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## Uniqueness Criteria for the Conduction-Diffusion Solution of the Boussinesq Equations

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Energy stability theory gives sufficient conditions for the exponential stability of basic fluid motions [1]. If the basic motion is steady, the energy criterion is also sufficient for uniqueness [2]. However, since it is sufficient to guarantee the exponential decay of arbitrary disturbances at all times, it can be overly conservative as a criterion for uniqueness. In the class of steady solutions one can find less conservative criteria. But since the disturbance of a steady state need not itself be steady, these criteria are here established only for uniqueness (and not for stability) of steady states.

The results which follow continue the earlier work [3] of SHIR & JOSEPH. That work reports the best results possible relative to the simplest energy functionals for the disturbance of the motionless solution of the Boussinesq equations for a chemically inhomogeneous (say, salty) fluid. In this work, we add an additional integral constraint which leads to a variational problem whose solution gives a uniqueness criterion for the steady, nonlinear problem.

Whereas the old energy stability-uniqueness limit did not depend on the Prandtl and Schmidt numbers, the new limits do depend on these parameters, not as in the exact nonlinear problem on their separate values, but on their ratio.

In the earlier work [3] it was established that a motionless fluid supporting a uniform, stabilizing salt gradient and a uniform, destabilizing temperature gradient normal to stress-free, horizontal boundaries is sublinearly stable in the limit of very large Schmidt numbers. Now, this result can be shown to hold in the limit of very small Schmidt numbers or very large Prandtl numbers.

We want to study the stability and uniqueness of the motionless conduction-diffusion solution of the Boussinesq equations.\* It is an elementary consequence of these equations that a motionless state is possible only when the gravity-gradient and density-gradient vectors are collinear. We assume that this is the case. Then, the partial differential equations which govern the disturbance of the motionless state are [3]:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p - (\mathcal{R}\theta - \mathcal{C}c) \mathbf{H} + \Delta \mathbf{u}, \quad (1)$$

$$Pr \left( \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta \right) + \mathcal{R} \mathbf{u} \cdot \mathbf{H}_T = \Delta \theta, \quad (2)$$

\* The motionless state is studied for simplicity. No additional complication is introduced by starting with a nontrivial basic motion.

and

$$Sc \left( \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c \right) + \mathcal{C} \mathbf{u} \cdot \mathbf{H} c = \Delta c, \quad (3)$$

where

$$g \mathbf{H} = \mathbf{g}, \quad \beta_T \mathbf{H}_T = \nabla T \quad \text{and} \quad \beta_c \mathbf{H}_c = \nabla C$$

are the gravity, temperature gradient and concentration gradient for the motionless state, respectively, and  $\mathcal{R}^2 = \alpha_T g \beta_T l^4 / \nu \kappa_T$ ,  $\mathcal{C}^2 = \alpha_c g \beta_c l^4 / \nu \kappa_c$ ,  $Pr = \nu / \kappa_T$  and  $Sc = \nu / \kappa_c$  are, respectively, heat Rayleigh, solute Rayleigh, Prandtl number and Schmidt number. The letters  $\mathbf{u}$ ,  $\theta$ ,  $c$ ,  $p$  stand for disturbance velocity (solenoidal), temperature, concentration and pressure.

The equations (1), (2) and (3) are to hold on the arbitrary domain  $\mathcal{V}$ . Let  $\mathcal{S}$  be a part of the boundary  $\partial \mathcal{V}$  of  $\mathcal{V}$ . On  $\mathcal{S}$  we require

$$\mathbf{u} = 0 \quad \text{or} \quad \mathbf{u} \cdot \mathbf{N} = (\mathbf{N} \cdot \mathbf{d}) \times \mathbf{N} = 0, \quad (4)$$

where  $\mathbf{d}$  is the strain-rate tensor for the disturbance, and  $\mathbf{N}$ , the outward normal on  $\mathcal{S}$ . For the scalar field, we require that

$$\frac{\partial \theta}{\partial N} + Nu \ell(\mathbf{x}) \theta = 0, \quad (5)$$

and

$$\frac{\partial c}{\partial N} + Sh g(\mathbf{x}) c = 0, \quad (6)$$

where  $Nu$  and  $Sh$  are the Nusselt and Sherwood numbers, and  $\ell(\mathbf{x})$  and  $g(\mathbf{x})$  are piecewise continuous functions defined on  $\partial \mathcal{V}$ . Some appropriate combination of (4), (5) and (6) holds on each part  $\mathcal{S}$  of  $\partial \mathcal{V}$ .

