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Existence of Convective Solutions of the Generalized Bénard Problem Which are Analytic in Their Norm

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1. Introduction

The generalized nonlinear Bénard problem defined below, like the standard Bénard problem itself, possesses a unique, motionless, *conduction*-solution when the parameters lie within a restricted range. This solution, however, bifurcates at certain critical parameter values and *convective* motion with a finite norm ensues. We shall establish the existence of this bifurcation and the analyticity of the convective motion, not only in its own norm, but in parameters expressing the generalization as well. In the course of our demonstration we construct the analytic solution. This construction has been used elsewhere. It has led to comprehensive, though hitherto formal, results relative to the physics of nonlinear convection. The need to establish these results in a mathematically correct way motivates this work.

We shall begin, in this introductory chapter, with some remarks designed to explain the type of problem we shall be investigating. First we formulate the problem in the context of the Boussinesq equations, define the *conduction* solutions and set out equations governing possible *convective* solutions. We then review the high points of the mathematical theory of the classical Bénard problem and explain what is meant by expansions (Taylor series) in a norm of the (convective) solution. This is followed by an explanation of what we call a generalized Bénard problem and a description of the multiple parameter expansions (Taylor series).

Generalizations of the Bénard problem can be obtained by relaxing approximations which lead to the Boussinesq equations. These equations are the consequence of approximations of the compressible Navier-Stokes equations in which: (1) Variations of density are neglected everywhere except in the body (buoyant) force term of the momentum equation. (2) Material parameters are treated as constants. (3) The effect on the temperature of the conversion of mechanical work to heat is negligible. (4) The density ρ is a linear function of the temperature

$$\frac{\rho}{\rho_0} = 1 - \alpha(T - T_0) \tag{1.1}$$

where $\alpha = -\left(\frac{d\ln\rho}{dT}\right)_0$ and T_0 and ρ_0 are a reference temperature and density, respectively. These can be shown, formally (see [1] and [2]), to imply the following set of (Boussinesq) partial differential equations:

$$\nabla \cdot \boldsymbol{v}=0, \tag{1.2a}$$

$$\frac{d\boldsymbol{v}}{dt}=-\nabla q+[1-\alpha(T-T_0)]\,\boldsymbol{g}+\nu\Delta\boldsymbol{v}, \tag{1.2b}$$

$$\frac{dT}{dt}=\kappa\Delta T+S(P,t) \tag{1.2c}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla$$

and $P(x_1, x_2, x_3)$ is a point of the flow domain. Here $\boldsymbol{v}$ is the velocity, T is the temperature, q is the pressure, $S(P, t)$ is a prescribed heat source field, $\boldsymbol{g}$ is the gravity acceleration vector, ν is the kinematic viscosity and κ is the thermal diffusivity.

Certain very simple solutions of (1.2) can be found for the motionless ($\boldsymbol{v}\equiv 0$) state. Such solutions necessarily satisfy the equation

$$\nabla T\times\boldsymbol{g}=0 \tag{1.3}$$

which expresses the curl of (1.2b) when $\boldsymbol{v}\equiv 0$. Let x_3 increasing be antiparallel to gravity $\boldsymbol{g}=-\boldsymbol{i}g$. Then a family of solutions corresponding to the state of no motion may be found, when S is independent of t, as integrals of the equations

$$\frac{dq^*}{dx_3}=-[1-\alpha(T^*-T_0)]\,g, \tag{1.4a}$$

and

$$\frac{\partial T^*}{\partial t}=\kappa\frac{\partial^2 T^*}{\partial x_3^2}+S(x_3). \tag{1.4b}$$

The solution $T^*(x_3, t)$ of (1.4b) determines the distribution of density through (1.1) and the pressure through (1.4a). We are interested in steady solutions of (1.4b)

$$T^*(x_3)=A-\beta x_3-\int_0^{x_3}\int^{\eta}\frac{S(\eta')}{\kappa}\,d\eta'\,d\eta. \tag{1.5}$$

Equation (1.5) is a *conduction* solution of the Boussinesq equations.

Equations which govern the difference between solutions with motion ($\boldsymbol{v}\not\equiv 0$) and the conduction solution are formed by introducing the differences $T(P)-T^*(x_3)$, $q(P)-q^*(x_3)$ with $T^*(x_3)$ expressed by (1.5). In suitably rescaled variables we have

$$\nabla\cdot\boldsymbol{u}=0, \tag{1.6a}$$

$$\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\nabla\omega+p\Delta\boldsymbol{u}+\boldsymbol{i}p\theta, \tag{1.6b}$$

and

$$\frac{\partial\theta}{\partial t}+(\boldsymbol{u}\cdot\nabla)\theta=R(\boldsymbol{i}\cdot\boldsymbol{u})(1+\eta s(z))+\Delta\theta. \tag{1.6c}$$

Here $z=x_3/d$, $\boldsymbol{u}=(u_1, u_2, u_3)$ is the convective velocity, θ is the temperature difference, ω is a modified pressure, $\eta s(z_3)=\frac{1}{\kappa\beta}\int^{x_3} S(x_3)\,dx_3$, p is the Prandtl number and $R=\alpha\beta d^4 g/\nu\kappa$ is the Rayleigh number.

