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D. D. Joseph

On the Place of Energy Methods in a Global Theory of Hydrodynamic Stability

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By

D. D. Joseph

Imperial College of Science and Technology, London, U.K.

The point of departure for the global theory to be described is the system of the nonlinear Boussinesq equations (1, 2) governing the disturbance of some given motion. For simplicity, let $(\boldsymbol{U}, T, \Gamma)$ be a basic steady velocity, temperature and concentration (say, salt) field which satisfies the Boussinesq equations in a bounded domain (or in a period cell) and such that these fields take on prescribed values on $\partial\mathscr{V}$ (or periodic values on appropriate parts of $\partial\mathscr{V}$). Let $(\boldsymbol{u}, \theta, \gamma)$ be disturbances of the given motion which are induced at time zero as initial conditions. Subsequently [30]

$$(\boldsymbol{u}, \theta, \gamma)/\partial\mathscr{V} = 0, \qquad \operatorname{div} \boldsymbol{u} = 0/\mathscr{V}, \tag{1a, b}$$

$$\frac{d\boldsymbol{u}}{dt} + R(\boldsymbol{u} \cdot \nabla)\, \boldsymbol{U} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = -\nabla P - (\mathscr{R}\theta - \mathscr{C}\gamma)\, \boldsymbol{\eta} + \Delta \boldsymbol{u}, \tag{2a}$$

$$\mathscr{P}_T \frac{d\theta}{dt} + \mathscr{P}_T\, \boldsymbol{u} \cdot \nabla\theta + \mathscr{R}\boldsymbol{u} \cdot \boldsymbol{\eta}_T = \Delta\theta, \tag{2b}$$

and

$$\mathscr{P}_\Gamma \frac{d\gamma}{dt} + \mathscr{P}_\Gamma\, \boldsymbol{u} \cdot \nabla\gamma + \mathscr{C}\boldsymbol{u} \cdot \boldsymbol{\eta}_\Gamma = \Delta\gamma, \tag{2c}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + R\boldsymbol{U} \cdot \nabla.$$

Here, the vector fields $\boldsymbol{\eta}$, $\boldsymbol{\eta}_T$, $\boldsymbol{\eta}_\Gamma$ are respectively, the normalized gravity, temperature and salt gradient vectors for the basic motion. $\mathscr{P}_T$ and $\mathscr{P}_\Gamma$ are Prandtl numbers for heat and salt and R, $\mathscr{R}$ and $\mathscr{C}$ are the Reynolds number and square root of the Rayleigh numbers for heat and salt, respectively. Instead of three stability parameters R, $\mathscr{R}$ and $\mathscr{C}$ we work with one polar parameter ϱ introduced by the transformation

$$R = \varrho \cos a, \qquad \mathscr{R} = \varrho \sin a \sin b, \qquad \mathscr{C} = \varrho \sin a \cos b .$$

The polar angles a and b are preassigned and we solve a stability problem on a fixed ray from the origin.

The stability of the basic motion (U, T, Γ) is judged by the behaviour of the disturbance "energy"

$$E(t, \lambda_T, \lambda_\Gamma) = \tfrac{1}{2} \langle |\boldsymbol{u}|^2 + \mathscr{P}_T \lambda_T \theta^2 + \lambda_\Gamma \mathscr{P}_\Gamma \gamma^2 \rangle,$$

where λ_T and λ_Γ are any preassigned positive numbers and the angle bracket denotes integration over $\mathscr{V}$. A *stable motion* we define as one for which $E(t)/E(0) \to 0$ as $t \to \infty$ (asymptotic stability in the mean). An *unstable motion* is not stable. A motion is *globally stable*[1] if it is stable for all bounded initial energies $E(0) < \infty$. A value ϱ_G is a *global stability limit* if we have global stability for all $\varrho < \varrho_G$. A motion is *linearly stable* if stable for sufficiently small $E(0)$. A value ϱ_L is a *linear stability limit* if we have linear stability for all $\varrho < \varrho_L$. A *sublinearly unstable motion* is unstable but linearly stable, that is, unstable for $\varrho_L > \varrho \geq \varrho_G$.

Since sublinear instabilities are commonplace in fluid mechanics we cannot expect the stability criterion of linear theory to have predictive value. But a motion which is unstable in the linearized theory is unstable so that the linear theory of *instability* has a place in a global theory, that is, we have instability if $\varrho \geq \varrho_L$.

