = B_o/U_o(\rho\mu)^{1/2}$ is the Alfven number and the other symbols are as before. We will now work only with dimensionless variables which, for economy, are also designated as $\underset{\sim}{u}, \underset{\sim}{b}$, etc. In dimensionless variables we have

$$\frac{d\mathcal{E}}{dt} = RI_1 - \langle \nabla\underset{\sim}{u}:\nabla\underset{\sim}{u} + \nabla\underset{\sim}{b}:\nabla\underset{\sim}{b} \rangle ,$$

where

$$I_1 = -\langle \underset{\sim}{u}\cdot\underset{\sim}{D}\cdot\underset{\sim}{u} \rangle + P_m(\langle \underset{\sim}{b}\cdot\underset{\sim}{D}\cdot\underset{\sim}{b} \rangle + 2\langle \underset{\sim}{b}\cdot\underset{\sim}{\Omega}_b\cdot\underset{\sim}{u} \rangle),$$

$$\mathcal{E} = \frac{1}{2}\langle |\underset{\sim}{u}|^2 + P_m |\underset{\sim}{b}|^2 \rangle, \qquad (5)$$

$$R = \frac{U_o l}{\nu}, \quad R_m = RP_m .$$

Here the magnetic Reynolds number is ratio of the convection rate to the diffusion rate of the magnetic field. Large R_m implies a thin boundary layer in which dissipation occurs. Outside this region the magnetic field and the flow are "frozen" together. Small R_m, on the other hand, implies that the total magnetic field of the flow is essentially equal to the imposed one, so that the induced field is small.

The magnetic Prandtl number P_m is a measure of the ratio of the rate of diffusion of vorticity to the rate of diffusion of the magnetic field.

A direct consequence of (5) and the estimates given in [2] is the following theorem of unconditional stability of CARMI & LALAS.

Let $\mathcal{V} = \mathcal{V}(t)$ be a bounded domain. Let $\underline{U}$ and $\underline{B}$ be the velocity and magnetic flux density vectors satisfying prescribed condition on $\partial\mathcal{V}$. Then $\mathcal{E} = \frac{1}{2}\langle |\underline{u}|^2 + P_m|\underline{b}|^2 \rangle$ satisfies the inequality

$$\mathcal{E}(t) < \mathcal{E}(o) \exp(-2MNt), \qquad (6)$$

where

$$M = \hat{\Lambda} - (\hat{R} + \hat{R}_m), \quad \hat{R} = \max[R, P_m R], \quad \hat{R}_m = \mathcal{b}_m R_m,$$

$$\mathcal{b}_m = \max_{\substack{i=1,2,3,\\ \underline{x} \in \mathcal{V} \\ o \leq t' \leq t}} \{\text{curl } B\}_i, \quad N = \min(1, P_m^{-1}),$$

and $\hat{\Lambda}(\mathcal{V})$ is the decay constant defined by (5). If $M>0$ for all t them $\mathcal{E}(t) \to 0$ as $t \to \infty$ and the flow is globally and monotonically stable.

This is a theorem of a type first given by SERRIN [6]. CARMI & LALAS, following SERRIN, also show that if

$$\frac{1}{R} > \max_{H} \frac{I_1}{\langle |\nabla \underline{u}|^2 + |\nabla \underline{b}|^2 \rangle}$$

where H is the space of solenoidal vectors vanishing on $\partial\mathcal{V}$ then $\mathcal{E} \to 0$ exponentially. This criterion also implies the uniqueness of steady flow.

There are a number of energy identities besides (3) which are of some value in treating the stability of hydromagnetic flows. For example, the three equations (2b) for the components of $\underset{\sim}{b}$ each give an energy equation. This contrasts with the equations (2a) for the components of u which involve the pressure. Energy equations for the components of b are like the separate equations (3a) and (3b) in that they involve cubic nonlinearities. Energy analysis here requires special procedures, like those used in [5] to handle the cubic nonlinearities.

One special identity, the correlation identity, merits special attention. This identity is formed from the sum $\langle \underset{\sim}{b} \cdot (2a) \rangle + \langle \underset{\sim}{u} \cdot (2a) \rangle$ and leads after integration by parts to

$$\frac{d}{dt} \langle \underset{\sim}{u} \cdot \underset{\sim}{b} \rangle = -2 \langle \underset{\sim}{u} \cdot \underset{\sim}{\Omega}_U \cdot \underset{\sim}{b} \rangle - \langle \underset{\sim}{u} \cdot \underset{\sim}{D}_B \cdot \underset{\sim}{u} \rangle + \frac{1}{\rho\mu} \langle \underset{\sim}{b} \cdot \underset{\sim}{D}_B \cdot \underset{\sim}{b} \rangle - (\nu + \frac{1}{\sigma\mu}) \langle \nabla \underset{\sim}{b} : \nabla \underset{\sim}{u} \rangle. \qquad (7)$$

The correlation identity is striking because it depends on the basic flow only through the measures $\underset{\sim}{D}_B$ and $\underset{\sim}{\Omega}_U$ of the "strain rate" of the magnetic flux and the vorticity of the basic motion; these measures are completely absent from the energy identity (4).

