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Linear Instability of Asymmetric Flow in Channels

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A study of the linear stability of asymmetric channel flows is presented. Three one-parameter families of basic velocity which possess, respectively, no, one, and two inflection points are treated. The competing effects of stabilizing asymmetry and destabilizing vorticity distributions are discussed. An inviscid wave speed theorem which extends a result of Stuart to flows with two inflection points is proved.

I. INTRODUCTION

For nearly parallel motions the Orr–Sommerfeld system is viewed as an approximation to the linearized Navier–Stokes equation.[1] The eigenvalues and eigenfunctions of this equation are very sensitive to changes in the basic flow distribution. This distribution makes itself known in the mathematical system through the coefficients of the Orr–Sommerfeld equation. To understand the behavior of the Orr–Sommerfeld system it is necessary to study this equation in terms of its coefficients and our study is a preliminary step in this direction. Three families of profiles have been invented which approximate known laminar motions for particular values of the parameters but for other values they do not. A simple approach (but completely heuristic), using the well-known single turning point approximations[2,3] has been adopted for our work. This simple procedure seems adequate for flows with two bounding surfaces but may fail for unbounded regions (say, free convection boundary layers). A direct numerical scheme is employed to provide an independent check on the asymptotic solutions. The many details of the finite difference scheme as well as a complete exposition of the asymptotic methods are to be found in Ref. 4.

II. WAVE SPEED BOUNDS FOR THE RAYLEIGH EQUATION

We wish to know how the stability limit $R(c_r, \alpha)$ associated with the Orr–Sommerfeld problem

$$(D^2 - \alpha^2)^2\phi = -i\alpha \operatorname{Re} [(w - c) \cdot (D^2 - \alpha^2)\phi - (D^2w)\phi] \qquad (1)$$

and

$$\phi = D\phi = 0 \quad \text{at} \quad y = 0, 2$$

varies with changes in w (here $\alpha \geq 0$, $\operatorname{Re} \geq 0$ are preassigned and c_r is the eigenvalue). The symbols have conventional meanings.

Our first result is a simple extension of a result of Stuart[5] and gives eigenvalue (wave speed) bounds for the Rayleigh problem associated with (1); i.e., (1) with $\alpha\operatorname{Re} = \infty$.

Theorem: Given $w(y)$ such that:

(i) $w(y) > 0, 0 < y < 2$.
(ii) $w(0) = w(2) = 0$.
(iii) There is only one value y_0 such that $w'(y_0) = 0$.

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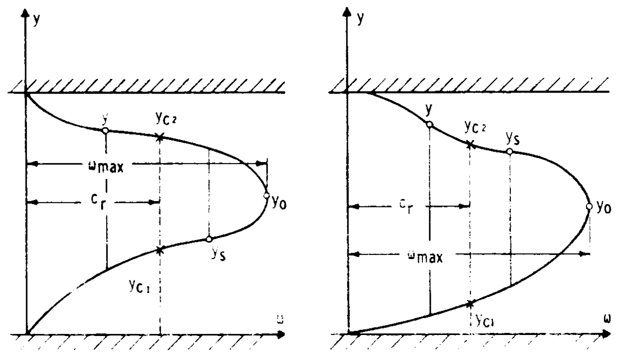

FIG. 1. Asymmetric velocity profiles with two inflection points in channels.

(iv) w'' changes sign at y_s and y_t, $y_s < y_t$, and not elsewhere in (0, 2); $w''(y_s) = w''(y_t) = 0$.

(v) A neutrally stable solution of Rayleigh's equation with eigenvalue c_r exits.

Then,

$$\min [w(y_s), w(y_t)] \leq c_r \leq \max [w(y_s), w(y_t)]. \quad (2)$$

Proof: If a neutrally stable solution exists, then $0 < c_r < w_{\max}$ where $w_{\max} = w(y_0)$. This follows from results of Lin[1].