We shall seek steady solutions of (1.6) on any (rescaled) domain Ω on which (1.5) holds and for which suitable boundary conditions can be set. We call such solutions *convective* if they are not identically zero. Let $\partial\Omega$ be the boundary of Ω. In most applications, one or the other of the following conditions on the velocity are required to hold on each part of the boundary (see [3] for a discussion):

$$\boldsymbol{u}=0 \quad \text{(rigid surface to which the fluid adheres)} \tag{1.7a}$$

or

$$\boldsymbol{u}\cdot\boldsymbol{N}=0 \quad\text{and}\quad (\nabla\boldsymbol{u}+\text{Transpose}\,\nabla\boldsymbol{u})\cdot\boldsymbol{N}\times\boldsymbol{N}=0 \tag{1.7b}$$

(free surface with outward normal $\boldsymbol{N}$ has no normal motion and allows no tangential tractions).

For θ it is required that

$$A\frac{\partial\theta}{\partial N}+B\theta=0 \quad (A \text{ and } B \text{ are piecewise continuous on the boundary}). \tag{1.7c}$$

It is conventional to associate the name "Bénard problem" with the following special situation: the domain Ω is a horizontal layer, $S(x_3)\equiv 0\equiv\eta s(z)$ and $A\equiv 0$. By a generalized Bénard problem we shall understand any generalization of the classical problem, including generalizations of the Boussinesq equations, for which a conduction solution exists. We are interested in proving the existence and analyticity of convective solutions.

The classical Bénard problem is one of the most extensively studied and best understood problems in the theory of hydrodynamic stability (see SEGEL [4] and GÖRTLER & VELTE [5] for comprehensive reviews). We know, for example, that the conduction solution is unique when the Rayleigh number is less than the smallest eigenvalue, R_0, of the linearized Boussinesq equations*. We know that R_0 is a point from which convective solutions with finite norms bifurcate. The existence of such convective solutions can be inferred by topological degree arguments ([6] and VELTE [9]), by constructive procedures involving successive approximations (RABINOWITZ [10]) or by Liaponouv-Schmidt series expansions (JUDOVICH [11]; see also KIRCHGÄSSNER [12] for an application to Couette flow). These results all lay claim to mathematical rigor.

Still more comprehensive results are known through formal procedures which involve the determination of the coefficients of the formal series expansions

$$\boldsymbol{u}(P,\varepsilon)=\varepsilon\sum_{i=0}\boldsymbol{u}_i(P)\,\varepsilon^i, \qquad \theta(P,\varepsilon)=\varepsilon\sum_{i=1}\theta_i(P)\,\varepsilon^i, \qquad R(\varepsilon)=\sum_{i=0}R_i\,\varepsilon^i$$

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* This result was established for arbitrary domains and any of boundary conditions (1.7) by JOSEPH [3]. The uniqueness of the conduction solution when $R<R_0$ among all *steady* solutions was proved by UKHOVSKII & JUDOVICH [6] and SANI [7]. HOWARD [8] has proved this result for the wider class of all statistically stationary solutions.

of solutions of (1.6,7). Here R_0 is the smallest eigenvalue of the linear ($\varepsilon \to 0$) Boussinesq system and $(\boldsymbol{u}_0, \theta_0)$ is its vector-valued eigenfunction. The coefficients $(\boldsymbol{u}_i, \theta_i, R_i)$ may be obtained as solutions of inhomogeneous boundary-value problems which arise upon substitution of the series in the original problem. In our work ε is the preassigned L_2 norm

$$\varepsilon^2 = \int (|\boldsymbol{u}|^2 + |\theta|^2)\, dP.$$

Relative to the classical Bénard problem, the most systematic and comprehensive analysis, based on such series, has been given by SCHLÜTER, LORTZ & BUSSE [13].* These authors examine the entire manifold of periodic solutions in fluid layers. They conclude that not all solutions of the linearized problem $(\boldsymbol{u}_0, \theta_0, R_0)$ are zero-amplitude ($\hat{\varepsilon} \to 0$) limits of nonlinear solutions $(\boldsymbol{u}(P, \hat{\varepsilon}), \theta(P, \hat{\varepsilon}), R(\hat{\varepsilon}))$, where their $\hat{\varepsilon}$ is defined below. Moreover *all* three-dimensional solutions are unstable to disturbances of certain class. Two-dimensional (roll cell) solutions are stable to disturbances of this class and also maximize the heat transport. The parameter $\hat{\varepsilon}$ is not identical to our ε, but is given (implicitly, see Section 6 for a full discussion) by

$$\hat{\varepsilon} = \frac{\langle \boldsymbol{u}; \Delta \boldsymbol{u}_0 \rangle}{\langle \boldsymbol{u}_0; \Delta \boldsymbol{u}_0 \rangle}$$

and is related to ε by an invertible analytic transformation.

