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NONLINEAR DIFFUSION INDUCED BY NONLINEAR SOURCES*

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I. Introduction. In the published literature dealing with a number of diverse scientific and technological problems, one encounters the mathematical system

$$\nabla^2\psi + \lambda g(\mathbf{x})\Phi(\psi) = 0 \quad \text{in} \quad R, \tag{1a}$$

$$\partial\psi/\partial n + Bh(\mathbf{x})\psi = 0 \quad \text{on} \quad S. \tag{1b}$$

A physically relevant generalization of the foregoing, the characteristics of which have yet to be explored, is

$$\nabla\cdot[f(\psi)\nabla\psi] + \lambda g(\mathbf{x})\Phi(\psi) = 0 \quad \text{in} \quad R, \tag{2a}$$

$$f(\psi)\ \partial\psi/\partial n + Bh(\mathbf{x})\psi = 0 \quad \text{on} \quad S. \tag{2b}$$

R is a closed region bounded by surface S, with n denoting the outer normal to S. The physical content of Eqs. (1) and (2) will now be discussed in broad terms. A survey of physical situations to which these equations are relevant is presented later.

All of the quantities appearing in Eqs. (1) and (2) are dimensionless. ψ denotes a scalar field such as the temperature or mass concentration, the gradient of which induces the transfer of quantities such as heat or mass. The term $\lambda\Phi(\psi)$ represents a ψ-dependent source of the transferred quantity, with λ denoting the intensity level of the source[1] and $\Phi(\psi)$ its dependence on ψ. Throughout this paper, λ will be termed the source intensity and Φ the source distribution function.

The factor f denotes a ψ-dependent diffusion coefficient, for instance, the thermal conductivity or the coefficient of mass diffusion. In general, $f(\psi) > 0$. Eqs. (1) correspond to the case of a ψ-independent diffusion coefficient, so that, with proper normalization, f does not appear. B is a constant whose magnitude is an indication of the ease of transfer from the boundary surface S to the environment outside R relative to the ease of transfer within R itself. When $B = \infty$, Eqs. (1b) and (2b) reduce to $\psi = 0$ on S. The functions $g(\mathbf{x})$ and $h(\mathbf{x})$ are required to be positive, with g having two continuous derivatives and h being piecewise continuous.

The major concern of the paper is with *nonlinear* source distributions $\Phi = \Phi(\psi)$ and with ψ-dependent diffusion coefficients $f = f(\psi)$.

In physical problems involving distributed sources, the quantity $\bar{\psi} = \max_{\mathbf{x}\in R}\psi(\mathbf{x})$ is of considerable practical interest and may be used as an independent parameter. It frequently happens that in problems characterized by nonlinear sources, the solution $\psi(\mathbf{x}, \bar{\psi})$,

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[1] Typically $\lambda \sim L^2\dot{q}$, where L^2 is a characteristic length and $\dot{q}$ is the intensity of the source.

$\lambda(\bar{\psi})$ is uniquely determined by $\bar{\psi}$, but $\bar{\psi}(\lambda)$ may not be single-valued and the solution $\psi(\mathbf{x}, \lambda)$ may not be unique. In fact, it is known that in many situations, there is a value $\hat{\lambda} = \sup_{\bar{\psi} \geq 0} \lambda(\bar{\psi})$ such that if $\lambda > \hat{\lambda}$, then steady-state solutions are not possible. For instance, in the case of (1), such behavior is encountered when $\Phi(\psi)$ is such that $\psi/\Phi(\psi)$ is bounded [1].

Secs. II and III of the present study are concerned with the λ, $\bar{\psi}$ relationship. For any given $\bar{\psi}$, an a priori upper bound on λ is deduced. The expression for the bound is applicable for any positive functions $f(\psi)$, $\Phi(\psi)$, $g(\mathbf{x})$ and $h(\mathbf{x})$, for regions R of arbitrary shape, and for arbitrary values of B. Furthermore, for a specific but physically relevant class of problems, an expression is deduced which directly gives an a priori upper bound for $\hat{\lambda}$.

The just-mentioned general theorems are illustrated by application to the problem of ohmic heating of a circular wire or a plane sheet. Consideration is given both to constant thermal conductivity (i.e., $f = 1$) and to temperature-dependent thermal conductivity as expressed by the Wiedemann–Franz–Lorenz law. The a priori bounds for λ as a function of $\bar{\psi}$ are compared with λ, $\bar{\psi}$ distributions determined from numerical solutions of the governing differential equations. Comparisons of the $\hat{\lambda}$ bounds with numerically determined values are also made. These comparisons suggest that the bounds are sufficiently accurate to serve as useful estimates of both the λ, $\bar{\psi}$ distributions and of $\hat{\lambda}$.

In Sec. IV of the paper, for system (1), we demonstrate the existence and give an analytic perturbation from the zero solution in powers of $\bar{\psi}$, λ, or any other equivalent parameter. For a one-dimensional version of (1), it is shown that the radius of convergence of the power series in $\bar{\psi}$ is determined by the first zero of $\Phi(\psi)$ (Sec. V).

Attention may now be turned to a survey of physical problems to which systems (1) and (2) are relevant. Consider first the group of problems in which ψ represents the temperature field. Within this group, one encounters a diverse range of physical processes, each characterized by a different heat source function $\lambda\Phi$. The passage of electrical current through a solid conductor or an electrolyte causes ohmic heating to occur, which, in turn, gives rise to temperature nonuniformities [2]–[4].[2] The temperature dependence of the source distribution function Φ is related to the variation of the electrical resistivity with temperature.

In materials fatigue studies, when a solid is subjected to a cycled load, a finite amount of heat is generated in each cycle [5]. Heat generation also occurs as a result of chemical reactions, for instance, in a porous catalyst particle [6]–[9], in a tubular reactor with axial diffusion [6], [7] and in connection with thermal self-ignition of a chemically active mixture of gases in a vessel [10], [11].

A flowing fluid undergoes a dissipation of mechanical energy into heat owing to the action of viscosity [12]–[17]. The resulting heating of the flow is an unwanted effect in viscosity measuring instruments and in lubricated machine parts. On the other hand, in the forming of thermoplastic materials by means of a screw extrusion device, the internal heating due to viscous dissipation enhances the workability of the material and facilitates rapid processing.

The mathematical systems (1) and (2) are also relevant to nuclear reactor dynamics, specifically, to the determination of the energy released in the reactor as a result of a power excursion [18]–[20]. In addition, these equation systems describe the distribution

[2] These literature citations refer to representative publications relevant to the process under discussion.

of partial pressure in an isothermal porous catalyst, wherein ψ is related to the partial pressure and Φ is proportional to the rate of the chemical reaction [21].