There is one maximum $w(y_0)$ by assumption. Then, each value $w(y)$ in $[0, w(y_0)]$ is attained for two values of y. These are the values y_{c_1}, y_{c_2}, and $y_{c_1} < y_0 < y_{c_2}$ $[w(y_{c_1}) = w(y_{c_2})]$. From the work of Stuart,[5] it is known that a necessary condition for inviscid instability is

$$\tau = |\phi_{c_2}|^2 \frac{w''_{c_1}}{w'_{c_1}} = |\phi_{c_1}|^2 \frac{w''_{c_2}}{w'_{c_2}}, \quad (3)$$

where τ, ϕ, w', and w'' represent the jump of Reynold stress, eigenfunction, first and second derivatives of basic velocity profile, and the subscripts c_1 and c_2 denote the function evaluated at each critical point y_{c_1} and y_{c_2} (see Fig. 1), respectively.

By hypothesis (iv)

$$\frac{w'_{c_1}}{w''_{c_2}} \equiv \frac{w'(y_{c_1})}{w'(y_{c_2})} < 0, \quad (4)$$

and

$$\frac{w''_{c_1}}{w''_{c_2}} \equiv \frac{w''(y_{c_1})}{w''(y_{c_2})} < 0. \quad (5)$$

Now, we define a set of values c_r such that Eq. (5) holds.

At y_0 we have $w''(y_0) < 0$. Moreover, it is clear for values $w < w(y_0)$ but larger than the largest value at which w'' changes sign, i.e., $w^* = \max [w(y_s), w(y_t)]$. Then, for $w > w^*$, $w'' < 0$ and Eq. (6) cannot hold. Now observe that w'' changes sign twice as y ranges over (0, 2). It follows that w'' changes sign in the neighborhood of both walls; indeed, in the region $0 < w^{**} < \min [w(y_s), w(y_t)]$. It follows from this that the set of values c_r consistent with Eq. (6) is $w^{**} \leq c_r \leq w^*$. This proves the theorem.

The theorem can be reduced to known results for two special cases: (i) if the velocity profile is symmetric ($y_s = y_t$), then c_r is equal to $w(y_s)$ or $w(y_t)$ (Rayleigh). (ii) For asymmetrical profiles with only one inflection point, say, y_s, $c_r \leq w(y_s)$.[5]

In our calculations we have required that c_r be bounded as in Eq. (2) (see Table I).

We now turn to the full Orr–Sommerfeld problem [Eq. (1)].

III. PROFILES WITH ONE SIGNED CURVATURE

The n family of profiles given below varies continuously from Poiseuille flow to Couette flow and has no inflection points

$$w = \frac{1}{w_{\max}} \frac{-n^2 y(2 - y)}{n(n-1)y - Q_n}, \quad (6)$$

where

$$w_{\max} = \frac{-n^2 y_{\max}(2 - y_{\max})}{n(n-1)y_{\max} - Q_n},$$

$$y_{\max} = \frac{2[Q_n - (Q_n)^{1/2}]}{Q_n - 1},$$

$$Q_n = 1 + 2n(n-1).$$

TABLE I. The inviscid instability results for asymmetric flows (m-sequence velocity profiles with $y_{\inf} = 0.3$).[a]

m	α	c	y_s	y_t	w_s	w_t
1.0000	1.6555	0.38119	0.30000	1.7000	0.38119	0.38119
0.9500	1.4563	0.34120	0.30000	1.7895	0.37411	0.29801
0.9000	1.2558	0.29219	0.30000	1.8889	0.36795	0.17438
0.8500	1.0371	0.22996	0.30000	2.0000	0.36256	0.00000
0.8000	0.8063	0.15847	0.30000	2.1250	0.35783	0.24112
0.7500	0.6059	0.09779	0.30000	2.2082	0.35365	0.57297
0.7000	0.45691	0.05819	0.30000	2.4286	0.34995	1.03153
0.6000	0.25579	0.01875	0.30000	2.8333	0.34369	2.57926

[a] The subscripts s and t refer to the lower and upper inflection points, respectively. Starred values are outside the region of flow.