We show, in Sections 2–4, that these formal series converge. More generally we prove the convergence of multiple parameter series in which ε appears as but one parameter. Multiple series have been introduced by BUSSE [16] and have been used by him and later by KRISHNAMURTI [17] to obtain results for generalized Bénard convection. Generalized problems may be formed by allowing temperature dependent material parameters, nonlinear equations of state [16] as well as the heat source solution [17]. In the generalized problem, unlike the standard problem, the solution may bifurcate "downward" (this means convection for R less than its critical value R_0).

In Section 6 we develop certain properties of the construction, show how our construction is equivalent to that used, say, in [15], [16] and [17] and show how to apply the two parameter results to establish the existence of downward bifurcation.

2. Formulation of the Convection Problem

For ease of exposition we shall first develop our results for bounded domains Ω. On $\partial\Omega$, the boundary of Ω, it is required that the velocity, u, and temperature difference, θ, vanish. In Section 5 we widen the class of domains and boundary conditions to which our results apply.

* A large number of papers which utilize these series have been published in the past fifteen years. The first efforts of this type seem to be those of GORKOV [14] and MALKUS & VERONIS [15]. A nearly complete bibliography on the subject is given in [4].

We seek steady solutions of equations (1.6). It is better to write these equations as a single matrix equation. To this end define:

$$U \equiv (U_1, U_2, U_3, U_4) \equiv (u_1, u_2, u_3, \theta), \quad \partial \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, 0\right),$$

$$D = \begin{pmatrix} p\Delta & & & 0 \\ & p\Delta & & \\ & & p\Delta & p \\ 0 & & R & \Delta \end{pmatrix},$$

and

$$Q_{ij} = \delta_{i4}\,\delta_{j3}\, s(z), \qquad (i, j = 1, 2, 3, 4).$$

Equations (1.6) can be reexpressed as

$$D \cdot U - \partial\omega + \mu Q \cdot U = (U \cdot \partial)\, U|_{\Omega}, \qquad \partial \cdot U = 0|_{\Omega},$$

with $\mu = R\eta$ and

$$U = 0|_{\partial\Omega}. \tag{2.1}$$

The system (2.1), though general enough for the heat source problem [17], does not allow, for example, for a quadratic equation of state or for the temperature dependence of certain of the material parameters [16]. For this reason it is useful (and is possible) to treat the problem

$$\begin{gathered} D \cdot U - \partial\omega = B_0(U, U) + \sum_{i=1}^{q} (\mu_i A_i U + \mu_i B_i(U, U)\big|_{\Omega}, \\ \partial \cdot U = 0|_{\Omega}, \qquad U = 0|_{\partial\Omega}. \end{gathered} \tag{2.2}$$

Here the A_i are any matrices whose elements are linear differential operators of order 0 or 1 with smooth coefficients bounded in Ω. (See Section 5 about the possibility of including second derivatives.) The $B_i(U, U)$ are vectors whose elements are sums of quadratic differential expressions in the components of U with smooth coefficients bounded in Ω. Each term of these sums is of the form $a(P)\, U_j \partial_k U_l$ or $a(P)\, U_j U_l$ so that the total number of differentiations in each term of the vector does not exceed one.

It will suffice to seek solutions of (2.2) for $q=1$. The extension to $q>1$ is trivial. We want a solution of (2.2) of a certain size. The problem is to find a value $R(\varepsilon, \mu)$ and functions $U(P, \varepsilon, \mu)$, $\omega(P, \varepsilon, \mu)$ satisfying (2.3) for preassigned $|\mu| < \rho$ and $|\varepsilon| < \rho$ for some $\rho > 0$, and

$$\varepsilon^2 = \int_{\Omega} |U|^2\, dP. \tag{2.3}$$

This search is successful (and the result analytic in ε and μ) when ρ is small enough and when $R(0,0)$ can be chosen as a simple eigenvalue of the linearized problem.

The form of (2.3) suggests a change of variable

$$v = \varepsilon^{-1} U, \qquad \omega = \varepsilon^{-1} \omega$$

which leads to a problem with unit norm

$$\int_\Omega |\boldsymbol{v}|^2 dP = 1 \tag{2.4}$$

in which ε appears explicitly in the coefficients of the differential equations.