From equations (1) through (6), one can find the integral identities

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{V}} |\mathbf{u}|^2 = - \int_{\mathcal{V}} (2\mathbf{d} : \mathbf{d} + (\mathcal{R}\theta - \mathcal{C}c) \mathbf{H} \cdot \mathbf{u}), \quad (7)$$

$$\frac{Pr}{2} \frac{d}{dt} \int_{\mathcal{V}} \theta^2 = - \int_{\mathcal{V}} (\mathcal{R} \theta \mathbf{u} \cdot \mathbf{H}_T + \nabla \theta \cdot \nabla \theta) - Nu \oint_{\partial \mathcal{V}} \ell(\mathbf{x}) \theta^2, \quad (8)$$

$$\frac{Sc}{2} \frac{d}{dt} \int_{\mathcal{V}} c^2 = - \int_{\mathcal{V}} (\mathcal{C} c \mathbf{u} \cdot \mathbf{H}_c + \nabla c \cdot \nabla c) - Sh \oint_{\partial \mathcal{V}} g(\mathbf{x}) c^2, \quad (9)$$

and

$$\begin{aligned} Sc \frac{d}{dt} \int_{\mathcal{V}} \theta c + (1 + \tau) \int_{\mathcal{V}} \nabla \theta \cdot \nabla c + \oint_{\partial \mathcal{V}} (\tau Nu \ell(\mathbf{x}) + Sh g(\mathbf{x})) c \theta \\ + \mathcal{C} \int_{\mathcal{V}} \theta \mathbf{u} \cdot \mathbf{H}_c + \mathcal{R} \tau \int_{\mathcal{V}} c \mathbf{u} \cdot \mathbf{H}_T = 0, \end{aligned} \quad (10)$$

where  $\tau = Sc/Pr^*$ . Equations (4) through (9) are the basis of the energy results of SHIR & JOSEPH [3].

\*  $\tau$  is the reciprocal of the Lewis number  $Le = Pr/Sc$ . In the notation of [3],  $\gamma = Le = 1/\tau$ .

This paper is concerned with (10). To derive (10), we note that if  $\mathbf{u}$  is such that  $\nabla \cdot \mathbf{u} = 0$  and  $\mathbf{u} \cdot \mathbf{N} = 0|_{\partial\mathcal{V}}$ , then

$$\int_{\mathcal{V}} [\theta(\mathbf{u} \cdot \nabla)c + c(\mathbf{u} \cdot \nabla)\theta] = 0.$$

To find (10), we need only write out this equation using (2) and (3) and the boundary conditions.

Consider the set of all solutions of (1) through (6) such that time derivatives in (7) through (10) vanish. This includes all steady solutions. Define the functionals:

$$\mathcal{I}_0(\mathbf{u}, \theta, c) = \int_{\mathcal{V}} (\mathcal{R}\theta - \mathcal{C}c) \mathbf{H} \cdot \mathbf{u},$$

$$\mathcal{I}_T(\mathbf{u}, \theta) = \int_{\mathcal{V}} \mathbf{H}_T \cdot \mathbf{u} \theta,$$

$$\mathcal{I}_c(\mathbf{u}, \theta) = \int_{\mathcal{V}} \mathbf{H}_c \cdot \mathbf{u} \theta,$$

$$\mathcal{D}_0(\mathbf{u}, \mathbf{u}) = 2 \int_{\mathcal{V}} d : d,$$

$$\mathcal{D}_1(\theta, c) = \int_{\mathcal{V}} \nabla\theta \cdot \nabla c + \oint_{\partial\mathcal{V}} (\tau Nu \ell(\mathbf{x}) + Sh g(\mathbf{x})) \theta c,$$

$$\mathcal{D}_2(\theta, \theta) = \int_{\mathcal{V}} \nabla\theta \cdot \nabla\theta + \oint_{\partial\mathcal{V}} Nu \ell(\mathbf{x}) \theta^2,$$

$$\mathcal{D}_3(c, c) = \int_{\mathcal{V}} \nabla c \cdot \nabla c + \oint_{\partial\mathcal{V}} Sh g(\mathbf{x}) c^2,$$