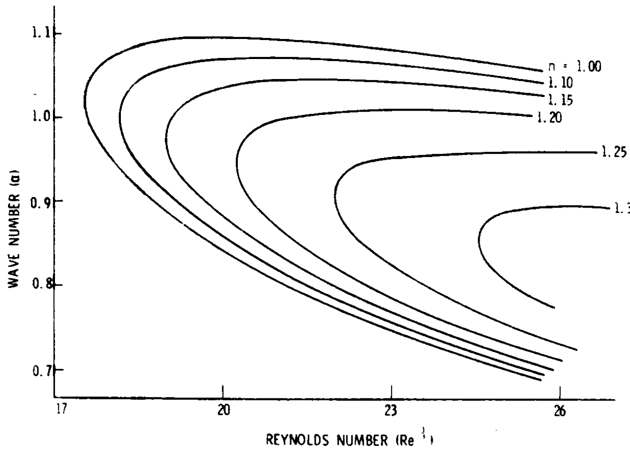

FIG. 2. Neutral stability curves for the n-sequence velocity profiles.

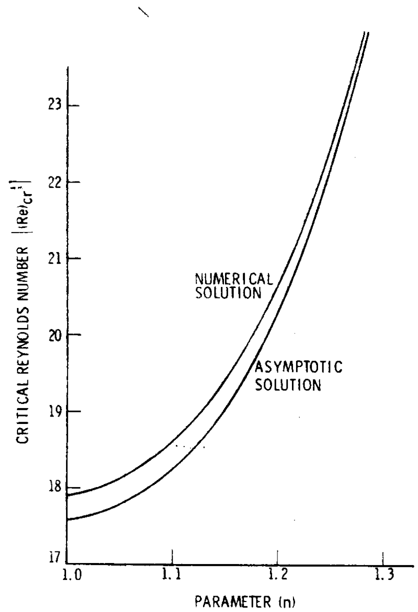

FIG. 3. Critical Reynolds numbers for n-sequence velocity profiles.

Therefore, we have the limiting cases:

(i) as $n \to 1$ then $y_{max} = 1$,

$$w_{max} = 1 \quad \text{and} \quad w = y(2 - y);$$

(ii) as $n \to \infty$ then $y_{max} \to 2$,

$$w_{max} = 2 \quad \text{and} \quad w \to y/2.$$

For case (i) we have Poiseuille flow and for case (ii) plane Couette flow. In the limit $n \to \infty$, the boundary condition $w(2) = 0$ is lost. This discontinuous limit, however, occurs at boundary points and not in the flow region. The n family of Orr–Sommerfeld equations valid in (0, 2) is analytic in y, uniformly in n. To examine the effect of asymmetry on the stability of the flow, the neutral stability curves for $N = 1.10$, 1.15, 1.20, 1.25, and 1.30 have been calculated by the asymptotic method. Figure 2 shows that as n increases, the neutral curves shift to the right and the value of $(\mathrm{Re})_{cr}$ shows a striking increase. It would appear that $(\mathrm{Re})_{cr} \to \infty$ as $n \to \infty$. In this limit the basic flow tends to plane Couette flow, or rather to a profile which coincides with plane Couette flow everywhere in (0, 2) but not at $y = 2$. In the theory of stability, plane Couette flow is absolutely stable $[(\mathrm{Re})_{cr} \to \infty]$. We stress that there is no inflection point in the n sequence and that only the asymmetry of the basic velocity affects the stability of the flow. It is suggested that the absolute stability which linear theory associates with plane Couette flow is not a singular result but is a limit result for profiles which are skewed toward one wall. A comparison of the asymptotic and numerical results is given in Fig. 3 where the variation of the critical Reynolds numbers with n is represented.