Various analytical aspects of Eqs. (1) have been examined in the literature. For certain source distributions $\Phi(\psi)$, there may be multiple solutions when λ lies in the range $\lambda^* \leq \lambda \leq \lambda^{**}$ and unique solutions when λ lies outside this range. Luss and co-workers [6]–[9] have given estimates of λ^* and λ^{**} and have examined the conditions on Φ to insure unique solutions. Keller and Cohen [22]–[24] considered a generalized version of system (1) (i.e., ∇^2 was replaced by a strongly elliptic operator and $g(\mathbf{x})\Phi(\psi)$ replaced with $F(\mathbf{x}, \psi)$) and, among other aspects, investigated the nonexistence of solutions.

The specific case of $\Phi(\psi) = e^{\psi}$ was examined by Gel'fand [10] in connection with an n-dimensional sphere ($n = 1, 2, 3$). One of Gel'fand's interesting findings is that the multiplicity of the solutions is of a different character when $n = 3$ than when $n = 1$ and 2. The existence of multiple steady-state solutions suggests the possibility that some are unstable; this aspect has been investigated in [4], by Luss and Lee [9], and by Fujita [11]. Within the knowledge of the present authors, analytical aspects of Eqs. (2) have not heretofore been explored in the literature directly relevant to this investigation.

II. Bounds on $\lambda(\bar{\psi})$. It is convenient to replace (2) with an equivalent problem

$$\nabla^2 u + \lambda g(\mathbf{x})\Phi[\psi(u)] = 0|_R \tag{3a}$$

and

$$\partial u/\partial n + Bh(\mathbf{x})\psi(u) = 0|_S \tag{3b}$$

which is induced by the Kirchhoff transformation

$$u = \int_0^{\psi} f(\psi')\, d\psi'.$$

We also make use of the comparison linear problem

$$\nabla^2\phi + \mu g(\mathbf{x})\phi = 0|_R \tag{4a}$$

and

$$\partial\phi/\partial n + B\chi h(\mathbf{x})\phi = 0|_S \tag{4b}$$

where

$$\chi = \oint_S h(\mathbf{x})\psi(\mathbf{x})\phi(\mathbf{x})\, dS \Big/ \oint_S h(\mathbf{x})u[\psi(\mathbf{x})]\phi(\mathbf{x})\, dS. \tag{5}$$

Multiply ϕ into (3a) and u into (4a) to find, after integration, that

$$\lambda = \mu \int_R g(\mathbf{x})\phi(\mathbf{x})u[\psi(\mathbf{x})]\, dR \Big/ \int_R g(\mathbf{x})\phi(\mathbf{x})\Phi[\psi(\mathbf{x})]\, dR.$$

Let $\mu(B\chi)$ be the principal eigenvalue of (4). It is known that the principal eigenfunction ϕ is one-signed (take it nonnegative).

Consider positive solutions ψ of (2). If ψ is positive, then, since $f(\psi) > 0$, u is a positive solution of (3). Let $\Phi(y) > 0$ when $y \geq 0$. Then,

$$\chi = \frac{\oint_S h(\mathbf{x})u[\psi(\mathbf{x})]\phi(\mathbf{x})(\psi(\mathbf{x})/u[\psi(\mathbf{x})])\, dS}{\oint_S h(\mathbf{x})u[\psi(\mathbf{x})]\phi(\mathbf{x})\, dS} = \frac{\psi(\bar{\bar{\mathbf{x}}})}{u[\psi(\bar{\bar{\mathbf{x}}})]} \tag{6}$$

where $\bar{\bar{\mathbf{x}}} \in S$, and

$$\lambda = \mu \frac{\int_R g(\mathbf{x})\phi(\mathbf{x})\Phi[\psi(\mathbf{x})](u[\psi(\mathbf{x})]/\Phi[\psi(\mathbf{x})])\, dR}{\int_R g(\mathbf{x})\phi(\mathbf{x})\Phi[\psi(\mathbf{x})]\, dR} = \mu \frac{u[\psi(\bar{\mathbf{x}})]}{\Phi[\psi(\bar{\mathbf{x}})]} \tag{7}$$

where $\bar{\mathbf{x}} \in R$.

Our first result is a simple consequence of (6) and (7).

THEOREM 1. *Let $\Phi(\psi) > 0$ when $\psi \geq 0$. Let ψ be a positive solution of* (2) *with maximum value $\bar{\psi}$. Then,*

$$\lambda(\bar{\psi}) \leq \mu[B\tilde{\chi}(\bar{\psi})]\Gamma(\bar{\psi}) \tag{8}$$

where

$$\tilde{\chi}(\bar{\psi}) = \max_{0 \leq \psi \leq \bar{\psi}} \psi/u(\psi)$$

and

$$\Gamma(\bar{\psi}) = \max_{0 \leq \psi \leq \bar{\psi}} u(\psi)/\Phi(\psi).$$

It is known that $\mu(\alpha)$ is an increasing function of its argument. Hence,

$$\mu(B\tilde{\chi}) \geq \mu(B\psi(\bar{\bar{\mathbf{x}}})/u[\psi(\bar{\bar{\mathbf{x}}})]),$$

and the proof follows from (6) and (7).

The estimate (8) is in principle (and in fact) easily computed. $\Gamma(\bar{\psi})$ and $\tilde{\chi}(\bar{\psi})$ are readily calculated from the given functions $\Phi(\psi)$ and $u(\psi)$, and $\mu(B\tilde{\chi})$ is found as the principal eigenvalue for the linear problem (4). Although exact values for $\mu(B\tilde{\chi})$ in arbitrary domains can be obtained only by extended numerical computations, upper estimates are easily determined by standard Ritz procedures. Any upper estimate of the values $\mu(B\tilde{\chi})$ naturally results in an upper estimate for the allowed values $\lambda(\bar{\psi})$ of the nonlinear problem. In particular, (8) holds with σ replacing μ, where

$$\mu(B\tilde{\chi}) \leq \sigma(B\tilde{\chi}, \theta) = [\int_R |\nabla\theta|^2\, dR + B\tilde{\chi} \oint_S h(\mathbf{x})\theta^2\, dS]/\int_R g(\mathbf{x})\theta^2\, dR,$$

and θ is any function with continuous first derivatives in R. Admissible functions θ should be selected to vanish at the same set of boundary points at which $h(\mathbf{x})B = \infty$.

To illustrate the nature of the results, a priori bounds evaluated from (8) will now be compared with λ, $\bar{\psi}$ distributions obtained by direct numerical solutions of (1) and (2) for the case of ohmic heating in a conductor of circular cross section. In the calculations, the relationship between the electrical resistivity ρ and the absolute temperature T was represented by the quadratic form $\rho = a + bT + cT^2$ which is realistic for a wide range of materials. If T_0 is the temperature of the environment in which the conductor is situated, then the ψ variable may be expressed in physical terms as $\psi = \beta(T - T_0)/\rho_0$ where $\beta = b + 2cT_0$ and ρ_0 is the resistivity at T_0.