For sufficient conditions for stability we use the energy method. The method starts from the evolution equation

$$\frac{dE}{dt} = \varrho I - D, \tag{3}$$

where

$$D = \langle |\nabla \boldsymbol{u}|^2 + \lambda_T |\nabla \theta|^2 + \lambda_\Gamma |\nabla \gamma|^2 \rangle$$

is the (stabilizing) dissipation and

$$I = TM + TH + TS$$

is the (destabilizing) energy source defined by

$$\begin{Bmatrix} TM \\ TH \\ TS \end{Bmatrix} = - \left\langle \begin{Bmatrix} \boldsymbol{u} \cdot \nabla U \cdot \boldsymbol{u} \cos a \\ (\boldsymbol{\eta} + \lambda_T \boldsymbol{\eta}_T) \cdot \boldsymbol{u} \theta \sin a \sin b \\ (-\boldsymbol{\eta} + \lambda_\Gamma \boldsymbol{\eta}_\Gamma) \cdot \boldsymbol{u} \gamma \sin a \cos b \end{Bmatrix} \right\rangle .$$

Equation (3) is mere restatement of the nonlinear problem (1, 2) and its general solution is beyond our ability. But if we consider (3) for a class of smooth functions which satisfy (1) and contain solutions of (1, 2) as a subset then on this class of kinematically admissible func-

[1] Globally stable motions are necessarily unique. For nonunique solutions one needs a more restricted stability concept. For example, in the global theory the long rod broadside down is *unstable* because there is a perturbation which stands the rod end up.

tions (called $\mathscr{H}$) one can establish that there exists positive numbers ϱ_E and ξ^2 such that

$$\frac{I}{D} \leq \frac{1}{\varrho_E(\lambda_T, \lambda_\Gamma)} = \max_{\mathscr{H}} \frac{I}{D}, \qquad \frac{E}{D} \leq \frac{1}{\xi^2} = \max_{\mathscr{H}} \frac{E}{D}. \qquad (4, 5)$$

If $\varrho < \varrho_E$, then

$$\frac{D - \varrho I}{E} \geq \sigma = \min_{\mathscr{H}} \frac{D - \varrho I}{E} \geq \xi^2\left(1 - \frac{\varrho}{\varrho_E}\right), \qquad (6, 7)$$

and we can integrate the inequality which arises from (3) and (6) to find a central estimate of energy theory

$$E(t) \leq E(0)\exp(-\sigma t). \qquad (8)$$

The value of ξ^2 in (7) is not smaller than π^2 in a domain $\mathscr{V}$ which can be contained in a layer of height one [24, 28, 31, 34].

When $\varrho < \varrho_E$ *every* kinematically admissible disturbance will decay so that global stability with exponential decay is assured. When $\varrho > \varrho_E$, however, one can find an initial disturbance whose energy will increase initially. For example, the vector that wins the competition (4) makes E increase. This vector is called the most persistent disturbance. The motion is called *strongly stable* if every disturbance of it decays from the start. Motions which are stable but not strongly stable can have persistent disturbances which make E increase for a time. The energy limit defines a necessary and sufficient condition for strong stability [1].

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[1] Energy methods are generally associated with the work of REYNOLDS [26] and ORR [23]. REYNOLDS derived the evolution equation (for the Navier-Stokes system) and used it to form stability criteria from guessed approximations to hydrodynamic solutions. ORR was the first to see that the proper procedure for drawing deductions from Reynold's integral was a variational method leading to eigenvalue problem for the variational equations. ORR did not prove that his criterion leads to strong stability, cf. Eq. (8), nor did he realise, as THOMAS [32] later proved, that these integrals could be estimated to establish the existence of strong stability in bounded domains. Despite ORR's insight in establishing a correct mathematical procedure for deducing stability limits from Reynolds' integrals I am not aware of a single stability limit calculated correctly by him. He has given incorrect values for plane Couette flow [12], for Hagen-Poiseuille flow [16] and HARRISON, on the urging of ORR, gave the incorrect limit for the stability of Couette flow between cylinders.

