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Contributions to the Nonlinear Theory of Stability of Viscous Flow in Pipes and Between Rotating Cylinders

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Contents

Three component disturbance vector fields of the title flows, which are constant along a distinguished direction, imply the existence of a component of disturbance velocity which is not driven by disturbance pressure. This fact implies the existence of two energy identities: one for the energy of the velocity component along the distinguished direction and one for the energy of the perpendicular components. We extract consequences of this assumed form for the nonlinear disturbance for (A) rectilinear motion of viscous fluids down pipes of arbitrary cross-section and for (B) Taylor vortex disturbances of Couette flow.

In Sections I–IV we consider Poiseuille flow down a pipe of arbitrary cross-section. The restriction to Poiseuille flow is for purposes of exposition only; in Section V, the results are extended to general parallel flows. It turns out that unstable motions which do not vary along the flow axis cannot be sustained in these flows. But in common cases, disturbances of this form are also the ones which make the disturbance energy increase initially at the smallest Reynolds number. The energy of such an initially increasing, but stable, disturbance must lead to sharp increases in the energy of the axial or longitudinal disturbance velocity component. This is the only way in which the total disturbance energy could increase while the combined energy of the transverse disturbances decays. This result may give a partial explanation of the fact that in the neighborhood of the wall layer in turbulent flow, the longitudinal RMS velocity is always considerably in excess of the RMS values of the transverse velocity components.

In contrast to rectilinear flow, the Couette flow between rotating cylinders can support secondary motions which do not vary along the direction (circles around the cylinder) of the basic flow. One point of novelty here is the search (Sections VI–VII) for the "optimal" linear combination of the separate energies. The solution of this problem gives a stability boundary for nonlinear axisymmetric disturbances which is barely distinguishable from TAYLOR'S (linear theory) boundary when the angular velocity ratio is not too negative. For this optimum stability boundary we have guaranteed nonlinear stability, however, not for all initial values, but only for those whose energy is smaller than a finite "critical amplitude" which we estimate explicitly from below.

I. Kinematically Admissible Disturbances of Poiseuille Flow

Consider the flow of an incompressible viscous fluid (density ρ) driven by a spatially constant pressure gradient $\rho\hat{P}$ down an infinitely long $(-\infty<x_3<\infty)$ straight pipe of arbitrary cross-section A. The resulting basic Poiseuille flow $U_3(x_1, x_2)$ satisfies the Navier-Stokes equations in the form

$$O=\hat{P}+\nu\partial^2_{\alpha\alpha}U_3 \tag{I.1}$$

where $(\alpha=1,2)$, the summation convention holds, and

$$U_1\equiv 0\equiv U_2 \text{ in } A \quad\text{and}\quad U_3=0|_{\partial A}. \tag{I.2}$$

Consider next an alternative solution of the Navier-Stokes equations which differs from Poiseuille flow initially. This alternative solution is a solenoidal vector field $U_i^a(x_i, t)$, $(i=1,2,3)$ which vanishes at the boundary ∂A of A and which, together with the bounded part $\rho\pi(x_i, t)$ of the pressure $\rho(\hat{P}x_3+\pi)$, is almost periodic in the coordinate x_3.

The disturbances of Poiseuille flow $u_i=U_i^a-\delta_{i3}U_3$ and π are governed by the following initial boundary value problem (IBVP):

$$\frac{\partial u_i}{\partial t}+U_3\partial_3 u_i+u_\alpha\partial_\alpha(\delta_{i3}U_3)+u_j\partial_j u_i=-\partial_i\pi+\nu\nabla^2 u_i, \tag{I.3a}$$

$$\partial_i u_i=0, \tag{I.3b}$$

$$u_i=0|_{\partial A}, \tag{I.3c}$$

and

$$u_i=f_i(\boldsymbol{x})|_{t=0} \tag{I.3d}$$

where u_i, f_i and π are almost periodic functions of x_3.

The IBVP is to be defined for kinematically admissible initial conditions. These are defined as the set of vector valued functions $f_i(\boldsymbol{x})$ which are almost periodic in x_3 and such that

$$\partial_i f_i=0,$$

and

$$f_i=0|_{\partial A}.$$

II. The Form of the Disturbance Whose Energy Grows Initially at the Smallest Value of the Reynolds Number (R)

Every solution of (I.3) necessarily satisfies the energy equation

$$\frac{d\mathscr{E}}{dt} = \mathscr{I} - \nu \mathscr{D} \tag{II.1}$$

where

$$\mathscr{E} = \tfrac{1}{2}\langle |\boldsymbol{u}|^2 \rangle, \qquad \mathscr{D} = \langle |\nabla \boldsymbol{u}|^2 \rangle,$$

and

$$\mathscr{I} = -\langle u_3 u_\alpha \partial_\alpha U_3 \rangle.$$

Here angle brackets are volume averaged integrals over the entire pipe.

Let us examine the possible values which $d\mathscr{E}/dt$ can take as u_i ranges over the set f_i of kinematically admissible initial conditions. Define

$$\nu_{\mathscr{E}} = \max_{f_i} \frac{\mathscr{I}}{\mathscr{D}}. \tag{II.2}$$

If $\nu > \nu_{\mathscr{E}}$ then $d\mathscr{E}/dt < 0$ for every initial condition. If $\nu < \nu_{\mathscr{E}}$ then an initial condition can be found for which $d\mathscr{E}/dt > 0$. The initial condition which makes $\mathscr{E}$ grow at the largest ν (smallest R) is the kinematically admissible vector which solves (II.2).*

Now examine the values which $d\mathscr{E}/dt$ can take as u_i varies over the set of initial conditions

$$u_i(x_1, x_2, x_3, t=0) = g_i(x_1, x_2), \tag{II.3}$$

which depend only on the transverse coordinates. We will call disturbances in the form (II.3) *longitudinal vortices.*

We may write the energy equation (II.1) as

$$\frac{1}{\mathscr{D}} \frac{d\mathscr{E}}{dt} = \mathscr{F}[u_i] - \nu \tag{II.4}$$

where

$$\mathscr{F}[u_i] = \mathscr{I}/\mathscr{D}$$

is a homogeneous functional of degree zero.

Consider the range of $\mathscr{F}$ when its domain is restricted to longitudinal vortices.

Lemma 1. *Given any kinematically admissible vortex $\boldsymbol{g} = (g_1, g_2, g_3)$, we have*

$$\mathscr{F}[g_1, g_2, g_3] = -\mathscr{F}[g_1, g_2, -g_3] = -\mathscr{F}[-g_1, -g_2, g_3]. \tag{II.5}$$

Moreover,

$$\tilde{\nu}_{\mathscr{E}} = \max_{g_i} \mathscr{F} < \infty, \tag{II.6}$$

nd

$$-\tilde{\nu}_{\mathscr{E}} < \mathscr{F}[g_i] < \tilde{\nu}_{\mathscr{E}}.$$

* This theorem was known to Orr [1907]. He says that the "numbers I have found are true least values—, that below them every disturbance must automatically decrease, and that above them it is possible to prescribe a disturbance which will increase for a time." Orr's study proceeds from a linearization of Reynold's [1895] equation for the mean-motion and fluctuation-motion. For this reason, most subsequent authors have regarded Orr's work as valid only for small disturbances. In this paper, we follow Serrin [1959] in treating the energy of the difference motion rather than the fluctuation energy.

The symmetry properties (II.5) follow from the fact that the indicated sign changes do not alter the conditions of admissibility (since, in any case, $\partial_\alpha g_\alpha = 0$) nor do they change the value of $\mathscr{D}$. Hence, the sign of $\mathscr{F}$ is the sign of $\mathscr{I}$, and $\mathscr{I}$ changes sign as is indicated in (II.5). The functional $\mathscr{F}$ is bounded from above. In fact,

$$\frac{\mathscr{I}}{\mathscr{D}} = \frac{-\langle g_3 g_\alpha \partial_\alpha U_3\rangle}{\langle \partial_\alpha g_i \partial_\alpha g_i\rangle} \leqq K \frac{\langle |g_i|^2\rangle}{\langle |\partial_\alpha g_i|^2\rangle} \leqq \frac{K l^2}{\hat{\Lambda}}$$

where

$$\frac{l^2}{\hat{\Lambda}} = \max_{g_i} \frac{\langle |g_i|^2\rangle}{\langle |\partial_\alpha g_i|^2\rangle} \leqq \frac{l^2}{\Lambda} = \max_{g_3} \frac{\langle g_3^2\rangle}{\langle |\partial_\alpha g_3|^2\rangle},$$

K depends only on the basic flow, l is the maximum diameter of A and

$$\langle |g_i|^2\rangle = \lim_{L\to\infty} \frac{1}{2LA} \int_{-L}^{L} \int_A \int |g_i|^2 \, dx_1 \, dx_2 \, dx_3 = \frac{1}{A} \iint_A |g_i|^2 \, dx_1 \, dx_2.$$

Lemma 2. *The energy of any given longitudinal vortex* (g_1, g_2, g_3) *or that of its mates* $(g_1, g_2, -g_3)$ *or* $(-g_1, -g_2, g_3)$ *will increase initially if R is large enough (ν is small enough).*

Suppose that an initial disturbance of the type (II.3) is given. We let $\nu < |\mathscr{F}[g_1, g_2, g_3]|$ and use (II.5) to prove the result.