$$\boldsymbol{D}\cdot\boldsymbol{v} - \partial\omega = \varepsilon\boldsymbol{B}_0(\boldsymbol{v},\boldsymbol{v}) + \mu\boldsymbol{A}_1\boldsymbol{v} + \mu\varepsilon\boldsymbol{B}_1(\boldsymbol{v},\boldsymbol{v})|_\Omega, \tag{2.5}$$

$$\partial\cdot\boldsymbol{v} = 0|_\Omega, \tag{2.6}$$

$$\boldsymbol{v} = 0|_{\partial\Omega} \tag{2.7}$$

If a solution of (2.4–2.7) exists and is analytic in ε and μ when $\varepsilon^2+\mu^2<\rho^2$ then

$$\boldsymbol{v} = \sum_{m,n\geqq 0} \varepsilon^m \mu^n \boldsymbol{v}^{mn}(P), \tag{2.8a}$$

$$\omega = \Sigma\varepsilon^m\mu^n\omega^{mn}(P), \tag{2.8b}$$

$$R = \Sigma\varepsilon^m\mu^n R^{mn}, \tag{2.8c}$$

where

$$\boldsymbol{v}^{mn} = \frac{1}{m!\,n!}\left.\frac{\partial^{m+n}\boldsymbol{v}(P)}{\partial\varepsilon^m\partial\mu^n}\right|_{\varepsilon=\mu=0}, \text{ etc.} \tag{2.8d}$$

On the other hand, if formal series (2.8) can be found, are convergent, and satisfy (2.4–7) then the solution exists, is analytic and (2.8) are its Taylor series. Our paper develops this last plan.

We employ the following convenient notation: If u^{mn} and v^{mn} are any quantities depending on two indices m and n, then

$$\Sigma^{mn} u v \equiv \Sigma u^{\alpha\beta} v^{\gamma\delta},$$

where the summation is over all indices α, β, γ, δ such that $\alpha+\gamma=m$ and $\beta+\delta=n$. In the same way, $\Sigma^{mn}(u\cdot\partial)v$ represents the same type of summation $\Sigma(u^{\alpha\beta}\cdot\partial)v^{\gamma\delta}$. Similar meanings will be attached to such symbols as $\Sigma^{mn}Dv$, etc. Finally, the symbol $\hat{\Sigma}^{mn}uv$ denotes the same summation $\Sigma u^{\alpha\beta}v^{\gamma\delta}$, except that the quantities of "highest order" are omitted: specifically,

$$\hat{\Sigma}^{mn} u v \equiv \Sigma^{mn} u v - u^{mn} v^{00} - u^{00} v^{mn}.$$

By definition any quantity with a negative index is automatically zero.

Substituting the formal expansion (2.8) into (2.4–7), and equating coefficients of $\varepsilon^m\mu^n$, we obtain

$$\begin{aligned} \boldsymbol{D}^{00}\boldsymbol{v}^{mn} - \partial\omega^{mn} = &-\boldsymbol{D}^{mn}\boldsymbol{v}^{00} - \hat{\Sigma}^{mn}\boldsymbol{D}\boldsymbol{v} + \Sigma^{m-1,n}\boldsymbol{B}_0(\boldsymbol{v},\boldsymbol{v}) \\ &+ \boldsymbol{A}_1\boldsymbol{v}^{m,n-1} + \Sigma^{m-1,n-1}\boldsymbol{B}_1(\boldsymbol{v},\boldsymbol{v}), \end{aligned} \tag{2.9a}$$

$$\partial\cdot\boldsymbol{v}^{mn} = 0, \tag{2.9b}$$

$$\int_\Omega |\boldsymbol{v}^{00}|^2 dP = 1, \tag{2.9c}$$

$$\int_\Omega (\Sigma^{mn}\boldsymbol{v}\cdot\boldsymbol{v})\,dP = 0, \qquad m+n\geqq 1, \tag{2.9d}$$

$$\boldsymbol{v}^{mn} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.9e}$$

where

$$D^{mn}=(m^{-1})!(n^{-1})!\,\partial^{m+n}D/\partial\varepsilon^m\partial\mu^n, \text{ etc.}$$

In Section 3 we shall indicate how such functions v^{mn}, ω^{mn} may be constructed; it will then be shown that the resulting series (2.8) represent a solution of the problem.

One technicality concerning the boundary values must be mentioned here. Since we do not assume the boundary to be smooth, it may contain so-called "irregular" points. These are defined as points at which a solution cannot in general satisfy the classical boundary condition (2.7). Irregular boundary points are well-known in connection with potential theory; such points may well occur in the present connection as well.

We should therefore replace the strict condition (2.7, 2.9e) by a weaker one. For this purpose we introduce a Hilbert space which will in fact be used throughout the following section. Let $\mathring{H}$ be the set of all smooth vector-valued functions $v=(v_1, v_2, v_3, v_4)$ such that v vanishes in a neighborhood of $\partial\Omega$ and $\partial\cdot v=\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\equiv 0$. Then the Hilbert space H is defined as the closure of $\mathring{H}$ with respect to the scalar product

$$(u, v)=R^{00}p\sum_1^3\int_\Omega(\nabla u_i\cdot\nabla v_i)\,dP+p\int_\Omega(\nabla u_4\cdot\nabla v_4)\,dP \tag{2.10}$$

(with associated norm $\|u\|^2=(u, u)$).