The modern version of energy theory dates from the work of SERRIN [28] which brings together the ideas of THOMAS and HOPF [10] on the existence of stability and ORR's variational method. SERRIN showed how to obtain an explicit criterion for strong stability of Navier-Stokes solutions by solving a variational problem for (essentially) the decay constant ξ^2, cf. Eq. (8). The proof that ORR's criterion implies strong stability I gave jointly with SERRIN [12]. For the existence of stability and for the important problem of the rate of decay of stable disturbances estimates of the value of the decay constant ξ^2 is the central problem and has been treated by SERRIN [28], VELTE [34], PAYNE and WEINBERGER [24] and SORGER [31].

The relevance of these stability definitions can be made clear by application to the problem of stability of Poiseuille flow. Theoretical and experimental results for this problem are shown in Fig. 1. The bottom line represents results of energy analysis [16]. Actually only the points on the curve are stability limits and the values of N give the azimuthal periodicity of the most persistent disturbance. Below the energy line the Poiseuille flow is strongly stable. The line ϱ_G gives the trend of experimental results for transition in cylindrical annuli from nine different authors. The top curve gives the result of a numerical integration of the linearized disturbance equations but only for axially symmetric disturbances [21]. The variation of the linear limit with the radius ratio η seems to have nothing common with experiments. On the other hand experimental and energy limits are alike with regard to their variation with η. For pipe flow ($\eta = 0$) the first (spiral) mode azimuthal periodicity of the most persistent disturbance is in qualitative agreement with the experimental observations of Fox, LESSEN and BHAT [6]. The integers on the energy line give the variation of azimuthal periodicity with radius ratio $N(\eta)$ and these are offered as hypotheses for experiment.

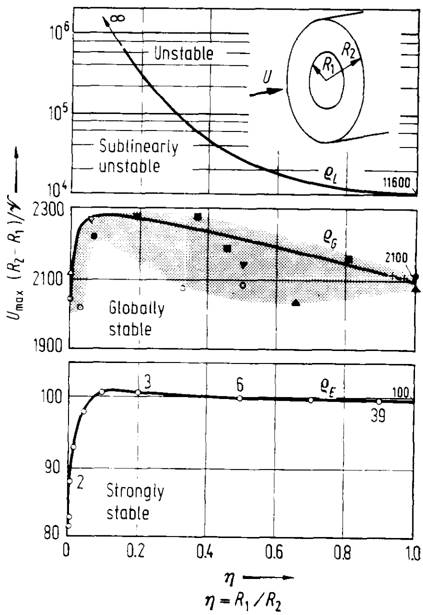

Fig. 1. Stability and instability limits for Poiseuille flow. The linear line ϱ_L is taken from a numerical calculation [21] for axisymmetric disturbance. The energy line ϱ_E [16] is also an outcome of numerical calculation, but over unrestricted periodic disturbances. The integers on ϱ_E indicate the azimuthal periodicity of the most unstable disturbance. The shaded band contains experimentally observed transition points compiled by R. HANKS [8] and the line called ϱ_G is an interpolation formula for the data. The line ϱ_G should roughly correspond to the as yet theoretically unknown global stability limit.

It is relevant to our considerations, but I cannot develop the idea here, that the linear limit collapses onto the energy line if the pipe is made to rotate ever faster about its symmetry axis [16, 25, 38].

For purposes of discussion let us assume that ϱ_G is the global stability limit. We want to know if the distinction made between "globally stable" and "strongly stable" is physically relevant. The experiments of DAVIES and WHITE [4] and LINDGREN [19, 20], in particular, but also of CAROTHERS [2], GRINDLEY and GIBSON [7] and NAUMANN [22] all suggest the strongly affirmative answer: In the region of strong stability decay of even very large norm disturbances is so rapid that the disturbance cannot be convected far downstream before it has decayed away. About their channel flow experiments (see Fig. 2) DAVIES and WHITE stress that there is:

"... a distinct deviation from true viscous flow if initial disturbing factors are present, and the influence of such disturbing factors does not disappear entirely until a second well-defined limit is reached, which has a value of about one tenth of the lower critical number" (global stability limit).