Lemma 3. *The longitudinal vortex disturbance which makes* $\mathscr{E}$ *increase initially at the smallest R (largest ν) is the vector field which solves* (II.6).

Suppose $\tilde{g}_i$ solves (II.6). Then $\mathscr{I}/\mathscr{D} = \tilde{\nu}_\mathscr{E}$ and, according to (II.4), $d\mathscr{E}/dt = (\tilde{\nu}_\mathscr{E} - \nu)\mathscr{D}$. If $\nu < \tilde{\nu}_\mathscr{E}$, $d\mathscr{E}/dt > 0$.

Of course, the restriction of the class of admissibility to longitudinal vortices could not raise the maximum in (II.2). Hence,

$$\nu_\mathscr{E} \geqq \tilde{\nu}_\mathscr{E}. \tag{II.7}$$

It is of considerable interest that existing analysis supports the conclusion that the maximizing vector for (II.2) *is* a longitudinal vortex for important special cases. For these cases, equality holds in (II.7). This is the case for plane Couette flow (JOSEPH [1966]), plane Poiseuille flow (BUSSE [1969]) and Poiseuille flow down an annulus (JOSEPH & CARMI [1969]). Though the maximizing vector for Hagen-Poiseuille flow is a *spiral vortex* (i.e., a longitudinal vortex whose axis spirals down the pipe), a longitudinal vortex very nearly solves this problem as well (see Eq. (II.7)); here, $\nu_\mathscr{E}/\tilde{\nu}_\mathscr{E} = 82.89/81.49$.

We will show in the next section that longitudinal vortices can never persist and the energy of the transverse velocity components must decay from the initial instant at every Reynolds number (all ν). On the other hand, these vortices can lead to an increase of the total energy when $\nu < \nu_\mathscr{E}$. For these vortices, the energy of the axial component of velocity increases faster than the decrease of the energy of the transverse velocity components.

III. Absolute Global Stability of Poiseuille Flow to Disturbances Which do not Vary in the Streaming Direction

By absolute stability we mean that $\mathscr{E}(t)/\mathscr{E}(0)$ tends to zero as $t\to\infty$ for all R (any ν). By global stability we mean that this stability holds for all initial energies $\mathscr{E}(0)$.

We will speak of a longitudinal vortex disturbance $u_i(x_1, x_2, t)$ of Poiseuille flow. We want to show that the kinetic energy of the transverse velocity components of this vortex must decay monotonically. Suppose that a solution of the IBVP can be found in the form of a longitudinal vortex $U_i^a(x_1, x_2, t)$ and pressure $\rho P^a(x_1, x_2, x_3, t)$. We can decompose this motion into Poiseuille flow plus a disturbance which is itself a longitudinal vortex. Thus

$$U_i^a(x_1, x_2, t)=\delta_{i3}\, U_3(x_1, x_2)+u_i(x_1, x_2, t) \tag{III.1a}$$

and

$$P^a(x_1, x_2, x_3, t)=\hat{P}x_3+\pi(x_1, x_2, t). \tag{III.1b}$$

The alternative motion with the superscript a satisfies the Navier-Stokes equations

$$\frac{\partial}{\partial t}[u_i+\delta_{i3}\, U_3]+u_\alpha\,\partial_\alpha[u_i+\delta_{i3}\, U_3]=\hat{P}\,\delta_{i3}-\delta_{i\alpha}\,\partial_\alpha\pi+\nu\,\partial_\alpha^2[u_i+\delta_{i3}\, U_3], \tag{III.2a}$$

$$\partial_\alpha u_\alpha=0, \tag{III.2b}$$

$$u_i=U_3=0\,|_{\partial A}, \tag{III.2c}$$

and

$$u_i(x_1, x_2, 0)=g_i(x_1, x_2), \qquad U_3=F_3(x_1, x_2). \tag{III.3}$$

The equations for the disturbance u_i of U_3 are obtained from (III.2a) using (I.1). We have

$$\frac{\partial u_3}{\partial t}+u_\alpha\,\partial_\alpha[u_3+U_3]=\nu\,\partial_{\alpha\alpha}^2 u_3, \tag{III.4a}$$

$$\frac{\partial u_\alpha}{\partial t}+u_\beta\,\partial_\beta\, u_\alpha=-\partial_\alpha\pi+\nu\,\partial_{\beta\beta}^2 u_\alpha \tag{III.4b}$$

where

$$\partial_\alpha u_\alpha=0, \tag{III.4c}$$

$$u_i=0\,|_{\partial A}, \tag{III.4d}$$

and

$$u_i(x_1, x_2, 0)=g_i(x_1, x_2). \tag{III.4e}$$

An energy equation for the transverse components of velocity can be formed using (III.4b, c, d). In the usual way

$$\frac{1}{2}\frac{d}{dt}\langle u_1^2+u_2^2\rangle=-\nu\langle|\partial_\beta u_1|^2+|\partial_\beta u_2|^2\rangle. \tag{III.5}$$

It follows from the usual energy estimates and (III.4e) that

$$\langle u_1^2+u_2^2\rangle\leqq\langle g_1^2+g_2^2\rangle\exp\left\{-\frac{2\nu\Lambda}{l^2}t\right\}. \tag{III.6}$$

Eq. (III.6) proves that the *transverse velocity components of a longitudinal vortex must decay monotonically from the initial instant.* This is an improved version and proof of a result first given by JOSEPH & TAO [1963].

If the transverse components of the disturbances must decay, independent of the size of the disturbance or the Reynolds number, it would be reasonable to expect an identical behavior for the axial component. There are, however, some interesting differences which we shall now delimit.

Consider the total energy of the longitudinal vortex disturbance. We have, from (II.1) and the assumption that x_3 derivatives vanish,

$$\frac{1}{2}\frac{d}{dt}\langle u_1^2+u_2^2+u_3^2\rangle = -\langle u_3 u_\alpha \partial_\alpha U_3\rangle - \nu\langle|\partial_\alpha u_i|^2\rangle,$$

and, on taking account of (III.5) and (III.6),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle u_3^2\rangle &= -\langle u_3 u_\alpha \partial_\alpha U_3\rangle - \nu\langle|\partial_\alpha u_3|^2\rangle \\ &\leqq K\sqrt{\langle u_3^2\rangle}\sqrt{\langle u_\alpha^2\rangle} - \nu\frac{\Lambda}{l^2}\langle u_3^2\rangle \\ &\leqq K\langle g_1^2+g_2^2\rangle^{\frac{1}{2}}\exp\left\{-\frac{\nu\Lambda}{l^2}t\right\}\langle u_3^2\rangle^{\frac{1}{2}} - \frac{\nu\Lambda}{l^2}\langle u_3^2\rangle\end{aligned}$$

where K is the largest of two values $(\partial_1 U_3, \partial_2 U_3)$ in A up to time t. Introduction of the variables $\tau=\frac{\nu\Lambda}{l^2}t$ and $\delta=\langle u_3^2\rangle^{\frac{1}{2}}$ leads us to

$$\frac{d\delta}{d\tau}\leqq c\exp(-\tau)-\delta$$

where C is a constant. This inequality can be integrated forward from time $\tau=0$ to give

$$\int_0^\tau \exp(\tau')\left[\frac{d\delta}{d\tau'}+\delta\right]d\tau' = \int_0^\tau \frac{d}{d\tau'}(\delta\exp(\tau'))\,d\tau' \leqq c\tau.$$

Hence

$$\delta(\tau)\leqq\delta(0)\,e^{-\tau}+c\tau e^{-\tau}. \tag{III.7}$$

Eqs. (III.6) and (III.7) show that

$$\langle u_1^2+u_2^2+u_3^2\rangle < c^2\tau^2 e^{-2\tau}+O(\tau e^{-2\tau})$$

at large τ, and this is true for all R and every initial value. *Hence, Poiseuille flow is absolutely and globally stable to disturbances in the form of longitudinal vortices.*

IV. On the Growth, at Early Times, of the Energy of the Axial Component of Velocity

Now we shall restrict our considerations to the set of longitudinal vortices whose energy increases initially. We showed in Lemma 2 that if R is large enough, every longitudinal vortex has an initially increasing energy in the sense that if the energy of (g_1, g_2, g_3) decreases, the energy of $(g_1, g_2, -g_3)$ increases.