We distinguish between two physically relevant situations where, respectively, the voltage gradient e and the current density j are uniform across the section.[3] The corresponding definitions of λ and Φ are

$$\lambda = e^2 r_0^2 \beta/k_0\rho_0^2, \qquad \Phi = (1 + \psi + \gamma\psi^2)^{-1}$$

[3] The voltage gradient is cross-sectionally uniform in a homogeneous wire, while the current density is commonly taken to be cross-sectionally uniform in coils.

and

$$\lambda = j^2 r_0^2 \beta / k_0 , \qquad \Phi = (1 + \psi + \gamma\psi^2)$$

in which k_0 is the thermal conductivity at temperature T_0 , r_0 the radius of the conductor, and $\gamma = \rho_0 c/\beta^2$. The temperature dependence of the thermal conductivity k can be accurately represented for many pure metals and some alloys by the Wiedemann–Franz–Lorenz law [2, pp. 112-114], which gives

$$f(\psi) = (1 + \delta\psi)/(1 + \psi + \gamma\psi^2) \tag{9}$$

with $\delta = \rho_0/\beta T_0$. The parameter B appearing in Eqs. (1b) and (2b) is the Biot modulus.

In Fig. 1, we present results for $\lambda(\tilde{\psi})$ for the case in which the voltage gradient is cross-sectionally uniform. The solid lines represent direct numerical solutions of systems (1)

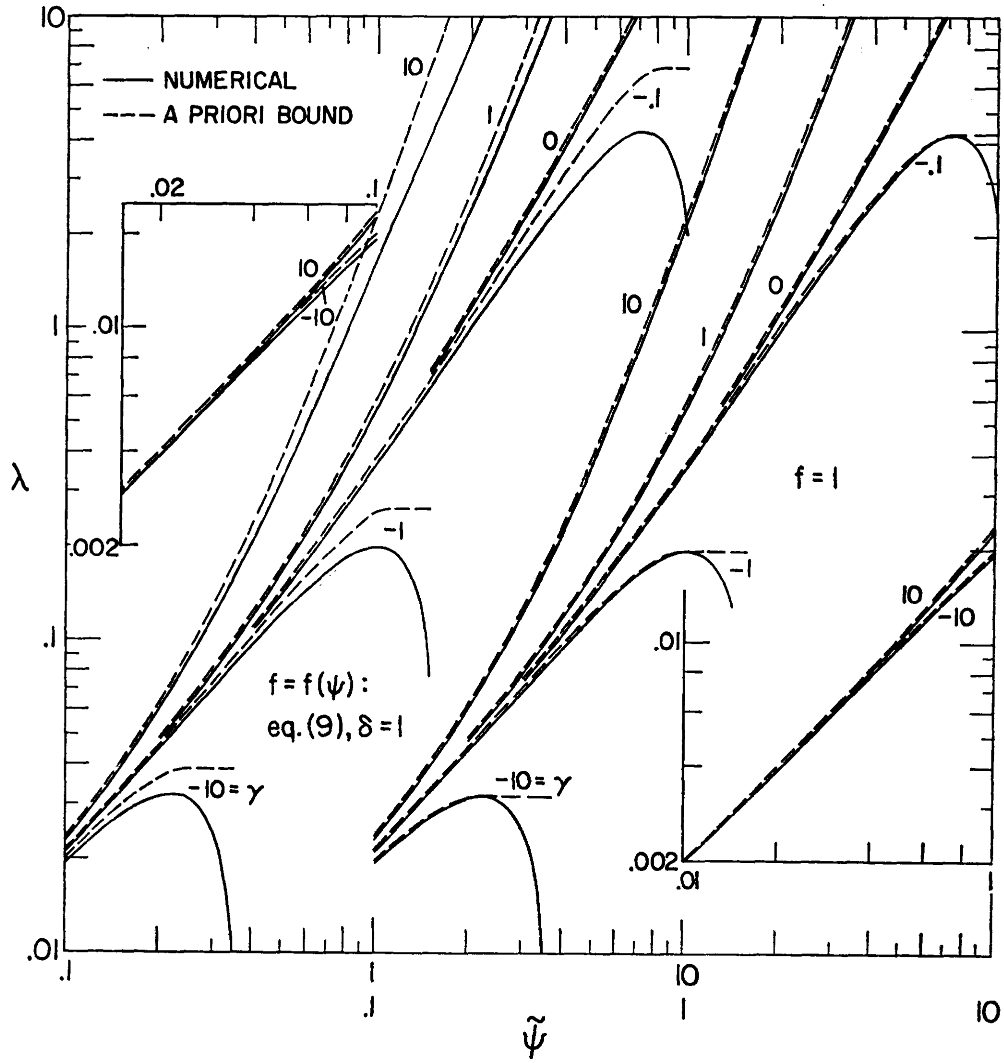

FIG. 1 Relationship between source intensity λ and maximum temperature $\tilde{\psi}$ for a conductor of circular cross section. The voltage gradient is cross-sectionally uniform.

and (2), while the dashed lines represent the a priori bounds expressed by Eq. (8). Results corresponding to ψ-dependent f, Eq. (9), and to ψ-independent f are respectively shown in the left-hand and right-hand portions of the figure. The various curves are parameterized by the quantity γ which appears in the expressions for Φ and f. The Biot modulus B has been assigned the physically realistic value of 0.1.

Inspection of the figure reveals a different behavior for $\lambda(\tilde{\psi})$ depending on whether or not $\psi/\Phi(\psi) = \psi(1 + \psi + \gamma\psi^2)$ is bounded for $\psi \geq 0$ on the domain of $\Phi(\psi) > 0$. When $\gamma \geq 0$, $\lambda(\tilde{\psi})$ increases monotonically, while when $\gamma < 0$, the curves display a maximum such that $\tilde{\psi}(\lambda)$ is double-valued. If $\hat{\lambda} = \sup_{\tilde{\psi}} \lambda(\tilde{\psi})$ for a given $\gamma < 0$, then steady-state solutions for that γ are not possible when $\lambda > \hat{\lambda}$. These same characteristics are in evidence both for $f = f(\psi)$ and for $f = 1$.