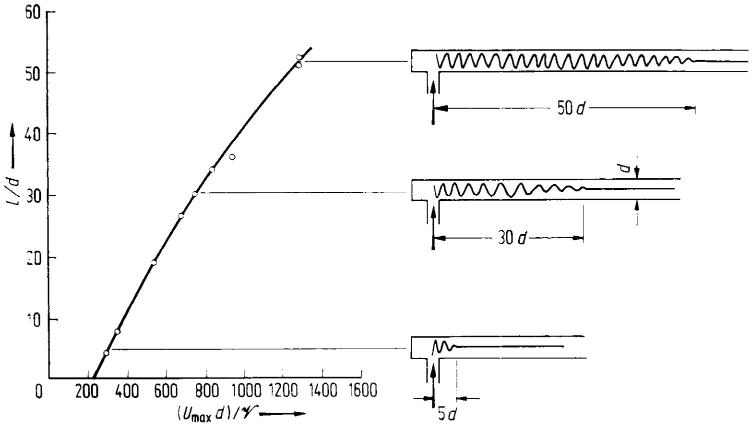

Fig. 2. Length of channel over which an entrance disturbance persists (after DAVIES and WHITE [4]). The experiment apparatus is a rectangular channel of large aspect ratio. The flow is disturbed at the entrance by turning a corner (as shown). The $R = 210$ intercept is an interpolation of the data and the true curve could be expected to pass through the origin. It is clear that the decay must become very rapid in the neighbourhood of $R = 200$. LINDGREN [19] says about this nearly identical result for round tubes that "... below a Reynolds number R of the order of 200 no disturbances were observed in the flow even quite near the tube inlet, however, strong disturbances were agitated in the entrance flow."

For the Boussinesq equations the problem of finding limits for stability is enriched by the presence of the numbers λ_T and λ_Γ. For fixed λ_T, λ_Γ the estimate (8) reduces the stability problem to a standard maximum problem (5) (this is equivalent to an eigenvalue problem for Euler's variational equations). Suppose (5) is solved. Then we choose λ_T and λ_Γ obtain the largest region of stability

$$\varrho_E = \max_{\lambda_T, \lambda_\Gamma} \varrho_E(\lambda_T, \lambda_\Gamma). \tag{9}$$

I call this the problem of the "optimum stability boundary". It can be resolved as an ordinary maximum problem. One finds that

$$\lambda_T \langle \boldsymbol{\eta}_T \cdot \boldsymbol{u}\theta \rangle = \langle \boldsymbol{\eta} \cdot \boldsymbol{u}\theta \rangle, \quad \lambda_\Gamma \langle \boldsymbol{\eta}_\Gamma \cdot \boldsymbol{u}\gamma \rangle = -\langle \boldsymbol{\eta} \cdot \boldsymbol{u}\gamma \rangle. \tag{10a, b}$$

The problem of the optimum stability boundary is relevant for Bénard convection [11, 12], convection in variable gravity spherical shells [15], convection with heat sources [14, 17], unsteady convection [29], thermohaline convection [30, 13], convection in porous materials [37], and for surface tension driven convection [3]. The heat source result is representative and is briefly reviewed below.

Consider the stability of the conduction solution in a homogeneous fluid layer $(a = b = \pi/2,\ \gamma = TM = TS = 0)$ heated from below ($\boldsymbol{\eta} = -\boldsymbol{k}$, $\boldsymbol{k}$ is a unit vector in the direction of increasing z) with constant heat sources of intensity ζ ($\boldsymbol{\eta}_T = -(1 + \zeta z)\,\boldsymbol{k}$). We consider periodic disturbances of the conduction solution and integrations are relative to the period rectangle. Also, for simplicity, we consider free strips on which

$$w = \boldsymbol{u} \cdot \boldsymbol{k} = \boldsymbol{k} \times \partial \boldsymbol{u}/\partial z = 0, \qquad z = \pm 1/2. \tag{11}$$

The results hold under much more general circumstances. For the energy limit we maximize $I/D = TH/D = \langle (1 + \lambda_T \zeta z)\, w\theta \rangle / D$ and find the best value $\lambda_T = \langle w\theta \rangle / \langle (1 + \zeta z)\, w\theta \rangle$. When $\zeta = 0$, $\lambda_T = 1$ and the maximum problem defines the classical linear problem for Bénard convection so that $\varrho_E = \varrho_L$ and no sublinear instabilities exist when $\zeta = 0$ [12]. For $\zeta \neq 0$ [14] there is a band of values which increases in size with the magnitude of ζ in which sublinear solutions are possible (Fig. 3). Sublinear steady convection (in hexagons) can be rigorously demonstrated to exist[1] when $\zeta \neq 0$ but the proof requires a sufficiently small norm E and source intensity ζ. The only available global result is that given in Fig. 3.