The point to be made about these initially increasing disturbances is that the energy of their transverse components must *always* decrease. Hence, the energy of the axial part of the disturbance must increase even more rapidly than the decrease in the transverse energy.

The situation with regard to $\langle u_3^2\rangle=\delta^2$ is already fairly well represented by the estimate (III.7). The function on the right of (III.7) first increases linearly with t and then decreases.

It is more revealing to note that at $t=0$, (II.4) and (III.5) combine into

$$\frac{1}{2}\frac{d\langle u_3^2\rangle}{dt}=(\mathscr{F}[g_i]-\nu)\langle|\partial_\alpha g_3|^2\rangle+\mathscr{F}[g_i]\langle|\partial_\alpha g_1|^2+|\partial_\alpha g_2|^2\rangle. \quad \text{(IV.1)}$$

When $\nu<\mathscr{F}[g_i]$ (R is large enough),

$$\frac{1}{2}\frac{d}{dt}\langle u_3^2\rangle\geqq\nu\langle|\partial_\alpha g_1|^2+|\partial_\alpha g_2|^2\rangle=-\frac{1}{2}\frac{d}{dt}\langle u_1^2+u_2^2\rangle \quad \text{(IV.2)}$$

at $t=0$.

The dissipation of the transverse motion bounds from below the growth of the energy of the axial motion at early times. The largest rate of growth of the axial motion is obtained for the initial condition which maximizes $\mathscr{F}(=\nu_g)$.

V. Extension to Parallel Flow, Summary and Discussion

It is easy to verify that all the deductions which follow from (III.4) are self-contained and apply to all parallel motions in the form $U_i=\delta_{i3}U_3(x_1,x_2)$. For instance, they would hold if in addition to Poiseuille flow, a Couette flow was induced by parallel sliding of one cylinder in another (the cylinders need not be circular). Hence,

Theorem 1. *Parallel flow with prescribed boundary values is globally and absolutely stable to disturbances of the form* (II.3). *The energy of the transverse components of such a disturbance must decay monotonically and satisfies the estimate*

$$\tfrac{1}{2}\langle u_1^2+u_2^2\rangle\leqq\tfrac{1}{2}\langle g_1^2+g_2^2\rangle\exp\left\{-\frac{2\nu\Lambda}{l^2}t\right\}. \quad \text{(V.1)}$$

The axial component of this same disturbance can increase initially and at time t satisfies the estimate

$$\tfrac{1}{2}\langle u_3^2\rangle\leqq\tfrac{1}{2}\{\langle u_3^2\rangle_{t=0}^{\frac{1}{2}}+ct\}^2\exp\left\{-\frac{2\nu\Lambda}{l^2}t\right\}. \quad \text{(V.2)}$$

If $\nu<\nu_g$, initially increasing disturbances exist. The axial energy of such increasing disturbances increases faster than the rate of decay of the energy of the transverse motion.

The first part of this theorem could be considered as a kind of nonlinear "Squire's theorem". It eliminates the possibility that a motion of the form (II.3) (a longitudinal vortex) could permanently replace parallel flow.

Much of the physical content of Theorem 1 is necessarily of a speculative character since the infinite domain theory is not a perfect model of natural nearly-parallel motions. We have in mind disturbances whose wave lengths in the direction of the motion are long relative to the transverse scale of disturbance motion. Many such motions are known, but one of the most striking is in the laminar sublayer of a tubulent boundary layer.

In the laminar sublayer at a solid wall (in a pipe or along a flat plate) the large scale motions (big eddies) are at present understood (from experiment) to be organized in longitudinal vortex-like structures. The role of these vortices are discussed in the papers of BAKEWELL & LUMLEY [1967] and KLINE, REYNOLDS, SCHRAUB & RUNDSTADLER [1967]. The view that flow in the laminar sublayer (near the wall) is a steady two-dimensional parallel motion is not borne out by experiments of these (and other) authors. Instead, a dominant motion in the form of rows (or pairs) of longitudinal vortices is said to prevail in the sublayer and in the viscous wall region beyond the sublayer. When hydrogen bubbles are inserted into the sublayer, as in the experiment of KLINE, *et al.*, these bubbles collect near the bottom wall along lines parallel to x_3 where the circulation velocities of adjacent vortices tend to sweep up fluid. The bubbles form elongated streaks which are clearly exhibited in Fig. 10 of KLINE, *et al.* (see also Fig. 15 (HAMA in CORRSIN [1957])).

At the same time, a student of turbulence is confronted with another outstanding experimental fact. It is well known from the experiments of REICHARDT [1938], LAUFER [1954] and KLEBANOFF [1955], among others (see SCHLICHTING, Chapter XVIII d) that the RMS component of the fluctuating longitudinal velocity is appreciably greater than the RMS value of either of the two other fluctuating components. The explanation for this fact is usually stated in terms of Reynolds averaged equations. These show that the energy received by the turbulence from the mean flow appears first as the RMS value of the longitudinal velocity (CORRSIN, equation (5), HINZE [1959], equations 4.4–4.7).

Our result (see Eq. (IV.2)) shows that the tendency of the longitudinal vorticity of a parallel motion to decay can be accompanied by a strong increase in the energy of the longitudinal fluctuation. Thus the continuous breakdown of longitudinal vorticity would appear as another source of the observed dominance of longitudinal fluctuations over transverse ones (cf. LUMLEY [1971] for a similar result, reached differently).

The above physical interpretation of our theorem is at best tentative. The observed "longitudinal" vortices are unsteady and they always have some three dimensional structure. On the other side it is well to note that though a stability theory for steady flow would be useless for these intermittent turbulent structures they can always be viewed as initial conditions, continuously renewed. Here an initial value analysis like that leading to IV.2 could have a physical content.

The longitudinal vortex is an efficient motion for the transport of disturbance momentum, but in rectilinear shear flows, it has no dynamic source to drive it and it decays. In the rotating flow which is considered next, the Taylor vortex disturbance can be driven by forces associated with the rotation. These vortices need not decay.

VI. The Basic Couette Flow and the Axisymmetric Disturbance Flow

The basic motion whose stability will now be studied is a shear flow between concentric rotating cylinders. Circumferential flow is induced by the shear transmitted by a differential rotation.

Let a, Ω_1 and b, Ω_2 be the radii and constant angular velocities, respectively, of the inner and outer cylinders. (r, θ, x) are polar co-ordinates and (w, v, u) are the corresponding disturbance velocity components.

A basic laminar solution of the Navier-Stokes equation of the form

$$(U_r, U_\theta, U_x) = (0, U_\theta(r), 0)$$

is readily obtained as

$$U_\theta = A r + B/r$$

and

$$A = \frac{b^2 \Omega_2 - a^2 \Omega_1}{b^2 - a^2}, \qquad B = \frac{-a^2 b^2 (\Omega_2 - \Omega_1)}{b^2 - a^2}. \tag{VI.1}$$

Our interest in this flow is in its stability to large disturbances; the basis for our global analysis is the method of energy. An energy analysis of the stability of the Couette flow between rotating cylinders was first successfully constructed by SERRIN (I) [1959]. An energy analysis of Couette flow between rotating and sliding cylinders has been constructed by JOSEPH & MUNSON (II) [1970]. The present work is a continuation of the work in these two references.

This and subsequent sections explore the consequences of assuming from the outset that the disturbance motion which replaces Couette flow has a Taylor vortex form. There is an important range of the parameters in which the axisymmetric form of the disturbance flow is just the one which is observed in experiments; in this parameter range the Taylor vortex form for the disturbance is theoretically "right" in the sense that, as far as the result of energy and linear analysis is known, both the instability (linear) limit and stability (energy) limit are taken on for axisymmetric disturbances.

