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Global Stability of the Conduction-Diffusion Solution

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This paper continues and, to a degree, completes the working out of an energy-stability theory for the thermosolutal conduction-diffusion solution of the Boussinesq equations [1, 2, 3]. The Boussinesq equations allow a steady conduction-diffusion solution in which solute and temperature fields are constant in horizontal planes and vary linearly in the vertical. This motionless solution exists for all values of the relevant Rayleigh numbers but is stable and unique when and only when these numbers are small enough. One seeks a global stability limit for the motionless solution which holds relative to disturbances of any size. This global limit is provided by energy analyses. To establish the *instability* of the conduction-diffusion solution, one uses linear theory [4, 5, 6, 7]. One can have instability when the Rayleigh numbers are below the critical values of linear theory. Such nonlinear instability is called sublinear, and the delineation of regions of possible sublinear instability is an aim of this paper.

Our main interest is in the motionless fluid destabilized by heating (below) and stabilized by solute (salty below). When both heat and salt are destabilizing, then the energy limit is both necessary and sufficient for stability [1]. For the more difficult problem with competing effects, we are obliged to study the stability of the null solution of the disturbance equations [1, 2].

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\nabla p+(\mathscr{R}\theta-\mathscr{C}c)\boldsymbol{k}+\Delta\boldsymbol{u}\,, \tag{1}$$

$$Pr\left(\frac{\partial\theta}{\partial t}+\boldsymbol{u}\cdot\nabla\theta\right)-\mathscr{R}w=\Delta\theta \tag{2}$$

and

$$Sc\left(\frac{\partial c}{\partial t}+\boldsymbol{u}\cdot\nabla c\right)-\mathscr{C}w=\Delta c \tag{3}$$

where $\boldsymbol{k}$ is a vertical unit vector (against gravity) and $\mathscr{R}^2=\alpha_T g\beta_T l^4/\nu\kappa_T$, $\mathscr{C}^2=\alpha_c g\beta_c l^4/\nu\kappa_c$, $Pr=\nu/\kappa_T$ and $Sc=\nu/\kappa_c$ are, respectively, heat Rayleigh, solute Rayleigh, Prandtl number and Schmidt number. Here, κ_T and κ_c are, respectively, thermal and salt diffusivity coefficients and α_T and β_T are defined by the Boussinesq equation of state

$$\rho=\rho_0\big(1-\alpha_T(T-T_0)+\alpha_c(C-C_0)\big)$$

where ρ, T and C are, respectively, the density, temperature and concentration, and the subscript zero designates a reference state. The constants $\beta_T=|\nabla T|$ and $\beta_c=|\nabla C|$ are associated with the linear temperature and salt distribution of the conduction-diffusion solution. The letters $\boldsymbol{u}$, θ, c, p stand for disturbance velocity (solenoidal), temperature, concentration and pressure and $w=\boldsymbol{u}\cdot\boldsymbol{k}$.

The equations (1), (2) and (3) are to hold on the arbitrary domain $\mathscr{V}$. Let $\mathscr{S}$ be that part of the boundary $\partial\mathscr{V}$ of $\mathscr{V}$ on which $\boldsymbol{u}=0$. We define conditions of kinematic admissibility:

$$\mathscr{k}\colon \{\theta=\gamma=0|_{\partial\mathscr{V}}, \quad \boldsymbol{u}=0|_{\mathscr{S}}, \quad \operatorname{div}\boldsymbol{u}=0|_{\mathscr{V}}\}. \tag{4}$$

On the rest of the boundary $\partial\mathscr{V}-\mathscr{S}$, we let stress-free boundary conditions hold

$$\boldsymbol{u}\cdot\boldsymbol{N}=(\boldsymbol{N}\cdot\boldsymbol{d})\times\boldsymbol{N}=0, \tag{5}$$

where $\boldsymbol{d}$ is the rate of strain tensor for the disturbance, and $\boldsymbol{N}$, the outward normal on $\partial\mathscr{V}$. In fluid layers, we treat periodic disturbances. Then $\mathscr{V}$ is a period cell, and we append periodicity conditions to $\mathscr{k}$.