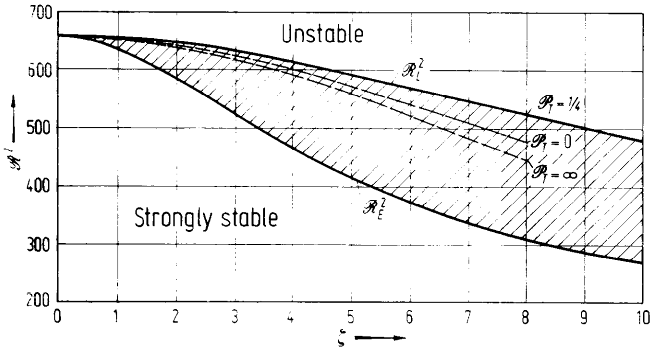

Fig. 3. Regions of stability and instability for a fluid layer with free surfaces heated from below and internally [17]. The principle of exchange of stability holds for the free surface problem even for $\varrho \neq 0$. Both the linear and energy limits are obtained from numerical integration. In the shaded region sublinear solutions of the Boussinesq equations cannot be excluded and at least one family is known to exist [5, 18] for sufficiently small norms and ζ, and is reported as observed [18].

[1] The proof is constructed using the method of Busse [1] and requires the use of a double series in parameters which Paul Fife and I [5] have proved converges. For the heat source problem the relevant calculation has been given, but incorrectly, by Krishnamurti [18] who also reports observing sublinear hexagons in experiments.

A similar situation holds for the salty fluid layer. Here we consider constant gradients of salt and temperature. Then, if the fluid is heated below ($\boldsymbol{\eta}_T = -\boldsymbol{k}$) and salty above ($\boldsymbol{\eta}_\Gamma = \boldsymbol{k}$) we find from (10) that $\lambda_T = \lambda_\Gamma = 1$ and again it is possible to completely exclude sublinear instabilities. But if the fluid is destabilized by heating and stabilized by salt[1] ($\boldsymbol{\eta}_\Gamma = \boldsymbol{\eta}_T = -\boldsymbol{k}$) then the linear limit can occur as either steady or oscillating convection [27, 35] and sublinear instabilities are possible, depending on the values $\mathscr{P}_\Gamma$ and $\mathscr{P}_T$. If one consults (10) one finds an acceptable value $\lambda_T = 1$ but also the unacceptable value $\lambda_\Gamma = -1$. It is not hard to show [30] that for this problem the maximum for (5) is taken on for $\gamma \neq 0$ (but $\theta \not\equiv 0$, $\boldsymbol{u} \neq 0$) so that (10b) holds but does not give the best value for λ_Γ. The criterion $\mathscr{R}^2 < 27\pi^4/4$ which guarantees linear stability when $\mathscr{C} = 0$ also holds globally when the salt field is *stabilizing* and $\mathscr{C} \neq 0$. This seems obvious enough from physical point of view but it is not at all obvious that any criterion better than $\mathscr{R}^2 \leq 27\pi^4/4$ will suffice for *strong stability*. There is, in fact, no better criterion which is independent of $\mathscr{P}_T$ and $\mathscr{P}_\Gamma$ because it is easy to show [30] that if $\mathscr{P}_T$ is fixed and $\mathscr{P}_\Gamma \to \infty$ then $27\pi^4/4$ is attained as the *linear limit*. In this way one concludes that $\mathscr{R}^2 \leq 27\pi^4/4$ is a necessary as well as sufficient condition for the stability of the conduction-diffusion solution when $\mathscr{P}_\Gamma \to \infty$.

It is clear then that if we are to have a global stability criterion which is better than $\mathscr{R}^2 < 27\pi^4/4$ it cannot be independent of $\mathscr{P}_\Gamma$ and $\mathscr{P}_T$. It is possible to have such a criterion in the class steady solutions of (1, 2). The criterion which we find below[2] does depend on $\mathscr{P}_T$ and $\mathscr{P}_\Gamma$ but not, as in (1, 2) on their separate values, but only on their ratio $\mathscr{P}_\Gamma/\mathscr{P}_T = \tau$.