Let $\partial'\Omega$ denote the "smooth" portion of $\partial\Omega$; i.e., the set of points where a normal is defined and is a differentiable function of position on $\partial\Omega$. We now replace (2.7) by

$$v=0 \quad \text{on} \quad \partial'\Omega, \quad \text{and} \quad v\in H. \tag{2.7'}$$

Correspondingly (2.9e) is replaced by

$$v^{mn}\equiv 0 \quad \text{on} \quad \partial'\Omega, \quad \text{and} \quad v^{mn}\in H. \tag{2.9e'}$$

Actually the condition that $v=0$ on $\partial'\Omega$ could be eliminated, because according to Lemma 1, it is implied already for solutions of (2.4, 5) by the requirement that $v\in H$.

In the following, "problem (2.9)" will be taken to mean the problem of determining a set v^{mn}, ω^{mn}, R^{mn}, for all $n, m\geqq 0$, satisfying (2.9a–d, 2.9e').

3. Existence of a Classical Solution

The plan will be, first, to show that the equations (2.9) can be solved uniquely for all $m, n\geqq 0$; then to show that the resulting series (2.8) actually converge to a solution of (2.4–7).

Theorem 1. *Suppose the number R^{00} is such that the homogeneous problem*

$$D^{00}\cdot\varphi-\partial\omega=0\big|_\Omega, \quad \partial\cdot\varphi\big|_\Omega=0, \quad \varphi=0\big|_{\partial\Omega}, \quad \varphi\in H \tag{3.1}$$

has exactly one (linearly independent) solution φ, ω_0. *(We take* φ *to be normalized so that* $\int_\Omega |\varphi|^2 dP = 1$.*) Then equations* (2.9) *with* $n=m=0$ *have exactly two solutions:* $\boldsymbol{v}^{00}=\varphi$ *and* $\boldsymbol{v}^{00}=-\varphi$. *For each of these two possibilities, equations* (2.9) *may then be solved recursively and uniquely for all* $n, m \geqq 0$.

Theorem 2. *Let* $\boldsymbol{v}^{mn}$, R^{mn} *be the solutions constructed in Theorem 1. Then there exists* $\varepsilon_0>0$, $\mu_0>0$ *such that for each pair* $\{\varepsilon, \mu: |\varepsilon|<\varepsilon_0, |\mu|<\mu_0\}$ *the series* (2.8a) *converges to a function* $\boldsymbol{v}(P, \varepsilon, \mu)$ *in the sense that*

$$\lim_{N\to\infty} \int_\Omega \Big| \sum_{0\leqq m+n\leqq N} \varepsilon^m \mu^n \boldsymbol{v}^{mn} - \boldsymbol{v}\Big|^2 dP = 0. \tag{3.2}$$

The series (2.8c) *also converges and its sum we call* $R(\varepsilon, \mu)$. *Moreover, for each given* ε, μ *there exists a pressure* $\omega(P, \varepsilon, \mu)$ *such that* $\{\boldsymbol{v}(P, \varepsilon, \mu), \omega(P, \varepsilon, \mu), R(\varepsilon, \mu)\}$ *is a solution of* (2.4–7).

Remark. The statement in Theorem 1 that $\boldsymbol{v}^{00}=\pm\varphi$ is obvious from the hypothesis, because for $n=m=0$ the equations (2.9a, b) reduce to (3.1). This implies that $\boldsymbol{v}^{00}=C\varphi$. Finally (2.9c) shows that C can only be ± 1.

For convenience in the proof of the theorems we introduce another scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle \equiv \sum_{i=1}^{3} R^{00} \int_\Omega u_i v_i \, dP + p \int_\Omega u_4 v_4 \, dP. \tag{3.3}$$

Let $\boldsymbol{h}^{mn}$ = the right side of (2.9a). Then we have

Lemma 1. *Problem* (2.9) *is equivalent to the problem of finding a series of functions* $\boldsymbol{v}^{mn} \in H$ *satisfying* (2.9c, d), $\boldsymbol{v}^{00}=\pm\varphi$, *and*

$$-(\boldsymbol{v}^{mn}, \boldsymbol{\Phi})_H + p R^{00} \int_\Omega (v_4^{mn} \Phi_3 + v_3^{mn} \Phi_4)\, dP = \langle \boldsymbol{h}^{mn}, \boldsymbol{\Phi}\rangle \tag{3.4}$$

for every $\boldsymbol{\Phi} \in H$.

Furthermore, problem (2.4–6, 2.7') *is equivalent to the problem of finding a function* $\boldsymbol{v} \in H$ *satisfying* (2.6) *and*

$$\begin{aligned} -(\boldsymbol{v}, \boldsymbol{\Phi})_H + p R^{00} \int_\Omega (v_4 \Phi_3 + v_3 \Phi_4)\, dP &= \langle (\boldsymbol{D}-\boldsymbol{D}^{00})\boldsymbol{v}, \boldsymbol{\Phi}\rangle + \mu \langle \boldsymbol{A}\,\boldsymbol{v}, \boldsymbol{\Phi}\rangle \\ &\quad + \varepsilon\mu \langle \boldsymbol{B}_1(\boldsymbol{v}, \boldsymbol{v}), \boldsymbol{\Phi}\rangle + \varepsilon \langle \boldsymbol{B}_0(\boldsymbol{v}, \boldsymbol{v}), \boldsymbol{\Phi}\rangle \end{aligned} \tag{3.5}$$

for every $\boldsymbol{\Phi} \in H$.