A linear combination of the energy identity (4) and the correlation identity (2) could form the basis for a modified energy analysis of the type considered in [4] for the convection problem in a fluid heated and salted from below.

After making (7) dimensionless, we may form a linear combination in the form

$$\frac{d\mathcal{E}_\lambda}{dt} = RI_\lambda - D_\lambda \,, \tag{8}$$

where

$$\mathcal{E}_\lambda = (\phi + \lambda)\,\mathcal{E} + 2\langle \underset{\sim}{u}\cdot\underset{\sim}{b}\rangle,$$

$$\phi = \frac{1}{P_m} + 1, \quad \lambda > 0,$$

$$I_\lambda = (\phi + \lambda)I_1 + 2I_2$$

$$I_2 = -2\langle \underset{\sim}{u}\cdot\underset{\sim}{\Omega}_U\cdot\underset{\sim}{b}\rangle - \langle \underset{\sim}{u}\cdot\underset{\sim}{D}_B\cdot\underset{\sim}{u}\rangle + P_m\langle \underset{\sim}{b}\cdot\underset{\sim}{D}_B\cdot\underset{\sim}{b}\rangle,$$

$$D_\lambda = \lambda\langle |\nabla\underset{\sim}{u}|^2 + |\nabla\underset{\sim}{b}|^2\rangle + \phi\,\langle |\nabla(\underset{\sim}{u}+\underset{\sim}{b})|^2\rangle .$$

We remark that for all fixed values of $\phi > 1$ and $\lambda > 0$ there exist values

$$\frac{1}{R_\lambda} = \max_H \frac{I_\lambda}{D_\lambda} \tag{9}$$

and

$$\frac{1}{\Lambda_\lambda} = \max_H \frac{\mathcal{E}_\lambda}{D_\lambda} \,. \tag{10}$$

These numbers define a stability limit and a decay constant in the energy stability theorem which is to be proved below. It is first necessary to establish a preliminary <u>Lemma: $\mathcal{E}_\lambda \geq 0$ for all $\lambda > 0$.</u>

To prove this, we note that

$$\frac{1}{2}(\phi + \lambda)\ (|\underset{\sim}{u}|^2 + P_m|\underset{\sim}{b}|^2) + 2\underset{\sim}{u}\cdot\underset{\sim}{b} \geq$$

$$\frac{1}{2}(\phi + \lambda)\ 2\ P_m^{1/2}\ |\underset{\sim}{u}\cdot\underset{\sim}{b}| - 2|\underset{\sim}{u}\cdot\underset{\sim}{b}|$$

$$= P_m^{1/2}\{(1 - \frac{1}{P_m^{1/2}})^2 + \lambda\}\ |\underset{\sim}{u}\cdot\underset{\sim}{b}| \geq 0$$

With this preliminary aside we may now establish the following:

<u>Energy theorem for hydromagnetic flow: Let</u>

$$R < R_\lambda \tag{11}$$

<u>for any $\lambda > 0$. Then</u>

$$\mathcal{E}_\lambda(t) < \mathcal{E}_\lambda(0)\ \exp\{-\Lambda_\lambda t[1 - \frac{R}{R_\lambda}]\}\ . \tag{12}$$

Proof: We may write (10) as

$$\frac{d\mathcal{E}_\lambda}{dt} = -D_\lambda\{-R\frac{I\lambda}{D_\lambda} + 1\} \leq -D_\lambda\{1 - \frac{R}{R_\lambda}\}$$

where we have used (9) in forming the last inequality.

If $R < R_\lambda$, by (10) we have

$$\frac{d\mathcal{E}_\lambda}{dt} < -\Lambda_\lambda \mathcal{E}_\lambda \{1 - \frac{R}{R_\lambda}\}$$

and (12) follows by integration.

<u>Corollary</u>: <u>The largest R domain of stability is associated with the "energy" $\tilde{\lambda} > 0$ where $\tilde{\lambda}$ is value of λ for which</u>

$$\frac{1}{R_{\tilde{\lambda}}} = \sup_{\lambda>0} \frac{1}{R_\lambda}$$

It is easy to prove that the initial condition which solves the maximum problem (9) is also the one which makes $\mathcal{E}_\lambda$ increase initially at the smallest R.

Computations for the criterion $R < R_{\tilde{\lambda}}$ for particular flows have not yet been carried out.

References

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