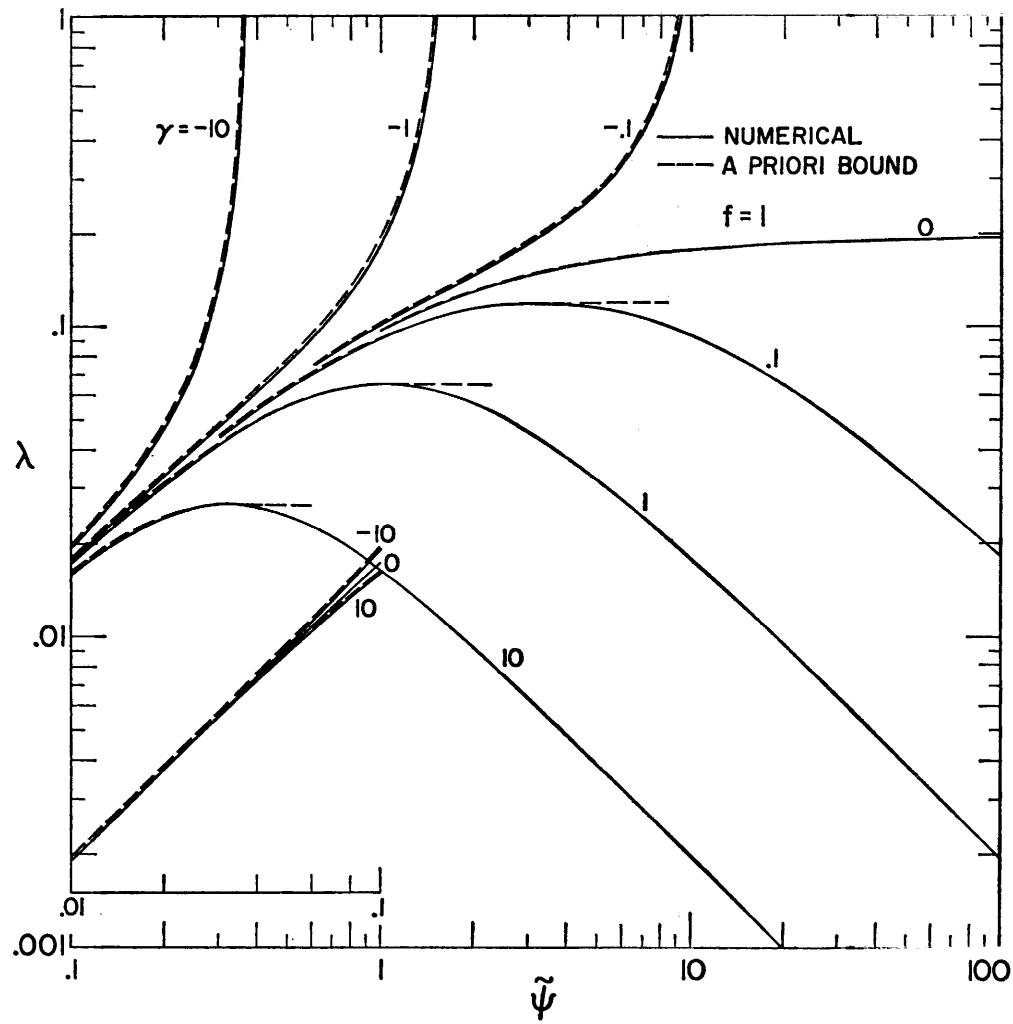

FIG. 2(a) Relationship between source intensity λ and maximum temperature $\tilde{\psi}$ for a conductor of circular cross section. The current density is cross-sectionally uniform and the thermal conductivity is temperature-independent.

The a priori bounds of Eq. (8) are, in general, numerically accurate representations of the true distributions $\lambda(\tilde{\psi})$ obtained from the numerical solutions, with slightly better agreement prevailing when $f = 1$. In those cases in which $\lambda(\tilde{\psi})$ has a maximum, the bound cannot simulate the descending part of the curve; however, the maximum value of the bound for each $\gamma < 0$ appears to be a close approximation to the corresponding numerically determined value of $\hat{\lambda}$.

Results for the case of cross-sectionally uniform current density are presented in Figs. 2(a) and 2(b), respectively for $f = 1$ and for the $f = f(\psi)$ of Eq. (9) with $\delta = 1$. The curves appearing therein are parameterized by γ, and $B = 0.1$. Once again, the behavior of the $\lambda(\tilde{\psi})$ distributions is related to whether or not $\psi/\Phi(\psi)$ is bounded, where now $\Phi = 1 +$

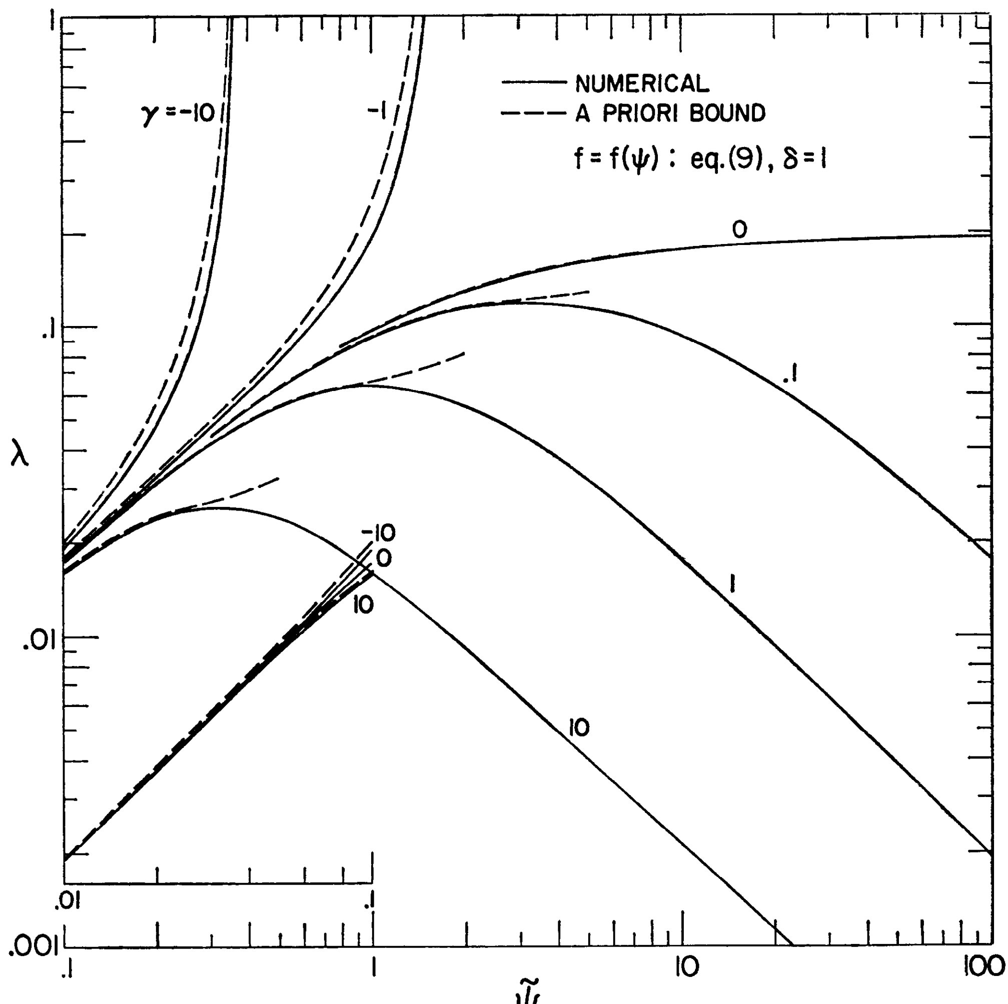

FIG. 2(b) Relationship between source intensity λ and maximum temperature $\tilde{\psi}$ for a conductor of circular cross section. The current density is cross-sectionally uniform and the thermal conductivity is temperature-dependent.

$\psi + \gamma\psi^2$. In particular, for each $\gamma \geq 0$, there is a value of $\hat{\lambda}$ such that steady-state solutions are not possible when $\lambda > \hat{\lambda}$.