Consider the steady disturbance equation (1, 2) for the constant gradient conduction-diffusion solution. Every solution of this problem is such that

$$\frac{1}{\varrho} = \frac{I}{D} \leq \max_{\mathscr{h}} \frac{I}{D}, \tag{12}$$

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[1] This configuration models a "solar Pond". The "Pond" is a contained fluid layer which is both heated and salted below, so that the upper fluid layers thermally insulate the lower. Like the Dead Sea, the pond is washed by fresh water at its free surface and salted at its bottom, ensuring the existence of a stabilizing salt gradient in the vertical. The dark bottom of the pond is an effective absorber of radiative energy of the sun, which has the effect of heating the pond from below. Without the stable salt gradient, the limit of heating that could be achieved in this way is determined by the stability condition for the onset of convective motions [30].

[2] The central equation (13) which makes the improved criterion possible is valid for any bounded domain provided only that $\boldsymbol{u} \cdot \boldsymbol{N} = 0/\partial\mathscr{V}$ and div $\boldsymbol{u} = 0$. The theory is given here only for zero boundary values (but see [13]).

where we have put $\lambda_T = \lambda_\Gamma = 1$. Moreover, for each such solution since div $\boldsymbol{u} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{k} = 0$, $z = \pm 1/2$, we have

$$0 = \langle \nabla \cdot (\boldsymbol{u}\theta\gamma) \rangle = \langle \theta (\boldsymbol{u} \cdot \nabla) \gamma + \gamma (\boldsymbol{u} \cdot \nabla) \theta \rangle. \tag{13}$$

Now, using (2b, c) we can write (13) as

$$\mathscr{I}(\varrho) = (1 + \tau) \langle \nabla\theta \cdot \nabla\gamma \rangle - \varrho \cos b \langle \theta w \rangle - \tau\varrho \sin b \langle \gamma w \rangle = 0. \tag{14}$$

If we require that functions competing for the maximum (12) in $\mathscr{k}$ also satisfy (14) then if $\cos b \neq 0$ (14) will not allow the previous unconstrained maximilizing function for which $\gamma \equiv 0$. It is also clear from (12) that values ϱ such that

$$\frac{1}{\varrho} > \max_{\mathscr{k}'} \frac{I}{D}$$

could not be associated with steady solutions of (1, 2).

Theorem: *The steady-conduction-diffusion solution, that is, the null solution of* (1, 2) *is unique provided only that*

$$\frac{1}{\varrho} > \max_{\mathscr{k}} \frac{I}{D} \equiv \frac{1}{\hat{\varrho}}, \tag{15}$$

where $\mathscr{k}'$ is the class kinematically admissible functions which also satisfy the side constraint $\mathscr{I}(\varrho) = 0$.

To find $1/\varrho$ we use variational calculus and require that $-\delta\{(I + 2\mu\mathscr{I})/D\} = 0$ where μ is a Lagrange multiplier. For the layer heated and salty below one finds that

$$\Delta \boldsymbol{u} + \varrho\{(\theta + \mu\tau\gamma) \sin b + \mu\theta \cos b\} \boldsymbol{k} + \nabla p = 0, \tag{16a}$$

$$\Delta\theta + \varrho(\sin b + \mu \cos b)\, w + \mu(1 + \tau)\, \Delta\gamma = 0, \tag{16b}$$

$$\Delta\gamma + \tau\varrho\mu \sin b w + \mu(1 + \tau)\, \Delta\theta = 0 \tag{16c}$$

for doubly periodic functions (in X, Y) such that div $\boldsymbol{u} = 0/\mathscr{V}$ and $\theta = \gamma = 0$, $z = \pm 1/2$. Equations (16b) and (16c) can be rearranged as

$$\Delta\theta + (\varrho\Phi w)/\chi = 0 \text{ and } \Delta(\Psi\theta + \Phi\gamma) = 0, \tag{17a, b}$$

where $\chi = 1 - \mu^2(1 + \tau)^2$, $\Phi = \mu \cos b + [1 - \mu^2\tau(1 + \tau)] \sin b$ and $\Psi = \mu \sin b + \mu^2 \cos b (1 + \tau)$. The boundary condition, (17b) and potential theory show that $\Psi\theta + \Phi\gamma = 0$ at each point in $\mathscr{V}$.