The present energy analysis differs from I and II in that, at the outset, in the governing nonlinear disturbance equations, a disturbance is assumed in axisymmetric form. In the analysis of I and II, no assumption about the form of the disturbance is made at the outset, but the *solution* vector field for the energy problem is an axisymmetric field.

In the restricted class of axisymmetric disturbances, one can achieve big improvements in the energy criteria for stability to nonlinear disturbances.

The basic equations for axisymmetric (Taylor vortex) disturbances of Couette flow are

$$\frac{\partial w}{\partial t} + (\boldsymbol{u} \cdot \nabla_2) w - \frac{v^2}{r} - 2v \left[A + \frac{B}{r}\right] = -\partial_r p + \nu \left(\nabla_2^2 - \frac{1}{r^2}\right) w, \tag{VI.2a}$$

$$\frac{\partial v}{\partial t} + (\boldsymbol{u} \cdot \nabla_2) v + \frac{w v}{r} + 2A w = \nu \left(\nabla_2^2 - \frac{1}{r^2}\right) v, \tag{VI.2b}$$

and

$$\frac{\partial u}{\partial t} + (\boldsymbol{u} \cdot \nabla_2) u = -\partial_x p + \nu \nabla_2^2 u, \tag{VI.2c}$$

$$\nabla_2 \cdot (r\boldsymbol{u})=0, \qquad \boldsymbol{u}=0|_{a,b}. \tag{VI.2d, e}$$

Here $\boldsymbol{u}=\boldsymbol{e}_r w+\boldsymbol{e}_x u$, and $\boldsymbol{u}$, v and p are periodic functions of x with period $2\pi/\alpha$

$$\nabla_2=\boldsymbol{e}_r\,\partial_r+\boldsymbol{e}_x\,\partial_x, \qquad \nabla_2^2=\frac{1}{r}\,\partial_r(r\,\partial_r)+\partial_{xx}^2.$$

The feature of these equations which forms the basis for our analysis is the absence of a pressure gradient term in Eq. (VI.2b).

VII. The Stability Criterion and the Critical Amplitude

A consequence of the absence of a pressure gradient in the circumferential direction is the existence of *two* energy equations

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle w^2+u^2\rangle-\left\langle\frac{wv^2}{r}\right\rangle-2\left\langle\left(A+\frac{B}{r^2}\right)wv\right\rangle\\&\qquad=-\nu\left\langle|\nabla_2 w|^2+|\nabla_2 u|^2+\left|\frac{w}{r}\right|^2\right\rangle\end{aligned} \tag{VII.1a}$$

and

$$\frac{1}{2}\frac{d}{dt}\langle v^2\rangle+\left\langle\frac{wv^2}{r}\right\rangle+2A\langle wv\rangle=-\nu\left\langle|\nabla_2 v|^2+\left|\frac{v}{r}\right|^2\right\rangle. \tag{VII.1b}$$

Here, the angle bracket is the volume integral over a period cell (see Eq. (VII.9)).

It is convenient to work with the linear combination

$$\text{(VII.1a)}+\lambda\text{(VII.1b)}$$

where λ is a positive coupling parameter. Then setting

$$\phi=\sqrt{\lambda}\,v,$$

we come to the single equation

$$\frac{d\mathscr{E}}{dt}+\left(1-\frac{1}{\lambda}\right)\left\langle\frac{w\phi^2}{r}\right\rangle=-\mathscr{H}[w,\phi,\lambda]-\nu\mathscr{D}_2 \tag{VII.2}$$

where

$$\mathscr{E}=\tfrac{1}{2}\langle w^2+u^2+\phi^2\rangle,$$

$$\mathscr{D}_2=\left\langle|\nabla_2 w|^2+|\nabla_2 u|^2+\left|\frac{w}{r}\right|^2+|\nabla_2\phi|^2+\left|\frac{\phi}{r}\right|^2\right\rangle,$$

and

$$-\mathscr{H}=\frac{2(1-\lambda)}{\sqrt{\lambda}}\,A\langle w\phi\rangle+\frac{2B}{\sqrt{\lambda}}\left\langle\frac{w\phi}{r^2}\right\rangle.$$

The stability theorem which we shall prove from (VII.2) is best stated in two parts. First we state the part about the *stability criterion.*

Theorem 2(a). *Let $\lambda>0$ be preassigned and suppose that $\nu>\nu_{\mathscr{E}}(\lambda)$ where*

$$\nu_{\mathscr{E}}=\max_{H_2}-\mathscr{H}/\mathscr{D}_2 \tag{VII.3}$$

and $\boldsymbol{H}_2$ *is the set of kinematically admissible axisymmetric vectors. Then, every axisymmetric disturbance of Couette flow satisfies the inequality*

$$\frac{d\mathscr{E}}{dt}+\frac{(\lambda-1)}{\lambda}\left\langle\frac{w\phi^2}{r}\right\rangle\leqq-(\nu-\nu_{\mathscr{E}})\mathscr{D}_2. \tag{VII.4}$$

Since axisymmetric solutions of the IBVP for (VII.4) are in $\boldsymbol{H}_2$, we must have that $-\mathscr{H}\leqq\nu_{\mathscr{E}}\mathscr{D}_2$ where $\mathscr{H}$ and $\mathscr{D}_2$ are evaluated for solutions. Combining this with (VII.2) we arrive at (VII.4).

The next and last part of the stability Theorem 2 introduces the "critical amplitude". We get nonlinear stability when the axially symmetric disturbances are initially below a certain finite size.

The decay constant $\hat{\Lambda}$ which will appear in the statement of the last part of the theorem is defined by

$$\frac{(b-a)^2}{\hat{\Lambda}}=\max_{\boldsymbol{H}_2}\mathscr{E}/\mathscr{D}_2. \tag{VII.5}$$

The constant $\hat{\lambda}$ is defined by

$$\frac{(b-a)^2}{\hat{\lambda}}=\max\langle\phi^2\rangle/\langle|\nabla_2\phi|^2\rangle$$

where the maximum is taken over functions ϕ which are $2\pi/\alpha$ periodic in x and vanish at $r=a$ and $r=b$. The constants $\hat{\Lambda}$ and $\hat{\lambda}$ are the positive zeros of certain Bessel functions.

Theorem 2(b). *Suppose that*

$$\nu>\nu_{\mathscr{E}}(\lambda) \tag{VII.6}$$

and

$$\mathscr{E}(0)<(F/G)^2 \tag{VII.7}$$

where

$$F=(\nu-\nu_{\mathscr{E}})\hat{\Lambda}^{\frac{1}{2}}/(b-a)\quad and\quad G=\frac{|\lambda-1|}{\lambda\sqrt{2a^3}}\sqrt{\frac{\alpha(b-a)}{\pi\hat{\lambda}^{\frac{1}{2}}}+1}.$$

Then $\mathscr{E}(t)$ *decays to zero monotonically according to the law*

$$\frac{\mathscr{E}(t)}{[F-G\mathscr{E}^{\frac{1}{2}}(t)]^2}\leqq\frac{\mathscr{E}(0)}{[F-G\mathscr{E}^{\frac{1}{2}}(0)]^2}\exp\left\{\frac{-F\hat{\Lambda}^{\frac{1}{2}}t}{(b-a)}\right\}. \tag{VII.8}$$

To prove the theorem, we first need to prove an imbedding inequality:

Lemma 4. *Let* $\phi(r,x)$ *be a smooth function which vanishes at* $r=a$ *and* $r=b$ *and which is periodic in x with period* $2\pi/\alpha$. *Then*

$$\begin{aligned}\langle\phi^4\rangle&\equiv\int_a^b\int_0^{2\pi/\alpha}\phi^4\,r\,dr\,dx\\&\leqq\frac{\alpha\langle\phi^2\rangle^{\frac{1}{2}}}{2\pi a}\left\langle\left|\frac{\partial\phi}{\partial r}\right|^2\right\rangle^{\frac{1}{2}}+\frac{\langle\phi^2\rangle}{a}\left\langle\left|\frac{\partial\phi}{\partial r}\right|^2\right\rangle^{\frac{1}{2}}\left\langle\left|\frac{\partial\phi}{\partial x}\right|^2\right\rangle^{\frac{1}{2}}\\&\leqq\frac{\langle\phi^2\rangle\langle|\nabla_2\phi|^2\rangle}{2a}\left\{\frac{\alpha(b-a)}{\pi\hat{\lambda}^{\frac{1}{2}}}+1\right\}.\end{aligned} \tag{VII.9}$$

The proof of (VII.9) is given in the Appendix.