The accord between the a priori bounds of Eq. (8) and the numerically determined $\lambda(\bar{\psi})$ distributions continues to be very good, with that in evidence in Fig. 2(a) being truly remarkable. As before, when the $\lambda(\bar{\psi})$ distribution has a maximum, the bound cannot follow the descending portion of the curve. In Fig. 2(b), the continuing ascent of the dashed curves for $\gamma > 0$ is due to the fact that for the specific $f(\psi)$ under consideration, $\tilde{\chi}(\bar{\psi})$ is monotonic increasing and, as a consequence, $\mu[B\tilde{\chi}(\bar{\psi})]$ increases toward the value $\mu(\infty) < \infty$.

On the basis of the foregoing comparisons, it may be concluded that the bounds given by (8) are sufficiently accurate to serve as useful estimates of the $\lambda(\bar{\psi})$ distribution. Other similar comparisons made by the authors suggest that comparable accuracy may be expected for nonpathological $\Phi(\psi)$.

III. Bounds on $\hat{\lambda}$. It is an obvious consequence of (8) and the fact that $\mu(B\tilde{\chi}) \leq \mu(\infty) < \infty$ that $\hat{\lambda}$ exists if $u(\psi)/\Phi(\psi)$ is bounded for all positive ψ. The estimate of $\hat{\lambda}$ which arises from this observation can have considerable numerical precision. As shown below, one can also obtain an even more precise estimate of $\hat{\lambda}$, but additional hypotheses are required.

THEOREM 2. *Let the hypotheses of Theorem* 1 *hold. There are no positive solutions of* (2) *when*

$$\lambda > \hat{\lambda} \geq \mu(B\chi^*)\Gamma^* \tag{10}$$

where

$$\chi^* = \max_{\bar{\psi}\geq 0} \chi(\bar{\psi}), \qquad \Gamma^* = \max_{\bar{\psi}\geq 0} \Gamma(\bar{\psi}).$$

Let $f(\psi)$ ($f(0) \equiv 1$) *be a nonincreasing function and let* $\psi(\bar{\mathbf{x}}) \geq \psi(\bar{\bar{\mathbf{x}}})$, *where* $\bar{\mathbf{x}}$ *and* $\bar{\bar{\mathbf{x}}}$ *are mean values* (*defined by Eqs.* (6) *and* (7)) *in the interior and on the boundary, respectively. The positive solutions of* (2) *do not exist when*

$$\lambda > \hat{\lambda} \geq \min\,[\mu(B\chi^*)\Gamma^*,\ \mu(B) \max_{\psi\geq 0} \psi/\Phi(\psi)]. \tag{11}$$

We note that for n-dimensional spheres, and for small perturbations from these, the inequality $\psi(\bar{\mathbf{x}}) \geq \psi(\bar{\bar{\mathbf{x}}})$ holds as a consequence of the maximum principle and need not be stated as an additional hypothesis. But for general domains, the inequality $\psi(\bar{\mathbf{x}}) \geq \psi(\bar{\bar{\mathbf{x}}})$ may not hold.

Proof of Theorem 2. Eq. (10) is obvious. We need to prove (11). From (7), one can find that

$$\lambda(B) = \mu(B\chi)\frac{u[\psi(\bar{\mathbf{x}})]}{\Phi[\psi(\bar{\mathbf{x}})]} = \frac{\chi(\bar{\bar{\mathbf{x}}})}{\chi(\bar{\mathbf{x}})}\frac{\mu[B\chi(\bar{\bar{\mathbf{x}}})]}{\chi(\bar{\bar{\mathbf{x}}})}\frac{\psi(\bar{\mathbf{x}})}{\Phi[\psi(\bar{\mathbf{x}})]}$$

where $\chi(\bar{\mathbf{x}}) = \psi(\bar{\mathbf{x}})/u[\psi(\bar{\mathbf{x}})]$ is a function of the form (6) defined through the interior mean value $\bar{\mathbf{x}}$. Moreover, since

$$\frac{d\chi}{d\psi} = \frac{d}{d\psi}\left[\frac{\psi}{u(\psi)}\right] = u^{-2}\left[\int_0^{\psi} f(\psi')\,d\psi' - \psi f(\psi)\right] = u^{-2}\psi[f(\bar{\psi}) - f(\psi)] \geq 0,$$

we have $\chi(\bar{\bar{\mathbf{x}}}) \leq \chi(\bar{\mathbf{x}})$ when $\psi(\bar{\mathbf{x}}) \geq \psi(\bar{\bar{\mathbf{x}}})$. Then

$$\lambda(B) \leq \max_{\chi \geq 1} (\mu(B\chi)/\chi) \max_{\psi \geq 0} (\psi/\Phi(\psi)). \tag{12}$$

We must show that

$$\max_{\chi \geq 1} (\mu(B\chi)/\chi) = \mu(B). \tag{13}$$

To prove (13), consider the problem

$$\nabla^2\phi + \mu(\alpha)g(\mathbf{x})\phi = 0|_R, \qquad \partial\phi/\partial n + \alpha h(\mathbf{x})\phi = 0|_S \tag{14}$$

and the perturbation problem

$$\nabla^2\phi^{(1)} + \mu(\alpha)g(\mathbf{x})\phi^{(1)} + \mu^{(1)}(\alpha)g(\mathbf{x})\phi = 0|_R, \qquad \partial\phi^{(1)}/\partial n + \alpha h(\mathbf{x})\phi^{(1)} + h(\mathbf{x})\phi = 0|_S \tag{15}$$

where

$$\phi^{(1)} = \partial\phi/\partial\alpha, \qquad \mu^{(1)} = d\mu/d\alpha.$$

The problem (15) has solutions if and only if

$$\oint_S h(\mathbf{x})\left(\phi^{(1)}\frac{\partial\phi}{\partial n} - \phi\frac{\partial\phi^{(1)}}{\partial n}\right) dS - \mu^{(1)}\int_R g(\mathbf{x})\phi^2\, dR$$
$$= \oint_S h(\mathbf{x})\phi^2\, dS - \mu^{(1)}\int_R g(\mathbf{x})\phi^2\, dR = 0. \tag{16}$$

The principal eigenvalues are given by

$$\mu(\alpha) = \left\{\int_R |\nabla\phi|^2\, dR + \alpha\oint_S h(\mathbf{x})\phi^2\, dS\right\}\Big/\int_R g(\mathbf{x})\phi^2\, dR$$
$$= \int_R |\nabla\phi|^2\, dR\Big/\int_R g(\mathbf{x})\phi^2\, dR + \alpha\mu^{(1)}.$$

Hence

$$\frac{d\mu(\alpha)/\alpha}{d\alpha} = -\frac{1}{\alpha^2}\frac{\int_R |\nabla\phi|^2\, dR}{\int_R g(\mathbf{x})\phi^2\, dR} \leq 0$$

and

$$\frac{d}{d\chi}\left[\frac{\mu(B\chi)}{\chi}\right] = B^2\frac{d}{d(B\chi)}\left[\frac{\mu(B\chi)}{B\chi}\right] \leq 0.$$

Since the minimum value of χ is one, we obtain (13), (12), and (11), thereby proving Theorem 2.