$$\mathcal{I} = \mathcal{I}_0 + \lambda_T \mathcal{R} \mathcal{I}_T(\mathbf{u}, \theta) + \lambda_c \mathcal{C} \mathcal{I}_c(\mathbf{u}, c),$$

$$\mathcal{D} = \mathcal{D}_0 + \lambda_T \mathcal{D}_2 + \lambda_c \mathcal{D}_3,$$

and

$$\mathcal{F} = (1 + \tau) \mathcal{D}_1 + \mathcal{C} \mathcal{I}_c(\mathbf{u}, \theta) + \tau \mathcal{R} \mathcal{I}_T(\mathbf{u}, c).$$

It is easy to see from (7, 8, 9) that

$$1 = -\frac{\mathcal{I}}{\mathcal{D}} \tag{11}$$

for every steady solution of disturbance equations (1)–(6). Clearly,

$$1 \leq \sup \left( -\frac{\mathcal{I}}{\mathcal{D}} \right), \tag{12}$$

where the supremum is over functions  $\mathbf{u}, \theta, c$  such that  $\mathbf{u} \cdot \mathbf{N} = 0|_{\partial\mathcal{V}}$ ,  $\nabla \cdot \mathbf{u} = 0|_{\mathcal{V}}$  and  $\mathbf{u}, \theta, c$  vanish on any measurable set of boundary points for which this is prescribed by (4), (5) and (6).

Let  $S(\mathcal{R}, \mathcal{C})$  be the set of values  $\mathcal{R} \geq 0, \mathcal{C} \geq 0$  for which (12) can be satisfied. It is shown in [3] that (12) cannot hold when  $(\mathcal{R}^2 + \mathcal{C}^2)$  is sufficiently small. Fix  $\mathcal{C} \in S$  and define

$$\tilde{\mathcal{R}}_\lambda(\lambda_T, \lambda_c, Nu, Sh, \mathcal{C}) = \inf_{\mathcal{R} \in S} \mathcal{R}.$$

Clearly, if for the given  $\mathcal{C}$ ,  $\mathcal{R} < \tilde{\mathcal{R}}_\lambda$ , then (11) cannot hold, and steady convection is not possible. In fact, only exponentially stable disturbances of the motionless state are possible.

It is known [3, Theorem 8] that when

$$-k = \mathbf{H} = \mathbf{H}_T = -\mathbf{H}_c \quad (13)$$

(salt and heat are destabilizing and  $k$  is up), then  $\lambda_T = \lambda_c = 1$  and  $\mathcal{R}_L = \tilde{\mathcal{R}}_\lambda = \tilde{\mathcal{R}}$  where  $\mathcal{R}_L$  is the linear limit. This limit is then both necessary and sufficient for stability. But when (13) does not hold, then it is not possible on the basis of energy and linear theory to exclude sublinear instabilities. For example, when

$$-k = \mathbf{H} = \mathbf{H}_T = \mathbf{H}_c \quad (14)$$

(salt is stabilizing and heat is destabilizing), stability can be guaranteed ([3] Theorem 9) when  $\mathcal{R} < \sqrt{Ra^*}$  where  $Ra^*$  is the linear limit for heat alone ( $Ra^* = 27 \pi^4/4$  between stress-free planes). It happens that when  $Sc \rightarrow \infty$ , the linear and energy limits coincide. Then, we have, once again, the coincidence of necessary and sufficient conditions for instability. Since the energy bound is independent of  $Pr$  and  $Sc$ , the coincidence shows the energy estimate is the best one possible.

The stability criterion  $\mathcal{R} < \sqrt{Ra^*}$  for (14) seems obvious enough on physical grounds. If the fluid is stable to small disturbances when there is no salt gradient, it might be expected to be stable to large disturbances when a stabilizing salt gradient is added. What is not at all obvious is that any criterion better than  $\mathcal{R} < \sqrt{Ra^*}$  will suffice to guarantee exponential decay for large disturbances. In fact, there is no better criterion if  $Sc \rightarrow \infty$ , and a much better criterion for other values of  $Pr$  and  $Sc$  may not exist. But if one does not ask for such strong stability and is content with a criterion of uniqueness\* alone, then equation (10) can be used and allows for considerable improvements in the criterion for uniqueness.