To obtain sufficient conditions for global stability, we turn to a study of the energy identities:

$$\frac{1}{2}\frac{d}{dt}\langle|\boldsymbol{u}|^2\rangle=-2\langle\boldsymbol{d}:\boldsymbol{d}\rangle+\mathscr{R}\langle\theta w\rangle-\mathscr{C}\langle c w\rangle, \tag{6}$$

$$\frac{Pr}{2}\frac{d}{dt}\langle\theta^2\rangle=-\langle|\nabla\theta|^2\rangle+\mathscr{R}\langle\theta w\rangle, \tag{7}$$

$$\frac{Sc}{2}\frac{d}{dt}\langle c^2\rangle=-\langle|\nabla c|^2\rangle+\mathscr{C}\langle c w\rangle \tag{8}$$

and

$$\frac{Sc}{1+\tau}\frac{d}{dt}\langle\theta c\rangle=-\langle\nabla\theta\cdot\nabla c\rangle+\frac{\mathscr{C}}{1+\tau}\langle\theta w\rangle+\frac{\mathscr{R}\tau}{1+\tau}\langle c w\rangle, \tag{9}$$

where $\tau=Sc/Pr$, and the angle bracket means integration over $\mathscr{V}$.

It is known [1] that if a criterion ($\mathscr{R}<\sqrt{Ra^*}$) holds, then (6), (7) and (8) imply that

$$E(t)<E(0)\exp\left\{-\xi^2\left(1-\frac{\mathscr{R}}{\sqrt{Ra^*}}\right)t\right\} \tag{10}$$

where

$$E(t)=\tfrac{1}{2}\langle|\boldsymbol{u}|^2+Pr\,\theta^2+Sc\,c^2\rangle,$$

$$\frac{1}{\sqrt{Ra^*}}=\max_{\mathscr{k}}\frac{2\langle w\theta\rangle}{\langle 2\boldsymbol{d}:\boldsymbol{d}+|\nabla\theta|^2\rangle}, \tag{11}$$

and ($a^2>0$)

$$\frac{1}{\xi^2}=a^2\max\frac{\langle|\boldsymbol{u}|^2+\theta^2\rangle}{\langle 2\boldsymbol{d}:\boldsymbol{d}+|\nabla\theta|^2\rangle}. \tag{12}$$

The stability-uniqueness criterion $\mathscr{R}<\sqrt{Ra^*}$ is independent of the parameters Pr, Sc and $\mathscr{C}$. It is shown in [1] that $\sqrt{Ra^*}$ is a *linear* limit which holds when $Sc\to\infty$ and $Pr\neq 0$. For these values, the energy and linear limits coincide, and the criterion $\mathscr{R}<\sqrt{Ra^*}$ is then necessary as well as sufficient for stability. It follows that there is no better stability criterion which is free from dependence on the parameters Pr and Sc and holds equally for all values of $E(0)$.

In [2], equation (9) is used to obtain an improved criterion for uniqueness of steady solutions. The criterion depends on Pr and Sc, not as in (1−5) on their separate values, but only through the ratio $\tau = Sc/Pr$. For steady flow, (9) is an orthogonality condition which defines a subspace in $\mathscr{H}$ in which one can solve the relevant maximum problem. The very considerable improvement in the uniqueness criterion which is implied by (9) is already evident from results given in [2], where it is proved, for example, that the nonexistence of sublinear steady solutions (which was established in [1] for $Sc \to \infty$) also holds for $Sc = \tau = 0$. But for intermediate values of τ, the results proved in [2] are not optimal, because the coupling constants λ_T and λ_c mentioned in the theorem proved in [2] are not optimally adjusted.

I have now found the way to choose these parameters optimally. As a result, we shall find the best possible criterion depending on a single parameter (τ) for stability and uniqueness.

Our energy results follow from analysis of the evolution equation

$$\frac{dE}{dt} + \frac{d\Psi}{dt} = -\langle |\nabla u|^2 + \nabla \phi^2 \rangle + \frac{\mathscr{R}}{\lambda_T}\left(1 + \lambda_T^2 - \frac{2\lambda_T \lambda_c \alpha}{1+\tau}\right)\langle w\phi \rangle, \tag{13}$$

which is formed as the linear combination,

$$(6) + \lambda_T^2(7) + \lambda_c^2(8) - 2\lambda_T\lambda_c(9),$$

of energy identities (6−9) with real coupling constants λ_T and λ_c. Here, $\mathscr{C} = \alpha\mathscr{R}$,