As an application of (11), $\hat{\lambda}$ values have been evaluated for ohmic heating in a circular conductor and in a plane sheet. The relevant physical parameters and the definitions of ψ, λ, Φ, and f have already been given in Sec. II. For the function $f = f(\psi)$ expressed by Eq. (9), the condition $f'(\psi) \leq 0$ of Theorem 2 is fulfilled for all $\psi > 0$ when $\delta \leq 1$ and $\gamma \geq 0$. Since $\lambda(\bar{\psi})$ is bounded when $\gamma \geq 0$ for the case of cross-sectionally uniform current density, it is for this situation that Eq. (11) will be applied. In the computations, δ was assigned a value of one.

In Table 1, numerical values of $\hat{\lambda}$ from Eq. (11) are compared with those from direct

TABLE 1
Comparison of λ from (11) with those from numerical solutions
(a) circular conductor

		$f = 1$		$f = f(\psi)$, Eq. (9)	
B	γ	Eq. (11)	Numerical	Eq. (11)	Numerical
0.1	10	0.0266	0.0266	0.0266	0.0262
0.1	1	0.0650	0.0650	0.0650	0.0642
0.1	0.1	0.1195	0.1195	0.1195	0.1188
∞	10	0.786	0.744	0.666	0.605
∞	1	1.928	1.838	1.686	1.569
∞	0.1	3.543	3.444	3.268	3.143

(b) plane sheet

		$f = 1$		$f = f(\psi)$, Eq. (9)	
B	γ	Eq. (11)	Numerical	Eq. (11)	Numerical
0.1	10	0.0132	0.0132	0.0132	0.0129
0.1	1	0.0323	0.0322	0.0323	0.0317
0.1	0.1	0.0593	0.0593	0.0593	0.0588
∞	10	0.339	0.326	0.284	0.269
∞	1	0.822	0.802	0.720	0.693
∞	0.1	1.511	1.489	1.395	1.366

numerical solutions of (1) and (2). The table is subdivided into two parts, respectively for the circular conductor and for the plane sheet. Inspection of the table reveals very good agreement of the a priori estimates for λ with the corresponding results of the numerical solutions. The accuracy of the a priori estimates is favored by decreasing values of B. Indeed, for the physically realistic condition $B = 0.1$, the estimates are essentially exact.

The foregoing comparison suggests that estimates of λ evaluated from (11) are of sufficient accuracy to be useful in practice.

IV. Existence of an analytic perturbation of the trivial solution. We have seen that it is sometimes convenient to use parameters other than λ, for example, $\bar{\psi}$, to characterize the solution. It makes very little difference as to which parameter characterizes the solution, provided that the parameters are unique functions of each other. But, as was observed earlier, it can happen that nonunique relationships exist between the parameters; for instance, $\bar{\psi}(\lambda)$ is a single-valued function in the examples of Sec. II, but $\lambda(\bar{\psi})$ is not unique.

For solutions with small norms, one can obtain the solution explicitly as a Taylor series in any of an infinity of small parameters. Although the choice of parameter is of no important consequence in the demonstration of the existence of the analytic solution, one can obtain a larger number of the set of all solutions by a proper choice of parameter (cf. Sec. V).

We now proceed to demonstrate the existence and to construct the analytic solution of system (1). It is convenient first to reformulate system (1) as an operator equation on the space of continuous functions. The reformulation allows us to introduce any of an infinity of parameters as preassigned.

Problem (1) can be expressed as an integral equation

$$\psi(\mathbf{x}) = \lambda \int_{R_0} G(\mathbf{x}, \mathbf{x}_0)\Phi[\psi(\mathbf{x}_0)]\, dR_0 \tag{17}$$

where $G(\mathbf{x}, \mathbf{x}_0)$ is the Green's function for the Laplace operator and boundary conditions (1b). Let $\mathfrak{N}[\psi]$ be a linear functional, defined for functions continuous on R and bounded in a maximum norm

$$|\mathfrak{N}[\psi]| \leq c_0 \,||\psi|| \equiv c_0 \max_{\mathbf{x}\in\bar{R}} |\psi(\mathbf{x})|. \tag{18}$$

We shall regard $A = \mathfrak{N}[\psi]$ as preassigned. The operator

$$m(\psi) = \int_{R_0} G(\mathbf{x}, \mathbf{x}_0)\Phi[\psi(\mathbf{x}_0)]\, dR_0 \tag{19}$$

maps continuous functions into continuous functions. Then, instead of (17), we can consider the problem

$$\psi\mathfrak{N}[m(\psi)] = Am(\psi). \tag{20}$$

Every solution of (20) for which $\mathfrak{N}[m(\psi)] \neq 0$ is also a solution of (17) with $\lambda = A/\mathfrak{N}[m(\psi)]$, and conversely.

One can choose the linear functional $\mathfrak{N}$ in different ways. Here are three ways: (i) A is the value of $\psi(\mathbf{x})$ at a point (say $\mathbf{x} = 0$). Then,

$$\mathfrak{N}[\psi] = \psi(0), \qquad \mathfrak{N}[m(\psi)] = \int_{R_0} G(0, \mathbf{x}_0)\Phi[\psi(\mathbf{x}_0)]\, dR_0 \,.$$

(ii) A is the total flux. Then,

$$\mathfrak{N}[\psi] = -\oint_S \frac{\partial\psi}{\partial n}\, dS = B \oint_S h(\mathbf{x})\psi(\mathbf{x})\, dS$$

and

$$\mathfrak{N}[m(\psi)] = B \oint_S h(\mathbf{x})\left\{\int_{R_0} G(\mathbf{x}, \mathbf{x}_0)\Phi[\psi(\mathbf{x}_0)]\, dR_0\right\} dS.$$

(iii) A is the heat source intensity. Then $\mathfrak{N}[\psi] = \lambda$, $\mathfrak{N}[m(\psi)] = 1$.

To prove the existence of solutions of (20) which are analytic in A, we first construct a formal power series. Then, we show that the series converges to a function $\psi(\mathbf{x}, A)$ and that this function satisfies (20).