The preceding remarks hold, of course, for situations which are less idealized than the stress-free layer.

**Theorem.** *Let  $\bar{\Omega}(\mathcal{R}, \mathcal{C})$  be the closure of that subset of the set  $S(\mathcal{R}, \mathcal{C})$  on which (12) holds and for which  $\mathcal{F} = 0$ . There are no steady solutions of (1) through (6) on the complement of  $\bar{\Omega}(\mathcal{R}, \mathcal{C})$ .*

The proof is obvious, because on the complement of  $\bar{\Omega}$  it is not possible to satisfy (11). To find  $\bar{\Omega}$ , we employ standard procedures of variational calculus and require that

$$-\delta \left( \frac{\mathcal{J} + 2\mu \mathcal{F}}{\mathcal{D}} \right) = 0, \quad (16)$$

where  $\mu$  is a multiplier. The values  $\mathcal{R}$  and  $\mathcal{C}$  are in  $\Omega$  if the inequality (12) holds, and  $\bar{\Omega}$  is defined by the equality. The multiplier  $\mu$  is determined for positive values  $\mathcal{R}$  and  $\mathcal{C}$  and maximizing functions such that  $\sup \left( -\frac{\mathcal{J}}{\mathcal{D}} \right) = 1$  by the condition that  $\mathcal{F} = 0$ .

\* If the limit flow of a disturbance of the conduction-diffusion solution is one for which time derivatives on the left of (7–10) are zero (sometimes called “statistically stationary”), then uniqueness and stability are two names for the same thing.

The Euler equations for the variational problem (16) are

$$2\nabla^2 \mathbf{u} = \theta \{ \mathcal{R} \mathbf{H} + 2\mu \mathcal{C} \mathbf{H}_c + \lambda_T \mathcal{R} \mathbf{H}_T \} \\ + c \{ \lambda_c \mathcal{C} \mathbf{H}_c - \mathcal{C} \mathbf{H} + 2\mu \tau \mathcal{R} \mathbf{H}_T \} + 2\nabla p, \quad (17)$$

$$2\nabla^2 \theta + 2\mu(1+\tau)\nabla^2 c = (\mathcal{R} \mathbf{H} + \lambda_T \mathcal{R} \mathbf{H}_T + 2\mu \mathcal{C} \mathbf{H}_c) \cdot \mathbf{u}, \quad (18)$$

and

$$2\nabla^2 c + 2\mu(1+\tau)\nabla^2 \theta = (\lambda_c \mathcal{C} \mathbf{H}_c - \mathcal{C} \mathbf{H} + 2\mu \tau \mathcal{R} \mathbf{H}_T) \cdot \mathbf{u}. \quad (19)$$

Here, the field  $p$  arises through the condition that  $\nabla \cdot \mathbf{u} = 0$ . The equations are to be solved relative to boundary conditions (4), (5) and (6).

The Theorem holds relative to any domain and class of functions for which the indicated operations are valid. Bounded domains are included and so are fluid layers, provided that the function class considered is restricted to periodic disturbances.

We consider next the fluid layer bounded by stress-free, conducting surfaces. This is the only problem for which even approximate nonlinear results exist, and we can solve it explicitly. The basic motionless state consists of linear temperature and concentration profiles satisfying (13) or (14). For (13), it is known that the motionless state is sublinearly stable [3]. Greater interest accrues to (14), in which the salt gradient is stabilizing. For this problem, we can write the Euler equations as

$$\nabla^4 w + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) [(\mathcal{R} + \mu \mathcal{C})\theta + \tau \mu \mathcal{R} c] = 0, \quad (20)$$

$$\nabla^2 \theta + (\mathcal{R} + \mu \mathcal{C})w + \mu(1+\tau)\nabla^2 c = 0, \quad (21)$$

and

$$\nabla^2 c + \tau \mu \mathcal{R} w + \mu(1+\tau)\nabla^2 \theta = 0, \quad (22)$$

with

$$w = \theta = c = \frac{\partial^2 w}{\partial z^2} = 0 \quad @ z = \pm \frac{1}{2}, \quad (23)$$

where

$$\lambda_T = \lambda_c = 1.$$

Equation (20) follows as the vertical ( $z$ ) component of the double curl of the equation for  $\mathbf{u}$ . Solutions of (20) to (23) can be found in the form