IV. ONE INFLECTION POINT

In order to examine the influence of a single inflection point in the basic flow on the stability of parallel flows, we first introduce a parameter γ^2 and obtain a cubic polynomial as the velocity profile in channels, which is

$$w(y) = -\frac{8(\gamma^2 + 3)^2(2\gamma^2 + 3)^2}{81(\gamma^2 + 3)^2} \cdot \left(\frac{\gamma^2}{3} y^3 - \frac{2(\gamma^4 - 3)}{(\gamma^2 - 2)} y^2 - \frac{4(2\gamma^2 + 3)}{3(\gamma^2 + 2)} y\right) \qquad (7)$$

channel, and L is the half-height of the channel.

The graph of Eq. (2) is asymmetric and may have one inflection point of the velocity profile for sufficiently large values of γ^2. The position of the inflection point ($w'' = 0$) can be shown to be

$$y_{inf} = 2(\gamma^4 - 3)/3\gamma^2(\gamma^2 + 2),$$

when $\gamma^2 \geq (3)^{1/2}$, then $y_{inf} \geq 0$; therefore, an inflection point exists inside the channel. When $\gamma^2 = 0$, Eq. (7) reduces to the usual parabolic velocity profile. Equation (7) gives the velocity profile of fluid flow with temperature-dependent viscosity, and γ^2 depends upon the temperature difference of the two walls and a thermal property of the fluid.

The stability results are represented in Fig. 4. The neutral stability curve, starting from $\gamma^2 = 0$, moves first to the right (the critical Reynolds number increases). However, the appearance of a point of inflection introduces certain changes in the neutral curve (notice the curve for $\gamma^2 = 1.6$), which begins

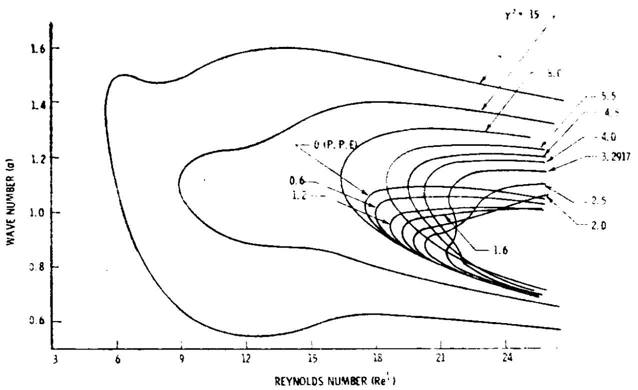

FIG. 4. Neutral stability curves for the γ^2 sequence.

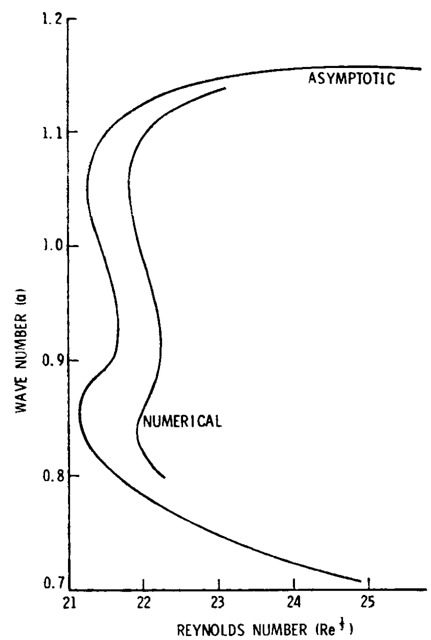

FIG. 5. Comparison of neutral curves ($\gamma^2 = 3.293$) calculated from turning point solutions and by numerical methods.

to distort. As γ^2 increases further, the stability curve distorts more and another local minimum of Re is formed on the upper branch. The new minimum, instead of moving to the right, moves to the left. For $\gamma^2 > 3.32$, the minimum on the upper branch represents the critical state and continuously decreases as γ^2 increases. The flow tends to be unstable at large wavenumbers. On the other hand, the old minimum of the lower branch is gradually smoothed out.