Differentiation of (20) ν times with respect to A using the product rule yields, after setting $A = 0$,

$$\psi^{(\nu)}\mathfrak{N}[m(0)] = \nu[m(\psi)]^{(\nu-1)} - \sum_{r=1}^{\nu-1} \binom{\nu}{r}\psi^{(r)}\mathfrak{N}[m(\psi)]^{(\nu-r)} \tag{21}$$

where

$$[m(\psi)]^{(\nu)} = \int_{R_0} G(\mathbf{x}, \mathbf{x}_0)(\Phi[\psi(\mathbf{x}_0)])^{(\nu)}\, dR_0$$

and

$$(\Phi[\psi])^{(\nu)} = \sum \frac{\nu!}{i!\,j!\,h!\cdots k!}\,\Phi^{(m)}(0)\left(\frac{\psi^{(1)}}{1!}\right)^{i}\left(\frac{\psi^{(2)}}{2!}\right)^{j}\left(\frac{\psi^{(3)}}{3!}\right)^{h}\cdots\left(\frac{\psi^{(l)}}{l!}\right)^{k}. \tag{22}$$

Here the superscript (ν) means $\partial^\nu/\partial A^\nu$ at $A = 0$ and the symbol $\sum$ indicates summation over all solutions in positive integers of the equations $i + 2j + 3h + \cdots + lk = \nu$ and $m = i + j + h + \cdots + k$. Define

$$\overline{\Phi^{(\nu)}(\psi)} = \sum \frac{\nu!}{i!\,j!\,h!\cdots k!}\,|\Phi^{(m)}(0)|\,\left\|\frac{\psi^{(1)}}{1!}\right\|^{i}\left\|\frac{\psi^{(2)}}{2!}\right\|^{j}\left\|\frac{\psi^{(3)}}{3!}\right\|^{h}\cdots\left\|\frac{\psi^{(l)}}{l!}\right\|^{k}. \tag{23}$$

The following estimates hold:

$$(\Phi(\psi))^{(\nu)} \le \overline{\Phi^{(\nu)}(\psi)}, \qquad [m(\psi)]^{(\nu)} \le c_1\overline{\Phi^{(\nu)}(\psi)}$$

and

$$\mathfrak{N}[(m(\psi))^{(\nu)}] \le c_0\,\|[m(\psi)]^{(\nu)}\| \le c_2\overline{\Phi^{(\nu)}(\psi)}$$

where

$$c_1 = \left\|\int_{R_0} G(\mathbf{x}, \mathbf{x}_0)\,dR_0\right\| \quad \text{and} \quad c_2 = c_1c_0\,.$$

From these estimates we find that

$$\|\psi^{(\nu)}\|\,\mathfrak{N}[m(0)] \le c_3\left\{\nu\overline{\Phi^{(\nu-1)}(\psi)} + \sum_{r=1}^{\nu-1}\binom{\nu}{r}\|\psi^{(r)}\|\,\overline{\Phi^{(\nu-r)}(\psi)}\right\} \tag{24}$$

where $c_3 = \max\,[c_1, c_0]$.

THEOREM 3.[4] *Let $\Phi(\psi)$ be regular analytic when $|\psi| < \omega > 0$ and $\Phi(0) \ne 0$. Let the functions $\psi^{(\nu)}(\mathbf{x})$ be defined recursively by* (21). *There exists $|A_0| > 0$ such that the Taylor series*

$$\psi(\mathbf{x}, A) = \sum_{\nu=1}^{\infty}\frac{1}{\nu!}\,\psi^{(\nu)}(\mathbf{x})A^\nu \tag{25}$$

converges when $|A| \le |A_0|$ and represents a solution of (20).

Proof. Since $\Phi(\psi)$ is analytic it is absolutely convergent when $|\psi| < \omega$. Let the analytic function defined by absolute convergence be $\phi(\psi)$. Define the analytic function $\tau(A)$ implicitly through the equation

$$H(\tau, A) = \beta\tau - A\phi(\tau) + \tau(\phi(\tau) - \phi(0)) = 0 \tag{26}$$

where

$$\phi(\tau) = \sum_{l=0}^{\infty}\frac{\phi^{(l)}}{l!}\,\tau^l, \qquad \phi^{(l)} = |\Phi^{(l)}(0)| \tag{26a}$$

and

$$\beta = \mathfrak{N}[m(0)]/c_3 > 0. \tag{26b}$$

[4] We are indebted to P. Fife, University of Arizona, for considerable help with the theorem given here and for calling our attention to the fact that the result could also be obtained by verifying the hypotheses of a theorem on implicit operators (Vainberg and Trenogin [25, Theorem 3.1, p. 19]). The details of our proof are essentially unaltered if $\Phi(\psi)$ is replaced with $\Phi(\mathbf{x}, \psi)$.

$H(\tau, A)$ is the sum of products of analytic functions. $H(0, 0) = 0$ and $\partial H(0, 0)/\partial\tau = \beta > 0$, so that the implicit function theorem guarantees the existence of the analytic function $\tau(A)$ such that $\tau(0) = 0$. The derivatives of this analytic function satisfy the relations

$$\beta\tau^{(\nu)} = \nu[\phi(\tau)]^{(\nu-1)} + \sum_{r=1}^{\nu-1} \binom{\nu}{r} \tau^{(r)} [\phi(\tau)]^{(\nu-r)} \tag{27}$$

with $\beta\tau^{(1)} = \phi(0) > 0$. Since $\phi^{(\nu)} > 0$, it follows from an easy induction that $\tau^{(\nu)} \geq 0$ for every ν. Since (24), (23), and (26a) imply that

$$\|\psi^{(1)}\|\beta = |\Phi(0)| = \phi(0) = \beta\tau^{(1)},$$

we find by comparison of (24) and (27) that $\|\psi^{(\nu)}\| \leq \tau^{(\nu)}$, so that, by majorization, the series

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu!} \psi^{(\nu)} A^{\nu}$$

converges uniformly in R. To show that this convergent series satisfies (20), rewrite (21) as

$$\sum_{r=0}^{\nu} \binom{\nu}{r} \psi^{(r)} \mathfrak{M}[m(\psi)]^{(\nu-r)} = \nu[m(\psi)]^{(\nu-1)}.$$

Then multiply the foregoing by $A^{\nu}/\nu!$ and sum from one. On the right

$$\sum_{\nu=1}^{\infty} \frac{A^{\nu}}{(\nu-1)!} [m(\psi)]^{(\nu-1)} = A \sum_{l=0}^{\infty} \frac{A^{l}}{l!} \int_{R_0} G(\mathbf{x}, \mathbf{x}_0)(\Phi[\psi(\mathbf{x}_0)])^{(l)}\, dR_0 = Am(\psi).$$

On the left

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu!} \sum_{r=0}^{\nu} \binom{\nu}{r} A^{r}\psi^{(r)} \mathfrak{M}[A^{\nu-r}(m(\psi))^{(\nu-r)}] = \sum_{l=1}^{\infty} \frac{A^{l}\psi^{(l)}}{l!} \sum_{\nu=0}^{\infty} \frac{A^{\nu}}{\nu!} \mathfrak{M}[(m(\psi))^{(\nu)}] = \psi\mathfrak{M}[m(\psi)],$$

thereby proving Theorem 3.