$$\begin{bmatrix} w(x, y, z) \\ \theta(x, y, z) \\ c(x, y, z) \end{bmatrix} = \begin{bmatrix} \hat{W} \\ \hat{\Theta} \\ \hat{C} \end{bmatrix} \cos n \pi z f(x, y),$$

where

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -a^2 f.$$

All the doubly periodic solutions have this last property. Let  $N^2 + a^2 = \sigma$ . The constants  $\hat{W}$ ,  $\hat{\Theta}$ ,  $\hat{C}$  satisfy the reduced forms of (20, 21, 22),

$$\sigma^2 \hat{W} - a^2 (\mathcal{R} + \mu \mathcal{C}) \hat{\Theta} - a^2 \tau \mu \mathcal{R} \hat{C} = 0, \quad (24)$$

$$(\mathcal{R} + \mu \mathcal{C}) \hat{W} - \sigma \hat{\Theta} - \mu(1 + \tau) \sigma \hat{C} = 0, \quad (25)$$

$$\tau \mu \mathcal{R} \hat{W} - \mu(1 + \tau) \sigma \hat{\Theta} - \sigma \hat{C} = 0, \quad (26)$$

and, to have  $\mathcal{F} = 0$ ,

$$\sigma(1 + \tau) \hat{C} \hat{\Theta} - \mathcal{C} \hat{\Theta} \hat{W} - \mathcal{R} \tau \hat{C} \hat{W} = 0. \quad (27)$$

Equations (24), (25) and (26) have a solution if

$$\begin{aligned} \gamma(1 - \mu^2(1 + \tau)^2) - (\mathcal{R}^2 + 2\mu\mathcal{C}\mathcal{R} + \mu^2\mathcal{C}^2 + \mu^2\tau^2\mathcal{R}^2) \\ + 2\mu^2\tau(1 + \tau)\mathcal{R}(\mathcal{R} + \mu\mathcal{C}) = 0, \end{aligned} \quad (28)$$

where  $\gamma = \sigma^3/a^2$ . These solutions are compatible with (27) provided that

$$\begin{aligned} \gamma(1 + \tau) \{ \tau \mathcal{R} \mu (\mathcal{R} + \mu \mathcal{C}) (1 + \mu^2 (1 + \tau)^2) \\ - \mu (1 + \tau) (\mathcal{R}^2 + 2\mu \mathcal{C} \mathcal{R} + \mu^2 \mathcal{C}^2 + \tau^2 \mu^2 \mathcal{R}^2) \} \\ = \{ \mu \mathcal{C}^2 - \mu \tau \mathcal{R}^2 + \mathcal{C} \mathcal{R} [1 - 2\tau \mu^2 (1 + \tau)] \} \\ \cdot \{ \mathcal{R}^2 + 2\mu \mathcal{C} \mathcal{R} + \mu^2 \mathcal{C}^2 + \tau^2 \mu^2 \mathcal{R}^2 - 2\mu^2 \tau \mathcal{R} (1 + \tau) (\mathcal{R} + \mu \mathcal{C}) \}. \end{aligned} \quad (29)$$

It will be convenient in equations (28) and (29) to set

$$\mathcal{C} = \alpha \mathcal{R} \quad (0 \leq \alpha \leq \infty).$$

Then, elimination of  $\gamma$  in (28) and (29) leads to

$$\begin{aligned} (1 + \tau) \{ (\tau \mu + \tau \mu^2 \alpha) (1 + \mu^2 (1 + \tau)^2) - (1 + \tau) (\mu + 2\mu^2 \alpha + \mu^3 \alpha^2 + \tau^2 \mu^3) \} \\ = (1 - \mu^2 (1 + \tau)^2) \{ \alpha + \mu \alpha^2 - \tau \mu - 2\mu^2 (1 + \tau) \tau \alpha \}. \end{aligned}$$