Let A^2 be the minimum eigenvalue for the problem $\Delta^3 w - A^2(\partial^2/\partial X^2 + \partial^2/\partial Y^2)\, w = 0$ for w doubly-periodic (in X, Y) satisfying rigid or free boundary conditions at $z = \pm 1/2$. By reducting the system (16a) and (17a, b) to this problem we show that

$$\left(\frac{A}{\varrho \sin b}\right)^2 \chi = (1 + 2\mu \cot^2 b + \mu^2 \cot^2 b + \mu^2\tau^2) - 2\mu^2\tau(1 + \tau)(1 + \mu \cot b). \tag{18}$$

To complete the problem we select the multiplier μ to satisfy the condition $\mathscr{I} = 0$. This is equivalent to the equation

$$\frac{-\cot b}{1+\cot^2 b} = \frac{\mu}{h(\mu^2, \tau)}, \tag{19}$$

where

$$h(\mu^2, \tau) = \mu^2 + \left(1 - \mu^2\tau\left(\frac{1}{2} + \tau\right)\right)^2 + \mu^4(1 + 2\tau^2 + 11\tau/4).$$

To find (19) set $\gamma = -\theta\Psi/\Phi$ in $\mathscr{I}$, then use (17a) to eliminate $\langle w\theta\rangle$ and rearrange.

The solution of the relatively simple algebraic problem (18, 19) is given in [13]. The results for free surfaces ($\Lambda^2 = 27\pi^4/4$) are these: when $\tau = 0$ the linear and constrained energy results coincide. There are therefore no steady sublinear solutions in either of the limits ($\mathscr{P}_\Gamma \to \infty$) or ($\mathscr{P}_T \to \infty$, $\mathscr{P}_\Gamma \to 0$). For other values of $\mathscr{P}_T$ and $\mathscr{P}_\Gamma$ a band of possible sublinear solutions cannot be excluded. For very large $\mathscr{C}$ the result guarantees uniqueness when $\mathscr{R}^4 < \frac{27}{4}\pi^4\mathscr{C}^2/\tau^2$. We have represented the result for the value $\tau = 100$ in Fig. 4. The estimates hold relative to periodic disturbances but are otherwise unrestricted. One might expect that the sublinear solutions, like the hexagons mentioned earlier and the sublinear turbulence in pipes, have a three-dimensional structure. The small scale sublinear turbulent motions observed by TURNER and STOMMEL [33] in their experiment on

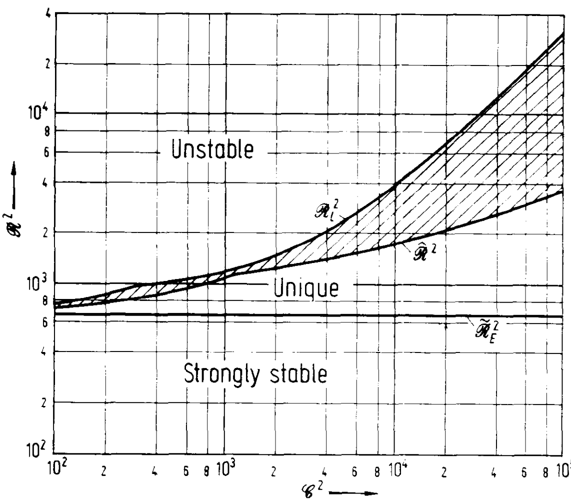

Fig. 4. Stability and uniqueness limits for the conduction-diffusion solution with a stabilizing solute ($\tau = \sqrt{10}$, $\mathscr{P}_T = 100$) gradient. For small $\mathscr{C}^2$ the linear limit $\mathscr{R}_L^2$ first occurs as exchange of stability and then, after the kink, as oscillations. The uniqueness limit $\hat{\mathscr{R}}^2$ also has a kink. The shaded region is open to steady sublinear solutions.

thermohaline convection in an open box are three-dimensional. On the other hand, two-dimensional, steady sublinear solutions have been calculated theoretically and by different (approximate) methods by SANI [27] and VERONIS [35, 36]. For these reasons it seems likely that sublinear solutions do exist and are perhaps very deep.

In closing, we note that the sense in which the uniqueness criterion of the theorem also implies stability has yet to be delineated. Moreover, though the result for $\tau = 0$ and $\tau = \infty$ cannot be improved it is likely that a better criterion can be found for intermediate values by an optimal selection of the coupling constants λ_T and λ_Γ.

Note added in proof: The problem mentioned in the last paragraph of this paper is solved in [39].

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