V. Some remarks about the circle of convergence. The statement and proof of Theorem 3 suggest that the condition $\mathfrak{M}[m(\psi)] \neq 0$ which is necessary for the existence of the analytic perturbation in powers of $A(=\mathfrak{M}[\psi])$ may also characterize the radius of convergence of the perturbation series. Although it seems difficult to carry out this idea for all A analytic solutions of (20), it is possible to make some definite statements about the convergence radius for the simplest class of problems governed by (20), that is, relative to the problem

$$\frac{d^2\psi}{dx^2} + \lambda(A)\Phi(\psi) = 0$$
$$\psi(0) = A, \qquad \frac{d\psi(0)}{dx} = 0, \qquad \psi(1) = 0. \tag{28}$$

The existence of the analytic solution

$$\psi(x, A) = \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \frac{\partial^{\nu}\psi(x, 0)}{\partial A^{\nu}} A^{\nu} \tag{29}$$

and

$$\lambda(A) = \sum_{\nu=1} \frac{1}{\nu!} \frac{d^{\nu}\lambda(0)}{dA^{\nu}} A^{\nu} \tag{30}$$

of (28) is guaranteed by Theorem 3 when A is small enough. The coefficients for the series (29) are constructed according to procedure given in the proof of Theorem 3. The coefficients for (30) are obtained by differentiating

$$\lambda = A/m[\psi(x, A)]$$

with respect to A at $A = 0$.

The system (28) has the energy integral

$$x(2\lambda)^{1/2} = \int_{\psi}^{A} \frac{dy}{(\Omega(A) - \Omega(y))^{1/2}}, \qquad 0 \leq x \leq 1 \tag{31}$$

where

$$\Phi(y) \equiv \frac{d\Omega}{dy} \quad \text{and} \quad (2\lambda)^{1/2} = \int_{0}^{A} \frac{dy}{(\Omega(A) - \Omega(y))^{1/2}}. \tag{32}$$

The integrals may be used to continue the series. In particular, *if* $\Phi(\psi)$ *is regular analytic for* $|\psi| \leq |\xi_0|$, *where* ξ_0 *is the first complex zero of* $\Phi(\xi) = 0$, *then the series* (29) *and* (30) *cannot converge when* $|A| \geq |\xi_0|$.

To prove this, note that when $A = \xi_0$

$$\Omega(\xi_0) - \Omega(y) = \frac{1}{2!} \frac{d\Phi}{d\xi} (\xi_0)(y - \xi_0)^2 + \cdots$$
$$= g(\xi_0, y)(y - \xi_0)^r$$

where $r \geq 2$ and $0 < |g(\xi_0, \xi_0)| < \infty$. Then, since

$$\lim_{\epsilon \to 0} \int_{\xi_0 - \epsilon}^{\xi_0} \frac{dy}{(\Omega(\xi_0) - \Omega(y))^{1/2}} = \lim_{\epsilon \to 0} \int_{\xi_0 - \epsilon}^{\xi_0} \frac{1}{g(\xi_0, y)} \frac{dy}{(y - \xi_0)^{r/2}}$$

diverges, the integrals (31) and (32) do not exist for $A = \xi_0$ and the series (29) and (30) cannot converge when $|A| \geq |\xi_0|$.

We can also show that *if* $|A| < |\xi_0|$, *then the series* (30) *converges*. Let $\Psi = \Psi(\Omega(A), y^2)$ be defined by the equation

$$y^2 = \Omega(A) - \Omega(\Psi) = \int_{\Psi}^{A} \Phi(\gamma)\, d\gamma. \tag{33}$$

Let $D(\Psi)$ be any domain on which $\Phi(\Psi) \neq 0$ and $\Phi(\Psi)$ is analytic and let $\Gamma(\Omega(A), y)$ be the image of D under (33). Since

$$d\Psi = d\Omega(A)/\Phi(\Psi) \quad \text{and} \quad d\Psi = -y\, dy/\Phi(\Psi),$$

$\Psi(\Omega(A), y^2)$ is an analytic function on Γ. We change variables in (32) and find that

$$(2\lambda)^{1/2} = \int_{(\Omega(A) - \Omega(0))^{1/2}}^{0} \frac{dy}{d\Omega} \frac{d[\Omega(y) - \Omega(A)]}{(\Omega(A) - \Omega(y))^{1/2}} = 2 \int_{0}^{(\Omega(A) - \Omega(0))^{1/2}} \frac{dy}{\Phi(\Psi[\Omega(A), y^2])}$$

where $\Psi(\Omega(A), 0) = A$ and $\Psi(\Omega(A), \Omega(A) - \Omega(0)) = 0$ are the end points of paths of integration in D. If $|A| < |\xi_0|$, we can integrate over paths lying entirely in Γ. It is, therefore, seen that the $\sqrt{\lambda}$ is an analytic function of $(\Omega(A) - \Omega(0))^{1/2}$, and $\lambda(\Omega(A) - \Omega(0))$ is also analytic for $|A| < |\xi_0|$.

There are a few functions $\Phi(\psi)$ for which exact solutions of (28) can be constructed.

Example 1. The function $\Phi(\psi) = e^\psi$ has no zeros in the finite ψ plane and, therefore, $\lambda(A)$ must be an entire function. One finds from (31) and (32) that

$$e^{\psi/2} = \frac{\cosh(\sqrt{\lambda}e^{A/2}/\sqrt{2})}{\cosh(x\sqrt{\lambda}e^{A/2}/\sqrt{2})} \quad \text{and} \quad e^{A/2} = \cosh(\sqrt{\lambda}e^{A/2}/\sqrt{2}),$$

so that $\psi(x, A)$ and $\lambda(A)$ are entire functions of A. Although this solution is uniquely determined by A, there is a maximum value $\hat{\lambda}(A^*) = \max_{A \geq 0} \lambda(A) = 0.893$ for $A^* = 1.18$. The estimate (11) gives $\hat{\lambda} < 0.91$. For each $\lambda < \hat{\lambda}$, there are exactly two solutions so that a power series for $\psi(x, \lambda)$ in powers of λ cannot converge to the solutions on the branch $A > 1.18$.

Example 2. The function $\Phi(\psi) = 1 + \psi + \gamma\psi^2$ has a smallest root given by

$$\xi_0(\gamma) = -\frac{1}{2\gamma}\left\{1 - (1 - 4\gamma)^{1/2}\right\}.$$

$|\xi_0(\gamma)|$ increases with γ from the value $\xi_0 = -1$ when $\gamma = 0$ to the value $\xi_0 = -2$ when $\gamma = \frac{1}{4}$. For values of $\gamma \geq \frac{1}{4}$, $|\xi_0| = 1/\sqrt{\gamma}$.

It follows that the perturbation series for $\lambda(A)$ never converges for $|A| \geq 2$ and the convergence radius tends to zero like $1/\sqrt{\gamma}$ for large γ. For real A, the solution has much in common with that given in the first example. Unique solutions exist for everv $A \geq 0$, but $\lambda(A)$ has a maximum $\hat{\lambda}$ and for each $\lambda < \hat{\lambda}$ there are two solutions with different values of A. An exact solution of this problem may be found in terms of elliptic functions of the first kind [15]. When $\gamma = 0$, the exact solution may be represented as

$$\psi + 1 = \cos\sqrt{\lambda}x/\cos\sqrt{\lambda} \quad \text{and} \quad 1/(A + 1) = \cos\sqrt{\lambda}.$$

$\lambda(A)$ and $\psi(x, A)$ are regular analytic functions in the unit circle.

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