Proof. First it should be shown that the integrals (the scalar products) on the right of (3.4, 5) exist. Consider (3.5) for example. The terms which are linear in $\boldsymbol{v}$ cause no difficulty. Moreover each term in the expressions $B_0(\boldsymbol{v}, \boldsymbol{v})$ and $B_1(\boldsymbol{v}, \boldsymbol{v})$ is a product of a function in H with a derivative of order 0 or 1 of a function in H. Such a product gives rise to an integral on the right of (3.5) of the form $\int v_i \partial_j v_k \Phi_l \, dP$ or $\int v_i v_k \Phi_l \, dP$. Functions in H have, by definition of H, first derivatives in $L_2(\Omega)$, and by Sobolev's imbedding theorem, are themselves in $L_6(\Omega)$. Thus we know that v_i and $\Phi_l \in L_6(\Omega)$, and $\partial_j v_k \in L_2(\Omega)$. The integrand will therefore be a function in $L_q(\Omega)$ where $\frac{1}{q} \geqq \frac{1}{2} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$, and hence is integrable. For future reference it is to be noted that such an integral is a linear functional of $\boldsymbol{\Phi}$ in H.

It can easily be shown that every strict solution of (2.9) is also a (weak) solution of (3.4). For suppose $\boldsymbol{v}^{mn}$, ω^{mn} satisfy (2.9). Then from (2.9a) with right hand side $\boldsymbol{h}^{mn}$ we find that

$$\langle \boldsymbol{\Phi}, \boldsymbol{D}^{00} \boldsymbol{v}^{mn}\rangle = \langle \boldsymbol{h}^{mn}, \boldsymbol{\Phi}\rangle, \quad \langle \boldsymbol{\Phi}, \partial\omega^{mn}\rangle = 0 \quad \text{(for solenoidal } \boldsymbol{\Phi}),$$

and, after integration by parts,

$$\langle \boldsymbol{\Phi}, \boldsymbol{D}^{00} \boldsymbol{v}^{mn}\rangle = \text{left hand side of (3.4)}.$$

Therefore the classical solution certainly satisfies (3.4). In the same way it is clear that a classical solution of (2.4–6, 2.7′) satisfies (3.5).

As for the converse statement, that weak solutions are classical, the essential details of this proof (for the case of smooth boundaries) were given by various authors, such as VOROVICH & JUDOVICH [18], LADYZHENSKAYA [19], and FUJITA [20]. The extension to boundaries not globally smooth was given by EDWARDS [21]. The proof amounts to showing that the weak solution is regular in the interior and on $\partial\Omega$ ($\partial'\Omega$ in EDWARDS' and our case), satisfies the boundary condition, and that there is a pressure function ω such that (2.9a) is satisfied. These properties were established by the above authors for weak solutions of the nonlinear Navier-Stokes equations without convective terms, that is, when $R^{00} = v_4 \equiv 0$. However, slight and obvious modifications in the argument yield the result in the case of the perturbed Boussinesq equations treated here. For example, the procedure for obtaining interior regularity is given in detail below in conjunction with the proof of Theorem 3.

Lemma 2. *Problem* (2.9) *is further equivalent to a system of operator equations in H of the form*

$$\boldsymbol{v}^{mn} + \boldsymbol{A}\boldsymbol{v}^{mn} = \boldsymbol{F}^{mn}, \tag{3.6a}$$

$$(\boldsymbol{v}^{mn}, \boldsymbol{B}\boldsymbol{v}^{00}) = \alpha^{mn}, \quad m+n \geqq 1, \tag{3.6b}$$

$$\boldsymbol{v}^{00} = \pm\boldsymbol{\varphi}, \tag{3.6c}$$

where $\boldsymbol{A}$ *is a compact self-adjoint operator,* $\boldsymbol{F}^{mn}$ *is the element in H such that*

$$(\boldsymbol{F}^{mn}, \boldsymbol{\Phi}) = \langle \boldsymbol{h}^{mn}, \boldsymbol{\Phi}\rangle \tag{3.7}$$

for all $\boldsymbol{\Phi} \in H$, α^{mn} *is defined by*

$$\alpha^{mn} = -\tfrac{1}{2}\int_\Omega \Sigma^{mn} \boldsymbol{v} \cdot \boldsymbol{v}\, dP, \tag{3.8}$$

and $\boldsymbol{B}$ *is the bounded operator such that*

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dP = (\boldsymbol{u}, \boldsymbol{B}\boldsymbol{v})_H = (\boldsymbol{B}\boldsymbol{u}, \boldsymbol{v})_H \quad (\boldsymbol{u}, \boldsymbol{v} \in H).$$

Proof. (Compare [11].) For any $\boldsymbol{v}^{mn} \in H$, the last integral on the left of (3.4) forms a linear functional of $\boldsymbol{\Phi}$, and as such, by the Riesz representation theorem, can be represented as a scalar product in H of the form $(\boldsymbol{A}\boldsymbol{v}^{mn}, \boldsymbol{\Phi})$. Such a representation would in fact be true of any element $\mathbf{v}^{mn}$ in the larger space $L_2(\Omega)$. By Rellich's lemma the imbedding of H into L_2 is compact. It follows that the operator A can be thought of as the composition of this compact imbedding operator with a bounded operator from L_2 back to H, the latter operator arising as a result of the Riesz representation theorem. Being such a composition, the operator A itself must be compact.