This interesting behavior of the neutral curve can be summarized as follows. For $0 \leq \gamma^2 \leq 1.6$, there is no noticeable effect of the inflection point and the neutral curve behaves as for the n sequence velocity profiles. The increasing asymmetry of velocity profiles with a one-sign curvature would seem to stabilize the flow. For $1.6 \leq \gamma^2 \leq 3.32$, however, even though the local minimum still denotes the critical state, the role of the inflection point seems significant. As $\gamma^2 > 3.32$, the inflection point of the basic flow completely dominates the stability of the flow. $(\mathrm{Re})_{cr}$ decreases monotonically as the inflection point moves farther inward. It would appear that competing effects of asymmetry and a local vorticity maximum are dominated by the latter which destabilizes the flow.

Figure 5 shows an example of neutral curve for $\gamma^2 = 3.293$ obtained by both numerical and asymptotic methods.

V. TWO INFLECTION POINTS

According to Rayleigh's theorem for a symmetrical velocity distribution, the condition of a vorticity maximum ($w'' = 0$ and $w''' < 0$) implies inviscid instability. For other profiles, this condition need not be sufficient and our calculations indicate that the condition is not sufficient. It is of interest to construct a velocity profile which has two inflection points, is then relocated, and the associated viscous and inviscid stability problems are examined.

Consider the following family of velocity profiles:

$$w = \frac{1}{w_{\max}}\,(d_1y^4 + d_2y^3 + d_3y^2 + d_4y), \qquad (8)$$

where

$$\begin{aligned}
w_{\max} &= d_1y_{\max}^4 + d_2y_{\max}^3 + d_3y_{\max}^2 + d_4y_{\max},\\
d_1 &= m,\\
d_2 &= 2(y_{\inf} - my_{\inf} - 2),\\
d_3 &= -6y_{\inf}(y_{\inf} - 2),\\
d_4 &= -8m - 8(y_{\inf} - my_{\inf} - 2) + 12y_{\inf}\\
&\quad\cdot(y_{\inf} - 2),
\end{aligned}$$

$y_{\max}$ is the point where the velocity is a maximum.

For $m = 1$, the velocity distribution is symmetrical with two inflection points at $y = y_{\inf}$. On the other hand, as $m \to 0$, one of the inflection points tends to infinity. The velocity distribution becomes a cubic polynomial and is essentially the same as the γ^2 sequence. However, in this case the parabolic velocity profile of plane Poiseuille flow can also be obtained by setting $y_{\inf}$ to infinity and $m = 1$. For $m \leq 0.85$, only one inflection point remains inside the channel.

First, let us consider the symmetric problem $m = 1$. We vary the inflection point $y_{\inf}$. If $y_{\inf}$ is inside the channel, inviscid instability is guaranteed

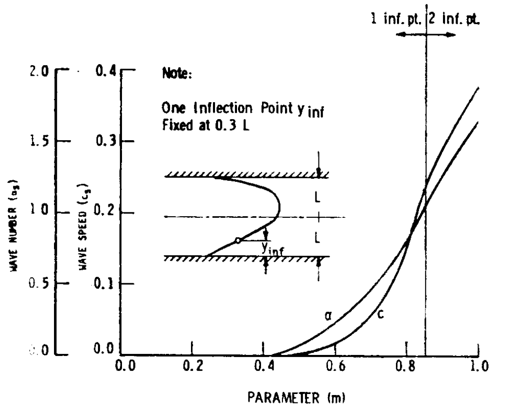

FIG. 6. The inviscid instability results for asymmetrical flows [m-sequence velocity profiles, Eq. (3) with y_{inf} = 0·3].