If we cancel terms and rearrange, this reduces to

$$-\frac{\alpha}{1 + \alpha^2} = \frac{\mu}{h(\mu^2, \tau)} \quad (30)$$

where

$$h(\mu^2, \tau) = \mu^2 + (1 - \mu^2 \tau (\frac{1}{2} + \tau))^2 + \tau \mu^4 (1 + 2\tau^2 + 11\tau/4)$$

is a concave function of  $\mu^2$ . Equation (30) determines two real negative roots  $\mu$  for each preassigned positive  $\alpha$ . There is a double root, however, where  $\alpha = 1$ , and here

$$\mu = -\frac{1}{1 + \tau}.$$

In the  $\alpha, \mathcal{R}$  variables, we can write (28) as

$$\frac{\gamma}{\mathcal{R}^2} (1 - \mu^2 (1 + \tau)^2) = (1 + 2\mu\alpha + \mu^2\alpha^2 + \mu^2\tau^2) - 2\mu^2\tau(1 + \tau)(1 + \mu\alpha). \quad (31)$$

To find the minimum value  $\mathcal{R}^2$ , we must take the smallest value of  $\gamma = (N^2 \pi^2 + a^2)^3/a^2$ , i.e.,  $\gamma = 27 \pi^4/4$ ,  $N = 1$ ,  $a^2 = \pi^2/2$ , and find the values of  $\mu$  for given  $\alpha, \tau$  which make  $\mathcal{R}^2$  smallest. For small  $\alpha$ , there are the roots of (30)  $\mu = -\alpha$  and  $\mu = -1/(\alpha\tau)^{1/3}(1 + \tau)$ . Inspection of (31) shows that  $-\alpha$  is the root which gives

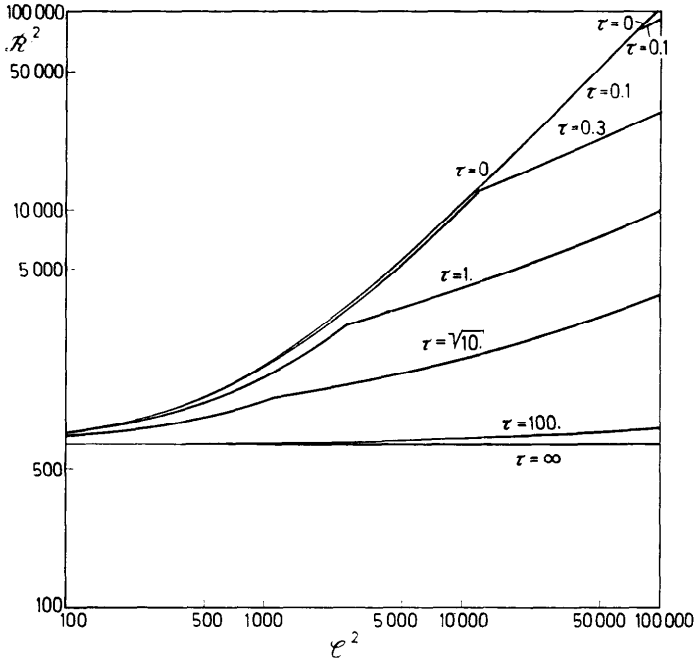


Fig. 1. Uniqueness limits for the conduction-diffusion solution of the nonlinear Boussinesq equations. The limits hold for periodic disturbances in a fluid layer bounded by stress-free, conducting surfaces and supporting a stabilizing, salt gradient and a destabilizing, temperature gradient. The kinks in the nonlinear limit appear also in linear analysis and are associated with time steadiness or unsteadiness in the instability limit [9]. For each  $\tau$ , there is only one steady solution when  $\mathcal{R}^2, \mathcal{C}^2$  are below the appropriate uniqueness boundary.

the smallest  $\mathcal{R}^2$ . The left and right of (31) are then positive when  $\alpha$  is small. The sign of the coefficient of  $\gamma/\mathcal{R}^2$  changes when we pass through the double root at  $\alpha=1$ . For  $\alpha < 1$ , we stay on the branch  $\mu(\tau)$  for which the root  $|\mu|$  is smallest. At  $\alpha=1$ , there is a crossover, and this branch becomes the larger of the two. But it is again the smallest absolute value of the two roots  $\mu$  which makes  $\mathcal{R}^2$  smallest, so that at  $\alpha=1$ , we must change branches. Just as in the linear theory (see [3] for a summary of the linear result of [9]) there is a kink in the uniqueness boundary  $\tilde{\mathcal{R}}^2(\mathcal{C}^2)$  (see Fig. 1). This kink moves to the right in  $\mathcal{R}^2, \mathcal{C}^2$  plane as  $\tau$  is decreased from infinity to zero. At  $\tau=0$ , the uniqueness boundary  $\tilde{\mathcal{R}}^2(\mathcal{C}^2, 0)$  is found from (31) as

$$\frac{27\pi^4}{4} = \tilde{\mathcal{R}}^2(1 - \alpha^2) = \tilde{\mathcal{R}}^2 - \mathcal{C}^2,$$

so that the crossover point  $\tilde{\mathcal{R}}^2 = \mathcal{C}^2$  no longer exists.