$$\Psi(t) = \frac{1}{2}\frac{Pr}{1+\tau}\langle \psi^2 \rangle,$$

$$E(t) = \frac{1}{2}\left\langle |u|^2 + \frac{Pr\,\tau}{1+\tau}\phi^2 \right\rangle,$$

$$\phi(x, t) = \lambda_T \theta(x, t) - \lambda_c c(x, t),$$

$$\psi(x, t) = \lambda_T \theta(x, t) - \tau \lambda_c c(x, t),$$

and we have set

$$\frac{1}{\lambda_T} + \lambda_T - \frac{2\alpha\lambda_c}{1+\tau} = \alpha\left(\frac{1}{\lambda_c} - \lambda_c\right) + \frac{2\lambda_T \tau}{1+\tau}. \tag{14}$$

The main result is

Theorem 1. *Let*

$$A^2 = \mathscr{R}^2 - \mathscr{C}^2 < Ra \tag{15}$$

when $\alpha < 1 \leqq 1/\tau$ *and when* $\alpha \geqq 1/\tau < 1$ *let*

$$A = (\tau^2 - 1)^{-\frac{1}{2}}(\mathscr{R}\tau - \mathscr{C}) < \sqrt{Ra^*}. \tag{16}$$

Then,

$$\frac{d}{dt}(E + \Psi) \leqq -\xi^2 (1 - A/\sqrt{Ra^*})\, E(t), \tag{17}$$

and $E(t)$ tends to zero in the following sense:

$$\lim_{t\to\infty} \int_0^t E(t)\,dt < \infty . \tag{18}$$

The criterion (15) *is necessary (and not only sufficient) for stability (in the sense of* (18)*).*

The stability criterion (15) is clearly the best one possible. The criterion (16), too, may be an optimal result for reasons to be made plain in the sequel. For now, I draw attention to the fact that the stability guaranteed by (15) and (16) is global in the sense that it holds for any initial value bounded in L_2. But the stability so guaranteed has not been shown to be strong in the sense of (10) (see [3] for a discussion of strong stability), and does not exclude the possibility that a very large, stable disturbance could be extraordinarily persistent.

Proof of Theorem 1. Let

$$2A \equiv \mathscr{R}\left(\frac{1}{\lambda_T}+\lambda_T-\frac{2\lambda_c\alpha}{1+\tau}\right).$$

Then we can write equation (13) as

$$\frac{dE}{dt}+\frac{d\Psi}{dt}=-\langle|\nabla \boldsymbol{u}|^2+|\nabla\phi|^2\rangle\left\{1-\frac{2A\langle w\phi\rangle}{\langle|\nabla \boldsymbol{u}|^2+|\nabla\phi|^2\rangle}\right\},$$

which is, by (11),

$$\leqq -\langle|\nabla \boldsymbol{u}|^2+|\nabla\phi|^2\rangle\left(1-A/\sqrt{Ra^*}\right)$$

and, by (12),

$$\leqq -\xi^2\left(1-A/\sqrt{Ra^*}\right)E .$$

Let $B=\xi^2(1-A/\sqrt{Ra^*})$ and integrate the last inequality

$$E(t)-E(0)+\Psi(t)-\Psi(0)\leqq -B\int_0^t E(t')\,dt' . \tag{19}$$

Since $E(0)$ and $\Psi(0)$ are bounded, it follows from (19) that $E(t)$ and $\Psi(t)$ are bounded uniformly in t. So too, then, must the integral $\int_0^t E(t')\,dt'$ be bounded uniformly and also in the limit as $t\to\infty$. Since $E(t)$ is integrable, the statement (18) holds when $A<\sqrt{Ra^*}$.

We next show that λ_T and λ_c can be selected so that A has the values given on the left of (15) and (16). We seek values of λ_T and λ_c which satisfy (14) and give the largest possible value (stability limit) on the right of the criterion

$$\mathscr{R}<2\sqrt{Ra^*}\Big/\left(\frac{1}{\lambda_T}+\lambda_T-\frac{2\lambda_c\alpha}{1+\tau}\right). \tag{20}$$

This problem of the "optimum" stability boundary is an ordinary maximum problem. To solve it, we consider (14) to define a function $\lambda_c(\lambda_T)$ and set the total derivative with respect to λ_T of the right of (20) to zero. Then, besides (14), we must have

$$(\lambda_T^2-1)(1+\lambda_c^2)(1+\tau)^2-4\tau\lambda_T^2\lambda_c^2=0. \tag{21}$$