Using (VII.9), we form the estimate

$$\frac{(\lambda-1)}{\lambda}\left\langle\frac{w\phi^2}{r}\right\rangle \leqq \frac{|1-\lambda|}{\lambda a}\sqrt{\langle w^2\rangle\langle\phi^4\rangle} \leqq G\sqrt{\langle w^2\rangle\langle\phi^2\rangle\langle|\nabla_2\phi|^2\rangle}$$
$$\leqq G\mathscr{E}\sqrt{\mathscr{D}_2}. \tag{VII.10}$$

The energy inequality can then be written as

$$\frac{d\mathscr{E}}{dt} \leqq [-(\nu-\nu_{\mathscr{E}})\sqrt{\mathscr{D}_2}+G\mathscr{E}]\sqrt{\mathscr{D}_2}.$$

When (VII.6) holds, this inequality may be written as

$$\frac{d\mathscr{E}}{dt} \leqq [-F\sqrt{\mathscr{E}}+G\mathscr{E}]\sqrt{\mathscr{D}_2}$$

where we have used (VII.5). Now, if in addition to (VII.6)

$$\mathscr{E}(0) < \left(\frac{F}{G}\right)^2 = \frac{2a^3\hat{\Lambda}(\nu-\nu_{\mathscr{E}})^2\lambda^2}{|1-\lambda|^2(b-a)^2}\Big/\left[1+\frac{\alpha}{\pi}\frac{(b-a)}{\hat{\lambda}^{\frac{1}{2}}}\right], \tag{VII.11}$$

we may again continue the inequality as

$$\frac{d\mathscr{E}}{dt} \leqq [-F\mathscr{E}+G\mathscr{E}^{\frac{3}{2}}]\hat{\Lambda}^{\frac{1}{2}}/(b-a). \tag{VII.12}$$

Eq. (VII.8) now follows from (VII.12) by integration.

The stability theorem gives rise to a stability criterion $\nu > \nu_{\mathscr{E}}$ and a side condition (VII.11) on initial conditions for which stability can be guaranteed. We note that large initial values $\mathscr{E}(0)$ are allowed by (VII.11) if $\nu > \nu_{\mathscr{E}}$ is fixed and either $a/b \to 1$ or $\lambda \to 1$. When $\lambda = 1$, the evolution equation (VII.2) is just the one which was considered by SERRIN (I). When $\lambda = 1$, axial symmetry is not required and the resulting stability criterion is global; it holds for all $\mathscr{E}(0)$ and for all possible disturbances. On the other hand, when $\lambda \neq 1$, Theorem 2 gives certain stability to axisymmetric disturbances with initial energies which tend to zero as $\nu - \nu_{\mathscr{E}} \to 0$.

In VIII we will compute, using the best λ, a conditional stability boundary (see Figs. 1, 2, 5) which is virtually indistinguishable from the linear stability boundary in the region

$$A < 0, \quad (a^2\Omega_1 > b^2\Omega_2 \geqq 0).$$

When $A = 0$ we have

$$-\mathscr{H} = \frac{2B}{\sqrt{\lambda}}\left\langle\frac{w\phi}{r^2}\right\rangle,$$

and since $-\sqrt{\lambda}\cdot(\mathscr{H}/\mathscr{D}_2)$ is bounded above, we may verify that

$$\nu_{\mathscr{E}}(\sqrt{\lambda}\to\infty)\to 0.$$

Hence, for $A = 0$ there is conditional stability to axisymmetric disturbances for all ν.

Recall RAYLEIGH'S inviscid stability criterion

$$A > 0, \quad (b^2\Omega_2 > a^2\Omega_1, \Omega_1 > \Omega_2)$$

for axisymmetric disturbances. The following conditional stability theorem for Couette flow holds when $A>0$ and $\Omega_1>\Omega_2$.

Let

$$\mathscr{E}_V(t)=\tfrac{1}{2}\left\langle A w^2+A u^2+\left(A+\frac{B}{r^2}\right) v^2\right\rangle,$$

$$\mathscr{D}_V=\left\langle A\left(|\nabla w|^2+|\nabla_2 u|^2+\left|\frac{w}{r}\right|^2\right)+(A r^2+B)\left|\nabla_2 \frac{v}{r}\right|^2\right\rangle,$$

$$G_V=\frac{B}{\sqrt{2}\, a^{\frac{1}{2}} A^{\frac{1}{2}} \Omega_1}\left\{\frac{\alpha}{\pi}\,\frac{b-a}{\hat{\lambda}^{\frac{1}{2}}}+1\right\}^{\frac{1}{2}},$$

and

$$\frac{(b-a)^2}{\hat{\Lambda}_V}=\max_{H_2}\frac{\mathscr{E}_V}{\mathscr{D}_V}.$$

Theorem 3. *Circular Couette flow is stable to arbitrary periodic (in x) axisymmetric disturbances when*

$$A>0, \qquad \Omega_1>\Omega_2, \tag{VII.13}$$

and

$$\mathscr{E}_V(0)<\left(\frac{\nu}{G_V}\right). \tag{VII.14}$$

Moreover, when (VII.13) *and* (VII.14) *hold*

$$\frac{\mathscr{E}_V(t)}{[\nu-G_V\mathscr{E}_V^{\frac{1}{2}}(t)]^2}\leqq\frac{\mathscr{E}_V(0)}{[\nu-G_V\mathscr{E}^{\frac{1}{2}}(0)]^2}\exp\left\{\frac{-\nu\hat{\Lambda}_V^{\frac{1}{2}} t}{(b-a)}\right\}. \tag{VII.15}$$

The stability criterion $A>0$, $\Omega_1>\Omega_2$ was established for axisymmetric solutions of the linearized stability equations by SYNGE (1938). The present result is a nonlinear extension of SYNGE's result.

Proof. To prove the conditional stability theorem for Couette flow we shall need to establish the following evolution equation

$$\frac{d\mathscr{E}_V}{dt}=-2B\left\langle\frac{w v^2}{r^3}\right\rangle-\nu\mathscr{D}_V. \tag{VII.16}$$

This equation is the sum of A. (VII.1a) and equation (VII.17) below. Equation (VII.17) follows from the integration of $\left\langle\frac{V}{r}v\text{(VI.2b)}\right\rangle$ over a period cell; in carrying out the integration we calculate

$$\left\langle\frac{V}{r}v(\boldsymbol{u}\cdot\nabla_2)v\right\rangle=B\left\langle\frac{w v^2}{r^3}\right\rangle,$$

$$\left\langle\frac{V}{r}v\left(\nabla_2^2-\frac{1}{r^2}\right)v\right\rangle=-\left\langle\frac{V}{r}|\nabla_2 v|^2\right\rangle+2B\left\langle\frac{v^2}{r^4}\right\rangle-\left\langle\frac{V}{r}\,\frac{v^2}{r^2}\right\rangle,$$

and

$$2B\left\langle\frac{v^2}{r^4}\right\rangle=2\left\langle\frac{V}{r}\,\frac{v}{r}\,\partial_r v\right\rangle.$$

Combining the last two equations we find that

$$\left\langle \frac{V}{r} v \left(\nabla_2^2 - \frac{1}{r^2} \right) v \right\rangle = -\left\langle \frac{V}{r} (\partial_x v)^2 \right\rangle - \left\langle \frac{V}{r} \left(r \partial_r \frac{v}{r} \right)^2 \right\rangle,$$

and

$$\frac{1}{2} \frac{d}{dt} \left\langle \frac{V}{r} v^2 \right\rangle + \left\langle \left(\frac{V}{r} + \frac{B}{r^2} \right) \frac{w v^2}{r} \right\rangle + 2A \left\langle \frac{V}{r} w v \right\rangle$$
$$= -\nu \left\langle \frac{V}{r} (\partial_x v)^2 + \frac{V}{r} \left(r \partial_r \frac{v}{r} \right)^2 \right\rangle. \qquad \text{(VII.17)}$$