The self-adjointness of A follows from the fact that the form

$$(A\mathbf{v}, \boldsymbol{\Phi}) = p R^{00} \int_{\Omega} (v_4 \Phi_3 + v_3 \Phi_4)\, dx \tag{3.8a}$$

is symmetric in $\mathbf{v}$ and $\boldsymbol{\Phi}$.

In the same way the term on the right of (3.4) can be written as $(\mathbf{F}^{mn}, \boldsymbol{\Phi})_H$ for some $\mathbf{F}^{mn} \in H$. Thus (3.4) becomes

$$(\mathbf{v}^{mn}, \boldsymbol{\Phi}) + (A\mathbf{v}^{mn}, \boldsymbol{\Phi}) = (\mathbf{F}^{mn}, \boldsymbol{\Phi}).$$

Since this holds for all $\boldsymbol{\Phi} \in H$ we obtain (3.6a).

Finally (3.6b, 3.8) are seen to be equivalent to (2.9d). This completes the proof.

Lemma 3. *Let A be a linear compact self-adjoint operator, and suppose that there is exactly one linearly independent nontrivial solution $\boldsymbol{\psi}$ of*

$$\boldsymbol{\psi} + A\boldsymbol{\psi} = 0. \tag{3.9}$$

Let $\mathbf{B}$ be a bounded positive definite operator. Let $\mathbf{F}$ be such that $(\mathbf{F}, \boldsymbol{\psi}) = 0$. Then for any scalar α the problem

$$\mathbf{U} + A\mathbf{U} = \mathbf{F}, \tag{3.10}$$

$$(\mathbf{U}, \mathbf{B}\boldsymbol{\psi}) = \alpha \tag{3.11}$$

has a unique solution. Furthermore there is a constant K, depending only on A and $\mathbf{B}$, such that

$$\|\mathbf{U}\| \leqq K(\|\mathbf{F}\| + |\alpha|). \tag{3.12}$$

Proof. The existence and uniqueness follow immediately from Fredholm theory. The estimate is established as follows. Let T be the Cartesian product of H with the reals, so that elements $X \in T$ are pairs $X = [\mathbf{F}, \alpha]$, $\mathbf{F} \in H$, and α is a real number. The solution of (3.10, 11) represents a linear mapping $\mathscr{L}$ from T into H, whose domain is a closed subspace, namely the space of pairs $[\mathbf{F}, \alpha]$ such that $(\mathbf{F}, \boldsymbol{\psi}) = 0$. By the closed graph theorem $\mathscr{L}$ must be bounded; in other words (3.12) holds.

Proof of Theorem 1. Lemma 2 reduces the problem to Equations (3.6). For definiteness we select the plus sign in (3.6c). Of course, when the minus sign is chosen, the same argument holds.

Lemma 3 further implies that (3.6a) is solvable if and only if

$$(\mathbf{F}^{mn}, \boldsymbol{\varphi}) = 0, \tag{3.13}$$

and in that case

$$\|\mathbf{v}^{mn}\| \leqq K(|\alpha^{mn}| + \|\mathbf{F}^{mn}\|). \tag{3.14}$$

We now examine the structure of h^{mn}, the right side of (2.9a). First one observes that $D_{ij}^{mn}=R^{mn}\delta_{i4}\delta_{j3}$ for $m+n\geqq 1$. In view of this, and $v^{00}=\varphi$, we may write

$$\begin{aligned} h_i^{mn}= & -\delta_{i4}R^{mn}\varphi_3-\delta_{i4}\hat{\Sigma}^{mn}Rv_3+(A_1 v)_i^{m,n-1} \\ & +\Sigma^{m-1,n}B_0(v,v)_i+\Sigma^{m-1,n-1}B_1(v,v)_i. \end{aligned} \tag{3.15}$$

Corresponding to this representation, we have

$$\begin{aligned} F^{mn} & =R^{mn}L_0\varphi+\hat{\Sigma}^{mn}RL_0 v+L_1 v^{m,n-1}+\Sigma^{m-1,n}Q_0(v,v)+\Sigma^{m-1,n-1}Q_1(v,v) \\ & =R^{mn}L_0\varphi+S^{mn} \end{aligned} \tag{3.16}$$

(here we define S^{mn} as the last three terms on the right), where L_i are bounded linear, and Q_i bounded bilinear, operators in H. For example L_0 arises through the Riesz representation theorem as

$$-p\int_\Omega \varphi_3\psi_4\,dP=(L_0\varphi,\psi)_H$$

for all $\psi\in H$.