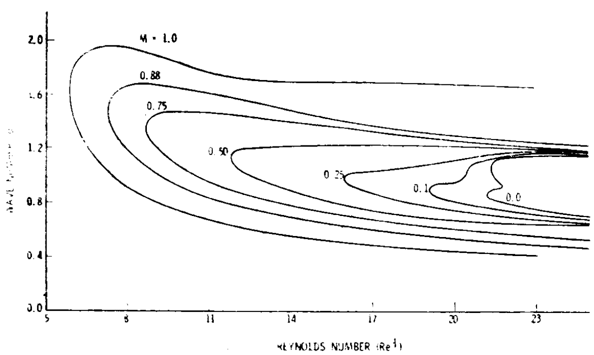

FIG. 7. The neutral stability curves for the m-sequence velocity profiles with one inflection point at 0.3.

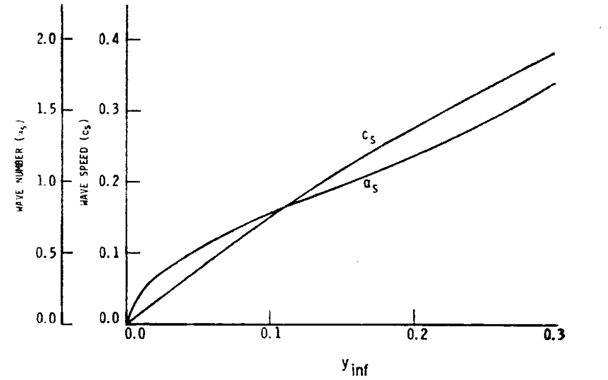

FIG. 8. Critical wave speed c_s as a function of the position of the inflection point for the neutral stability (inviscid case) of the m-sequence profiles with m = 1.

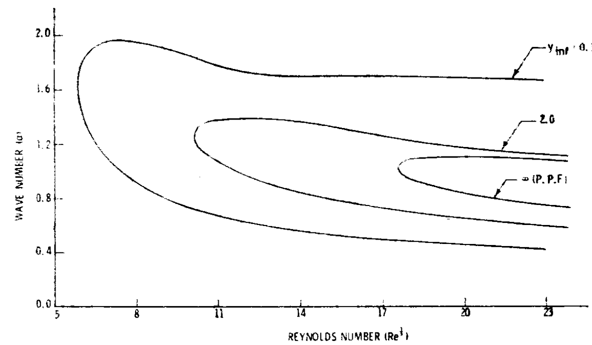

FIG. 9. Neutral curves for symmetric flows with two points of inflection [m-sequence Eq. (8) with m = 1].

and the eigenvalue c_s must take on the value of $w(y_{inf})$. Naturally, c_s must increase as y_{inf} moves toward the interior (Fig. 6). To what extent does inviscid instability make itself felt on the viscous problem (1)? Moving y_{inf} toward the channel center also shifts the nose of the neutral curve sharply toward lower Reynolds numbers (Fig. 7) and, therefore, induces instability.

We note that the results shown in Fig. 7 are obtained by turning point calculations and the results of Fig. 6 by numerical integration of Rayleigh's equation. But the inviscid and viscous results have nearly the same values of c_s and α_s for given y_{inf} as Re $\rightarrow \infty$ and evidently, as expected, inviscid results appears to be a limiting case of the viscous result.

Second, consider the asymmetric problem for increasing asymmetry (decreasing m). We fix y_{inf} = 3. The inviscid result is shown in Fig. 8. It develops that c_s and α_s decrease strongly with m. In fact $c_s = \alpha_s = 0$ when $m \leq 0.42$; i.e., the inviscid instability does not exist. Despite the fact that the flow satisfies Rayleigh's necessary condition for instability, it is inviscidly stable. These numerical results have been checked against the jump condition (3) and it can be seen from Table I that c_s lies between the velocity at the two inflection points in accord with Eq. (2).

The results for the viscous problem associated with the situation described in the above paragraph are shown in Fig. 9. The increasing asymmetry leads to sharply higher critical Reynolds numbers despite the presence of an inflection point.

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