We may simplify the computation by introducing variables

$$\frac{v}{r} = \phi, \quad \sqrt{A}\, w = \psi, \quad \sqrt{A}\, u = \gamma.$$

In these variables (VII.16) becomes

$$\frac{1}{2} \frac{d}{dt} \langle \psi^2 + \gamma^2 + (rV) \phi^2 \rangle + \frac{2B}{\sqrt{A}} \left\langle \frac{\psi \phi^2}{r} \right\rangle$$
$$= -\nu \left\langle |\nabla_2 \psi|^2 + |\nabla_2 \gamma|^2 + \left| \frac{\psi}{r} \right|^2 + (rV) |\nabla_2 \phi|^2 \right\rangle.$$

Since

$$\frac{d}{dr} (rV) = 2Ar > 0,$$

we have (since $A > 0$)

$$\max_r (rV) = b^2 \Omega_2 > \min_r (rV) = a^2 \Omega_1.$$

Hence, we may continue the inequality (VII.9) as

$$\langle \phi^4 \rangle \leqq \frac{\langle rV\phi^2 \rangle \langle rV |\nabla_2 \phi|^2 \rangle}{2a^3 \Omega_1^2} \left\{ \frac{\alpha}{\pi} \frac{(b-a)}{\lambda^{\frac{1}{2}}} + 1 \right\},$$

and the remainder of the proof leading to (VII.15) follows along the path leading from (VII.9) to (VII.12).

VIII. The Optimum Stability Boundary for Axisymmetric Disturbances of Couette Flow

Our task now is to find the stability limits $\nu_{\mathscr{E}}(\lambda)$. Here $\lambda > 0$ is a free parameter, which we can select to obtain the largest region for certain stability to axisymmetric disturbances. The value $\nu_{\mathscr{E}}(\tilde{\lambda}) = \tilde{\nu}_{\mathscr{E}}$ which yields the maximum region of stability, that is

$$\tilde{\nu}_{\mathscr{E}} = \min_{\lambda > 0} \nu_{\mathscr{E}}(\lambda), \qquad \text{(VIII.1)}$$

is called the "optimum stability boundary". It turns out that the critical value $\tilde{\lambda}$ can be so selected that $\tilde{\nu}_{\mathscr{E}}$ and the linear limit ν_L for Taylor vortices are virtually indistinguishable when $b^2 \Omega_2 < a^2 \Omega_1$, and $\Omega_2/\Omega_1 > 0$.

The following result will help to characterize the optimizing value $\lambda=\tilde{\lambda}$.

Lemma 5. *We have*

$$\tilde{\lambda}+1=\frac{-B\left\langle\frac{w\phi}{r^2}\right\rangle}{A\langle w\phi\rangle}. \qquad \text{(VIII.2a)}$$

When $b^2\Omega_2<a^2\Omega_1$ *and* $\Omega_2/\Omega_1>0$ *there exists an* $\bar{r}$, $a\leqq\bar{r}\leqq b$, *such that* $\left\langle\frac{w\phi}{r^2}\right\rangle=\langle w\phi\rangle/\bar{r}^2$. *Then*

$$\tilde{\lambda}+1=\frac{\eta^2\left(1-\frac{\Omega_2}{\Omega_1}\right)}{\left(\frac{\bar{r}}{b}\right)^2\left(\eta^2-\frac{\Omega_2}{\Omega_1}\right)}>1. \qquad \text{(VIII.2b)}$$

The formula (VIII.2b) expresses the requirement that $\lambda=\tilde{\lambda}$ when $\partial\nu_{\mathscr{g}}/\partial\lambda=0$. Here the limit $\nu_{\mathscr{g}}(\lambda)$ which solves (VII.3) is most conveniently found as the principal eigenvalue of Euler's equations for (VII.3)

$$\left\{\frac{B}{r^2\sqrt{\lambda}}-\frac{(\lambda-1)A}{\sqrt{\lambda}}\right\}\phi+\nu_{\mathscr{g}}\left(\nabla_2^2-\frac{1}{r^2}\right)w=\partial_r p, \qquad \text{(VIII.3a)}$$

$$\left\{\frac{B}{r^2\sqrt{\lambda}}-\frac{(\lambda-1)A}{\sqrt{\lambda}}\right\}w+\nu_{\mathscr{g}}\left(\nabla_2^2-\frac{1}{r^2}\right)\phi=0, \qquad \text{(VIII.3b)}$$

$$\nu_{\mathscr{g}}\nabla_2^2 u=\partial_x p, \qquad \text{(VIII.3c)}$$

$$\partial_r(rw)+\partial_x(ru)=0, \qquad \text{(VIII.3d)}$$

and

$$w=\phi=0|_{r=a,b}. \qquad \text{(VIII.3e)}$$

The equations (VIII.3) can be reduced to ordinary differential equations for the Fourier coefficient $\hat{w}(r)$ and $\hat{\phi}(r)$ by Fourier series (formally, set $(w,\phi,p)=(\hat{w}(r),\hat{\phi}(r),\hat{p}(r))\cos\alpha x$, $u=\hat{u}(r)\sin\alpha x$, etc., and eliminate $\hat{u}(r)$ and $\hat{p}(r)$ from the resulting set). The differential equation problem which results from this reduction is self-adjoint, and this shows that $\hat{w}(r)$ and $\hat{\phi}(r)$ may be presumed to have real values. Then the ratio on the right of (VIII.2a) reduces to

$$-\frac{B}{A}\frac{\int_a^b \hat{w}(r)\hat{\phi}(r)/r^2\cdot r\,dr}{\int_a^b \hat{w}(r)\hat{\phi}(r)\,r\,dr}.$$

To establish the existence of the mean value of $\bar{r}$, it will suffice to show that $\hat{w}(r)$ and $\hat{\phi}(r)$ are one-signed when $\lambda=\tilde{\lambda}$.

Let

$$\lambda+1=\frac{-B}{A\hat{r}^2}, \quad a\leqq\hat{r}\leqq b.$$

For these λ,

$$\frac{B}{r^2}-(\lambda-1)A=2A+B\left[\frac{1}{r^2}+\frac{1}{\hat{r}^2}\right]$$

$$=\frac{2\Omega_1}{1-\eta^2}\left\{\frac{\Omega_2}{\Omega_1}\left(1-\frac{\eta^2}{\rho^2}\right)-\eta^2\left(1-\frac{1}{\rho^2}\right)\right\}\geqq 0$$

where

$$\frac{1}{\rho^2}=\frac{b^2}{2}\left[\frac{1}{r^2}+\frac{1}{\hat{r}^2}\right] \quad \text{and} \quad \eta\leqq\rho\leqq 1.$$

It is clear from this inequality and equations (VIII.3) that the ordinary differential equations which govern the Fourier coefficients $\hat{w}(r)$ and $\hat{\phi}(r)$ can be converted into integral equations with oscillatory kernels (cf. YUDOVICH [1966]). Hence for these $\lambda(\hat{r})$, $\hat{w}$ and $\hat{\phi}$ are of one sign and $\langle\hat{w}\hat{\phi}/r^2\rangle/\langle\hat{w}\hat{\phi}\rangle=1/\bar{r}^2(\hat{r})$ with $a\leqq\bar{r}\leqq b$. Since $b^{-2}\leqq\hat{r}^{-2}\leqq a^{-2}$, there exists $\hat{r}\in[a,b]$ for which $\hat{r}=\bar{r}$. For this $\hat{r}$, (VIII.2a) holds and $\lambda=\lambda$.

The eigenvalue problem (VIII.3) is too hard to solve exactly though it is amenable to numerical analysis. It is convenient in summarizing the results to use the parameters $Y=\dfrac{b^2\,\Omega_1}{\nu}$, $X=\dfrac{b^2\,\Omega_2}{\nu}$ and to regard X as preassigned. The results of the numerical calculations are given in Figs. 1–6. The true values of $\tilde{\lambda}$, computed numerically, are given to good accuracy by (VIII.2) with $\bar{r}$ taken near the mean radius $\bar{r}/b=(1+\eta)/2$ when $\Omega_2/\Omega_1>0$ (cf. Figs. 3, 4, 6).