We find a continuous solution of (14, 21) on two branches. On the first branch, $\alpha<1\leqq 1/\tau$, and on it

$$\lambda_T=(\sqrt{1-\alpha^2\tau^2}-\tau\sqrt{1-\alpha^2})/(1-\tau),$$

$$\lambda_c=(\sqrt{1-\alpha^2\tau^2}-\sqrt{1-\alpha^2})/\alpha(1-\tau),$$

and from (20)

$$\mathscr{R}<\sqrt{Ra^*}/\sqrt{1-\alpha^2},$$

proving (15). On the second branch, $\alpha\geqq 1/\tau<1$,

$$\lambda_T=\lambda_c=\left(\frac{\tau+1}{\tau-1}\right)^{\frac{1}{2}}$$

and

$$\mathscr{R}<\frac{\sqrt{Ra^*(1-\tau^{-2})}}{(1-\alpha/\tau)},$$

proving (16).

Finally, it needs to be shown that (15) is a necessary condition for stability. To show this, we prove that (15) is a linear instability boundary, sufficient for instability and, therefore, necessary for stability. We need only any special solution of the linearized versions of (1), (2) and (3), and for our purpose steady solutions will do. For these, we must have

$$0=-\nabla p+(\mathscr{R}\theta-\mathscr{C}c)\boldsymbol{k}+\Delta\boldsymbol{u}, \tag{22}$$

$$0=\Delta\theta+\mathscr{R}w, \tag{23}$$

$$0=\Delta c+\mathscr{C}w, \tag{24}$$

for functions satisfying (4) or (5). Clearly, $\Delta(\theta/\mathscr{R}-c/\mathscr{C})=0$, and by potential theory $\mathscr{C}\theta=\mathscr{R}c$. Then,

$$0=-\nabla p+\frac{(\mathscr{R}^2-\mathscr{C}^2)}{\mathscr{R}}\boldsymbol{k}\theta+\Delta\boldsymbol{u}, \tag{25}$$

and (25), (23) and the boundary conditions determine the eigenvalue

$$Ra^*=\mathscr{R}^2-\mathscr{C}^2.$$

This completes the proof of the theorem.

The criterion (15) closes the question about the onset of instability when $\alpha<1\leqq 1/\tau$. But for $\alpha\geqq 1/\tau<1$, we have not excluded possible sublinearities. This, in any event, could be done only by comparison with the results of linear theory. Moreover, since it appears to be true that sublinear instabilities do occur in just those regions where we have not explicitly excluded them, it is of interest to give here a brief account of results which speak of the sublinear question.

Direct analysis of the nonlinear problem (1–4, 6) has been given by SANI [8] using a perturbation method and by VERONIS [5, 9], who uses a method of modal truncation and a computer. Both authors confine their attention, for simplicity and economy, respectively, to the free surface layer and to disturbances in two dimensions. For this problem, the results of the linear theory are known explicitly. Instability is guaranteed when

$$\mathcal{R}^2 \geqq \begin{cases} Ra^* + \mathcal{C}^2 & (\tau<1) \\ Ra^* + \mathcal{C}^2 & (\tau>1, \mathcal{C}/\mathcal{C}^*<1) \\ \dfrac{Ra^*(\tau+1)(1+Pr\,\tau)}{Pr\,\tau^2} + \dfrac{(1+\tau\,Pr)}{1+Pr}\dfrac{\mathcal{C}^2}{\tau^2} & \left(\tau>1, \dfrac{\mathcal{C}}{\mathcal{C}^*}>1\right) \end{cases} \qquad (26\text{a, b, c})$$

where

$$Ra^* = \frac{27}{4}\pi^4 = 657\,, \qquad \mathcal{C}^{*2} = \frac{Ra^*}{Pr}\,\frac{(Pr+1)}{\tau-1}\,.$$

The criterion (26c) is of interest for two reasons. First, for parameters in the given range, zero norm instability starts with overstable oscillations. Second, for large $\mathcal{C}$, instability can occur when the basic state is gravitationally stable. The basic state is said to be gravitationally stable if the density gradient for the conduction-diffusion solution is negative, that is, if

$$\boldsymbol{k}\cdot\nabla\rho = \frac{d\rho}{dz} = \rho_0\left(\alpha_c\frac{dC}{dz} - \alpha_T\frac{dT}{dz}\right) = \rho_0\, l\,(\alpha_T\beta_T - \alpha_c\beta_c) = \rho_0\, l\,\alpha_c\beta_c\left(\frac{\mathcal{R}^2\tau}{\mathcal{C}^2} - 1\right) < 0\,.$$

Since for large $\mathcal{C}$, overstable oscillations can occur when

$$\frac{\mathcal{R}^2\tau}{\mathcal{C}^2} = \frac{1+\tau\,Pr}{\tau(1+Pr)} \leqq 1\,,$$

the system can be actually unstable when gravitationally stable.