It is known [9, 4] that when  $\tau=0$ ,

$$\mathcal{R}_L^2 = \frac{27}{4} \pi^4 + \mathcal{C}^2,$$

where  $\mathcal{R}_L^2$  is the linear instability limit. Since the motionless conduction-diffusion solution is unique when  $\mathcal{R} < \tilde{\mathcal{R}}$  and since  $\tilde{\mathcal{R}} = \mathcal{R}_L$ , steady sublinear convection cannot exist when  $\mathcal{R} < \tilde{\mathcal{R}} = \mathcal{R}_L$ .

It is easy to show from (30) and (31), but is already known [3], that sublinear solutions can also be excluded when  $Sc \rightarrow \infty$ . We have proved that

*The steady motionless state of a Boussinesq fluid layer (the problem 20–23) heated and concentrated below is unique in the class of periodic functions provided that*

$$\mathcal{R} < \tilde{\mathcal{R}}(\mathcal{C}, \tau)$$

where  $\tilde{\mathcal{R}}(\mathcal{C}, \tau)$  is determined by (30) and (31). Moreover,

$$\tilde{\mathcal{R}}^2(\mathcal{C}, 0) = \mathcal{R}_L^2 = \frac{27\pi^4}{4} + \mathcal{C}^2$$

and

$$\tilde{\mathcal{R}}^2(\mathcal{C}, \infty = \infty/Pr) = \mathcal{R}_L^2 = \frac{27\pi^4}{4},$$

so that sublinear steady convection cannot exist when  $Sc \rightarrow 0$  or  $Sc \rightarrow \infty$ .

Equations (30) and (31) also hold for domains bounded by combinations of rigid and free surfaces provided  $c$  and  $\theta$  vanish on the boundary. This follows from the fact that (21) and (22) show that  $\mu\{\mathcal{R} + \mu\mathcal{C} + \tau\mu\mathcal{C}\}\theta + \{\mathcal{R} + \mu\mathcal{C} - \mu^2\mathcal{R}\tau(1+\tau)\}c$  is harmonic in  $\mathcal{V}$  and therefore vanishes everywhere in  $\mathcal{V}$ . Elimination of  $c$  then enables one to find (30) and (31) directly without the exact solution. Of course, the quantity  $\gamma$  in (31) is then the Bénard eigenvalue (1708 between rigid plates, etc.).

It is always a question, when the energy and linear limits do not coincide, whether it is possible for the nonlinear, steady problem (1–6) to have sublinear solutions. Such solutions are the outcome of formal calculations by VERONIS [4, 5] who uses a numerical method based on modal truncation and by SANI [6] who uses a perturbation method for small amplitudes. But both analyses rely heavily on the free surface boundary condition and are restricted to two dimensional solutions. The idealizations involved interdict comparison with the experiment of TURNER & STOMMEL [11] in which 3 dimensional convection was observed in a rigid box open on the top. It may be that as in the generalized Bénard problem the deepest sublinear steady convection is associated with the (three dimensional) convection in hexagons [7, 8]. Our result holds for all spatially periodic steady convection whether in two or three dimensions and is easily put into numbers for all those domains and boundary conditions for which the eigenvalue  $\gamma$  for the Bénard problem is known.

Without the constraint  $\mathcal{F} = 0$  energy procedures lead to strong stability statements in which exponential decay of an  $L_2$  norm for the disturbance is guaranteed from the initial instant. Eventually, if all of the eigenvalues of the linearized disturbance equations have negative real parts, the disturbance will decay exponentially [10]. But for finite times, such fast decay can generally be expected of



the nonlinear Boussinesq equations only when the stability parameters are low enough. For example, when a pipe flow is disturbed under but near its stability limit, the stable disturbance can be extraordinarily persistent.

It remains to determine whether the improved criterion  $\mathcal{R} < \tilde{\mathcal{R}}$  associated with the constraint  $\mathcal{F} = 0$  implies stability with fast decay.

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