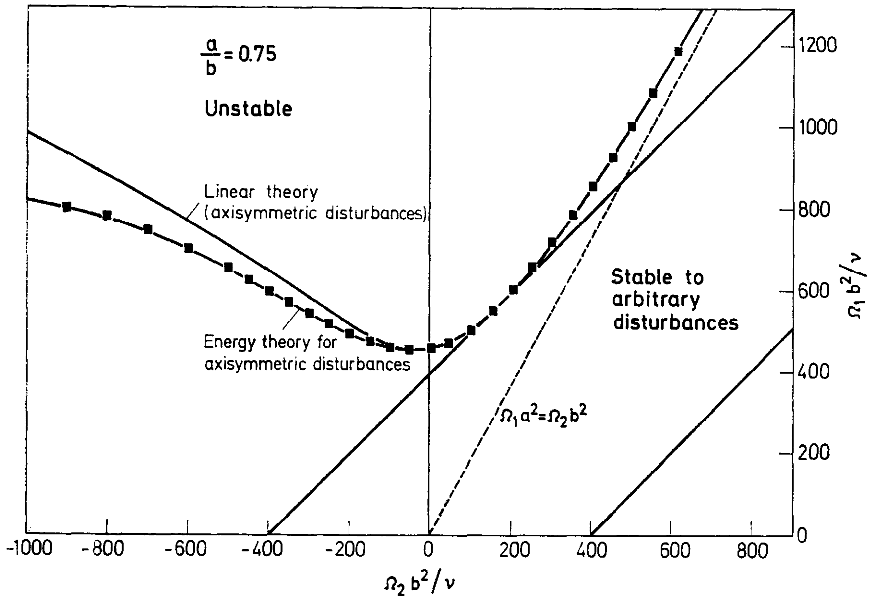

Fig. 1. Stability regions for Couette flow between rotating cylinders

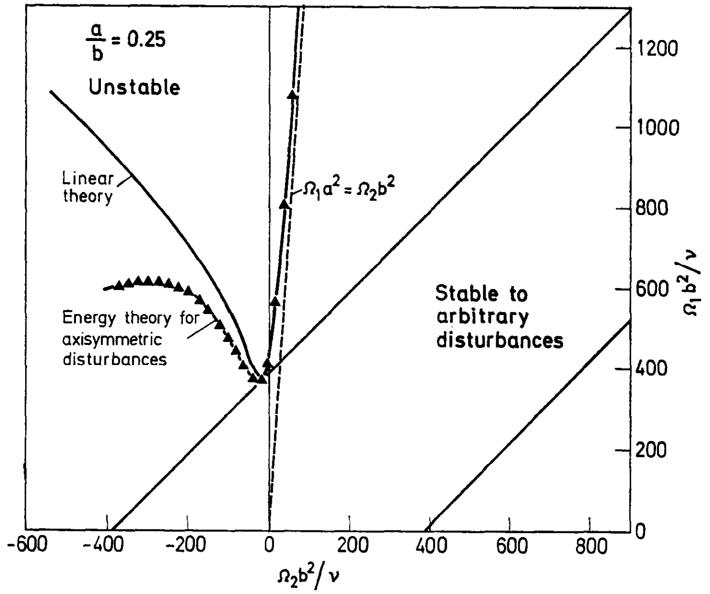

Fig. 2. Stability regions for Couette flow between rotating cylinders

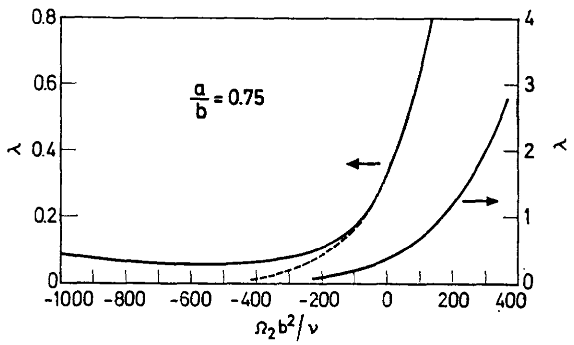

Fig. 3. The optimizing value $\lambda = \tilde{\lambda}$ for Eq. (VIII.2a). The dashed curve is the graph of Eq. (VIII.2) with $\bar{r} = 0.869$

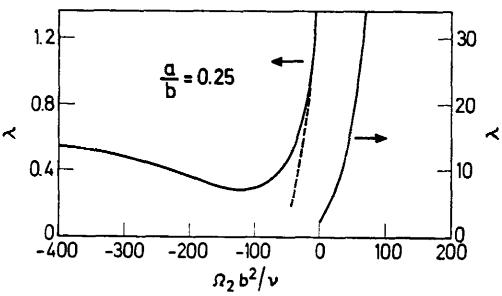

Fig. 4. The optimizing value $\lambda = \tilde{\lambda}$ for Eq. (VIII.2a). The dashed curve is the graph of Eq. (VIII.2b) with $\bar{r} = 0.574$

IX. Comparison of Linear and Energy Limits with Each Other and with Experiments

A most interesting aspect of the results of this section follows from comparing the linear and energy stability boundaries. TAYLOR's original calculation of the linear stability limit and subsequent ones assume exchange of stability as well as axial symmetry. The linear limit is then found as, say, the principal eigenvalue ν_L of

$$2\left\{A+\frac{B}{r^2}\right\}\frac{\phi}{\sqrt{\lambda}}+\nu_L\left(\nabla_2^2-\frac{1}{r^2}\right)w=\partial_r p, \tag{IX.1a}$$

$$\nu_L \nabla_2^2 u=\partial_x p, \tag{IX.1b}$$

$$-2A\sqrt{\lambda}\,w+\nu_L\left(\nabla_2^2-\frac{1}{r^2}\right)\phi=0, \tag{IX.1c}$$

subject to (VIII.3d, e).

From these equations it is easy to prove that within the mean-radius approximation*

$$\nu_L=\tilde{\nu}_{\mathscr{E}}.$$

Proof. In the mean radius approximation, we replace r with its arithmetic mean in the term

$$\frac{B}{r^2}=\frac{4B}{(a+b)^2}$$

but not elsewhere. Then each of the systems (VIII.3) and (IX.1) can be combined into a sixth-order problem of the form

$$-F^2\,\partial_{xx}^2 w+\nu^2\left(\nabla_2^2-\frac{1}{r^2}\right)^3 w=0 \tag{IX.2a}$$

where

$$w=\partial_r w=\left(\nabla_2^2-\frac{1}{r^2}\right)^2 w=0\Bigg|_{a,b}. \tag{IX.2b}$$

For the energy problem,

$$F^2=F_{\mathscr{E}}^2=\left(\frac{4B}{(a+b)^2\sqrt{\lambda}}-\frac{(\lambda-1)A}{\sqrt{\lambda}}\right)^2 \quad \text{and} \quad \nu=\nu_{\mathscr{E}}. \tag{IX.3a}$$

For the linear problem,

$$F^2=-A\left(A+\frac{4B}{(a+b)^2}\right) \quad \text{and} \quad \nu=\nu_L. \tag{IX.3b}$$

Denote the smallest eigenvalue of (IX.2) by

$$\frac{F^2}{\nu^2}=\underline{\Lambda}^2.$$

* The application to the linear stability problem (IX. 1) of the mean-radius approximation in the narrow-gap limit under conditions (VIII. 2) has been thoroughly discussed by CHANDRASEKHAR [pp. 299–315]. This approximation is good when $|\Omega_2/\Omega_1-1|$ is small. He finds that the errors introduced by the approximation do not exceed one per cent.

We consider the energy problem (IX.3a) and search for the optimum value $\lambda=\tilde{\lambda}$

$$\left(\frac{1}{\tilde{\nu}_{\mathscr{E}}}\right)^2=\underline{A}^2\max_{\lambda>0}F_{\mathscr{E}}^{-2}=\left(\frac{1}{\nu_{\mathscr{E}}(\tilde{\lambda})}\right)^2.$$

To find this value, set $\partial F_{\mathscr{E}}/\partial\lambda=0$. This leads us again to the relation (VIII.2)

$$\tilde{\lambda}=-\left\{\frac{4B}{(a+b)^2}+A\right\}\Big/A>0$$

which when inserted back into $F_{\mathscr{E}}$, gives

$$\left(\frac{1}{\tilde{\nu}_{\mathscr{E}}}\right)^2=\frac{A^2}{4A\left(A+\dfrac{4B}{(a+b)^2}\right)}=\left(\frac{1}{\nu_L}\right)^2, \tag{IX.4}$$

as asserted.