We note that the smallest value on the left of (26c) over all positive value of Pr for fixed τ is just $\mathcal{C}^2/\tau^2$ for large $\mathcal{C}$. This is, in fact, exactly the value given by the stability criterion (16), so that (16), too, is attained in the Pr (for $Pr\to 0$) family for each fixed τ. It follows that for the free surface problem, at any rate, there is a sense in which (16) is an optimal result. It is natural to wonder, though, whether or not it is possible to have the stability and instability limit (which coincide as $\mathcal{C}\to\infty$ when $Pr\to 0$) closer together for small $\mathcal{C}$. The question is equivalent to asking if sublinear solutions exist in the shaded region given in Fig. 1. The calculations of SANI [8], who uses the formal perturbation method of STUART and WATSON, are especially relevant here. SANI's results are in detailed agreement with (15) and (16). His calculations lead to sublinear solutions for all $\mathcal{C}>\mathcal{C}_1$ where $\mathcal{C}_1^2 = Ra^*/(\tau^2-1)$ is the value below which sublinear solutions are explicitly excluded by the energy considerations.

It would, therefore, appear that to have better criteria than (15) and (16), it will be necessary to find procedures which depend on the separate values Sc and

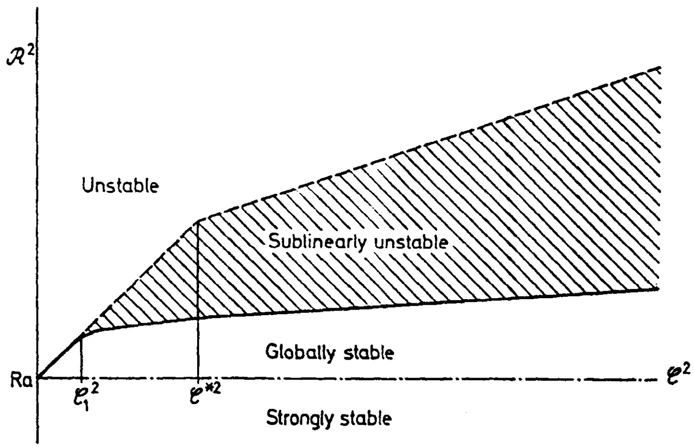

Fig. 1. Stability-instability boundaries for heated and salted below

This is a schematic sketch of the energy (stability) limit (15, 16) and the linear (instability) limit (26) for the layer between free surfaces when $\tau \geqq 1$. The top line (dotted) is the instability limit, and the heavy black line is the stability limit. Both limits have kinks; for the instability line the position of kinks and the slope of the line when $\mathcal{C} > \mathcal{C}^*$ depend on both *Pr* and *Sc*, but the energy limit depends only on *Sc*/*Pr*. The two limits coincide when $\mathcal{C}^2 < \mathcal{C}_1^2 = Ra^*/(\tau^2 - 1)$, but for larger $\mathcal{C}$, the most one could expect from the one (τ) parameter family of energy limits (16) is coincidence, say, when τ is fixed for some *Pr*. For τ fixed, $Pr \to 0$, the instability and stability limits do coincide both a $\mathcal{C} = 0$ and for large $\mathcal{C}$.

Only the shaded region is open to sublinear solutions. The perturbation calculation of Sani [8] indicates that two-dimensional solutions with small amplitudes fill the shaded region near the linear limit for $\mathcal{C} > \mathcal{C}_1$ where $\mathcal{C}_1$ is the above mentioned *energy* value.

In the region of strong stability, the estimate (10) holds, and the decay of a stable disturbance is very rapid.

Pr and not just on their ratio. On the other hand, the possibility that the decay of a globally stable disturbance satisfying (15) or (16) has a sense stronger than that given by (18) has not been excluded.

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