The agreement which (IX.4) asserts when the mean radius approximation is valid fails when Ω_2/Ω_1 is too negative (cf. CHANDRASEKHAR pp. 299–315). But (IX.4) already exhibits the main features of the numerical results. These numerical results are given graphically in Figs. 1, 2, 3, 4, 5, and 6. The energy theory result was calculated by HUNG [1971] and the linear theory result by SPARROW, MUNRO & JONSSON [1964]. The agreement between these two results is striking when $A<0$ and $\Omega_2/\Omega_1>0$. The agreement is good when Ω_2/Ω_1 is not too negative.

It is known through the work of BUSSE [1970] that when $(\Omega_2-\Omega_1)/(\Omega_2+\Omega_1)=-4(b-a)/(b+a)\to 0$ the criterion associated with (IX.4) is exact, and that it also applies to arbitrary nonaxisymmetric disturbances. In this limit the energy and linear theory give the same critical number. The linear theory number gives a limit for instability and the energy number is a limit for stability. Hence, the common limit solves the stability problem; it gives a necessary and sufficient condition for stability. The same is nearly true, as we see from (VII.4) and the numerical results, when $A<0$ and Ω_2/Ω_1 is not too negative. In this last case there is, however, an important proviso: the energy result is known to apply only to axisymmetric disturbance whose energy is not greater than some definite value.

In Figs. 1, 2, 5, we compare the present result with SERRIN'S. The energy $\mathscr{E}=\frac{1}{2}\langle u^2+v^2+w^2\rangle$ of every disturbance of Couette flow must decay monotonically when (Ω_1,Ω_2) are in SERRIN'S band. Outside this band one can find an axisymmetric disturbance whose energy increases initially. Our result guarantees monotonic decay of the energy $\frac{1}{2}\langle u^2+\lambda v^2+w^2\rangle$ to axisymmetric disturbances of restricted size.

The relation of the present criterion to experiments can be ascertained from Fig. 5. There we give the experimental results of D. COLES [1965], the linear (instability) limit and the energy (stability) limit. Both the linear and energy results are calculated only for axisymmetric disturbances.

Linear theory and experiments are apparently in disagreement when $\Omega_2/\Omega_1\ll 0$. There, the observed instability is evidently subcritical (under the critical number of linear theory). The analysis of KRUEGER, GROSS & DI PRIMA [1964] and of DI PRIMA & GRANNIK [1969] suggest that nonaxisymmetric disturbances are important when $\Omega_2/\Omega_1<-0.78$.

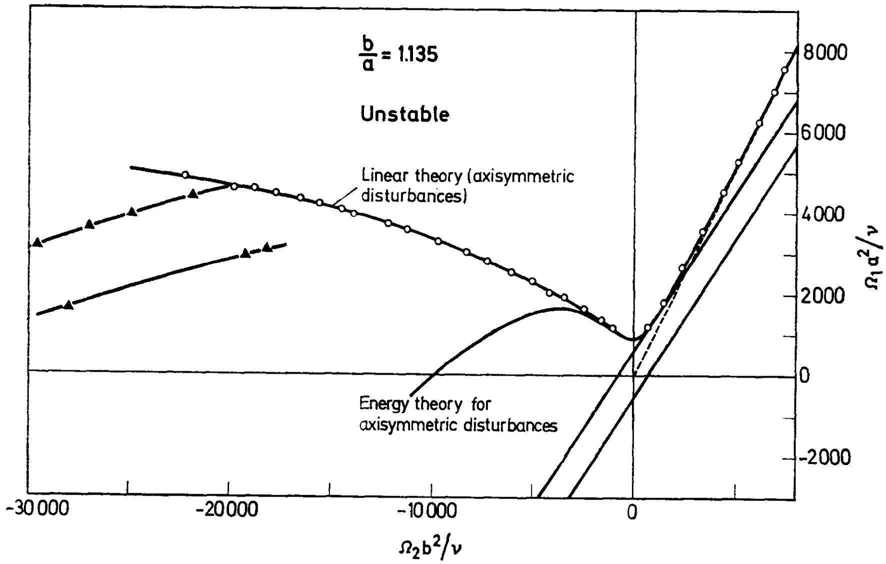

Fig. 5. Stability regions for Couette flow between rotating cylinders. The circles and triangles are observed points of instability in the experiments of D. COLES [1965]

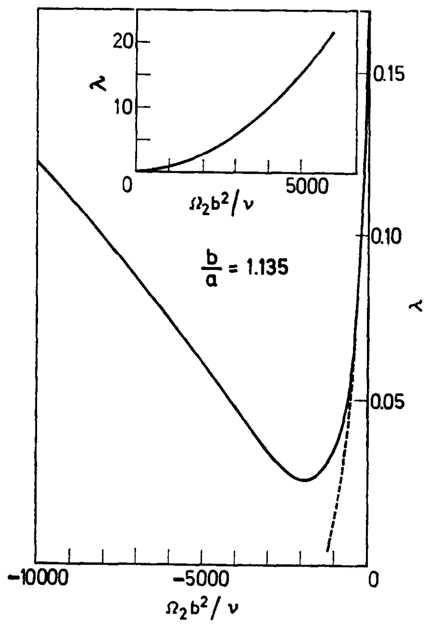

Fig. 6. The optimizing value $\lambda = \tilde{\lambda}$ for Eq. (VIII.2a). The dashed curve is the graph of Eq. (VIII.2b) with $\bar{r} = 0.9405$

A portion of the work in Sections VIII and IX comprise part of the doctoral thesis of W. HUNG. This work was partially supported under the NSF grant GK 1838.

Appendix: Proof of the Estimate (VII. 9)

Since $u(a, x)=u(b, x)=0$, we have

$$u^2(r,x)=2\int_a^r u(\xi,x)\frac{\partial u}{\partial \xi}(\xi,x)\,d\xi=-2\int_r^b u(\xi,x)\frac{\partial u}{\partial \xi}(\xi,x)\,d\xi,$$

and

$$u^2(r,x)\leqq \int_a^b |u(\xi,x)|\left|\frac{\partial u}{\partial \xi}(\xi,x)\right|d\xi.$$

In the same way, since $u(r, x_1)=u(r, x_2)$ where $x_2=x_1+2\pi/\alpha$, we find

$$u^2(r,x)\leqq u^2(r,x_1)+\int_{x_1}^{x_2}|u(r,\eta)|\left|\frac{\partial u}{\partial \eta}(r,\eta)\right|d\eta.$$

Thus,

$$\int_{x_1}^{x_2}\int_a^b u^4(r,x)\,r\,dr\,dx\leqq \int_{x_1}^{x_2}\int_a^b\left[u^2(r,x_1)\int_a^b|u(\xi,x)|\left|\frac{\partial u}{\partial \xi}(\xi,x)\right|d\xi\right]dx\,r\,dr$$
$$+\int_{x_1}^{x_2}\int_a^b\left[\int_a^b|u(\xi,x)|\left|\frac{\partial u}{\partial \xi}(\xi,x)\right|d\xi\cdot\int_{x_1}^{x_2}|u(r,\eta)|\left|\frac{\partial u}{\partial \eta}(r,\eta)\right|d\eta\right]dx\,r\,dr,$$

and after noting that $r\geqq a$ and using Schwarz's inequality, this is

$$\leqq\frac{1}{a}\langle u^2\rangle^{\frac12}\left\langle\left|\frac{\partial u}{\partial r}\right|^2\right\rangle^{\frac12}\int_a^b u^2(r,x_1)\,r\,dr+\frac{1}{a}\langle u^2\rangle\left\langle\left|\frac{\partial u}{\partial r}\right|^2\right\rangle^{\frac12}\left\langle\left|\frac{\partial u}{\partial x}\right|^2\right\rangle^{\frac12}.$$

Here the angle brackets denote the expression

$$\int_{x_1}^{x_1+2\pi/\alpha}\int_a^b f(x,r)\,r\,dr\,dx$$

where f is periodic in x. This integral is independent of x_1. Hence, we may integrate the whole inequality from $x_1=0$ to $x_1=2\pi/\alpha$ to find

$$\frac{a\langle u^4\rangle}{\langle u^2\rangle}\leqq\frac{\alpha}{2\pi}\langle u^2\rangle^{\frac12}\left\langle\left|\frac{\partial u}{\partial r}\right|^2\right\rangle^{\frac12}+\left\langle\left|\frac{\partial u}{\partial r}\right|^2\right\rangle^{\frac12}\left\langle\left|\frac{\partial u}{\partial x}\right|^2\right\rangle^{\frac12}.$$

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