We note from the definition (3.8a) of A and from (3.9) that

$$(L_0\varphi,\varphi)=-\tfrac{1}{2}(R^{00})^{-1}(A\varphi,\varphi)=\tfrac{1}{2}(R^{00})^{-1}(\varphi,\varphi)=\tfrac{1}{2}(R^{00})^{-1}. \tag{3.17}$$

Equation (3.16) now yields the following form for the orthogonality condition (3.13):

$$R^{mn}(L_0\varphi,\varphi)+(S^{mn},\varphi)=0. \tag{3.18}$$

By virtue of (3.17) we know that $(L_0\varphi,\varphi)\neq 0$, so this determines each R^{mn} uniquely in terms of S^{mn}, which involves only the $v^{\mu\nu}$ and $R^{\mu\nu}$ with $\mu+\nu<m+n$.

The theorem is thus established by an easy induction argument: once all the v^{mn}, R^{mn} with $m+n<s$ are known, (3.18) determines all the R^{mn} with $m+n=s$. Since (3.13) now holds, (3.6a) is solvable uniquely for all v^{mn} with $m+n=s$, completing the proof.

Proof of Theorem 2. The following estimates are immediately derived from (3.16) (which gives the definition of S^{mn}; note that the L_i and Q_i are bounded), (3.18), (3.6), and (3.14), for $n+m\geqq 1$:

$$\begin{aligned} & \|S^{mn}\|\leqq K_1(\hat{\Sigma}^{mn}|R|\,\|v\|+\|v^{m,n-1}\|+\Sigma^{m-1,n}\|v\|\,\|v\|+\Sigma^{m-1,n-1}\|v\|\,\|v\|) \\ & |R^{mn}|\leqq K_2\|S^{mn}\|, \\ & \|F^{mn}\|\leqq K_3(|R^{mn}|+\|S^{mn}\|)\leqq K_4\|S^{mn}\|, \\ & |\alpha^{mn}|\leqq\hat{\Sigma}^{mn}\|v\|\,\|v\|, \\ & \|v^{mn}\|\leqq K_5(\hat{\Sigma}^{mn}\|v\|\,\|v\|+\hat{\Sigma}^{mn}|R|\,\|v\|+\|v^{m,n-1}\|+\Sigma^{m-1,n}\|v\|\,\|v\| \\ & \qquad +\Sigma^{m-1,n-1}\|v\|\,\|v\|). \end{aligned}$$

Setting $\tau^{mn} = \|v^{mn}\| + |R^{mn}|$, we obtain

$$\tau^{mn} \leqq K_6(\hat{\Sigma}^{mn}\,\tau\tau + \tau^{m,n-1} + \Sigma^{m-1,n}\,\tau\tau + \Sigma^{m-1,n-1}\,\tau\tau), \qquad n+m \geqq 1, \tag{3.19}$$

and

$$\tau^{00} = 1. \tag{3.20}$$

Let $g(z_1, z_2)$ be the function of two complex variables defined implicitly by

$$\Phi(g, z_1, z_2) \equiv g - K_6[(g-1)^2 + z_2 g + z_1 z_2 g^2 + z_1 g^2] - 1 \equiv 0, \qquad g(0,0) = 1. \tag{3.21}$$

Since $\partial\Phi/\partial g = 1$ at $g=1$, $z_1 = z_2 = 0$, we know by the implicit function theorem that there exist positive numbers ε_0, μ_0 such that $g(z_1, z_2)$ is analytic for $|z_1| < \varepsilon_0$, $|z_2| < \mu_0$. Let the Taylor series expansion be

$$g(z_1, z_2) = \sum_{n,m \geqq 0} g^{mn} z_1^m z_2^n. \tag{3.22}$$

Substituting into (3.21), we have

$$\begin{aligned} g^{mn} &= K_6(\Sigma^{mn}\, g g - 2g^{mn} + g^{m,n-1} + \Sigma^{m-1,n-1}\, g g + \Sigma^{m-1,n}\, g g) \\ &= K_6(\hat{\Sigma}^{mn}\, g g + g^{m,n-1} + \Sigma^{m-1,n-1}\, g g + \Sigma^{m-1,n}\, g g), \qquad m+n \geqq 1, \end{aligned} \tag{3.23}$$

and $g^{00} = 1$. Comparing this with (3.19), we find

$$0 \leqq \tau^{mn} \leqq g^{mn}. \tag{3.24}$$

By majorization, the convergence of (3.22) now implies that of (2.8a) and (2.8c) in the sense of the norm of H, which implies (3.2).

We multiply (3.4) by $\varepsilon^m \mu^n$ and sum over m and n. The convergence in the norm of H of (2.8a), and the convergence of (2.8c), imply that the left side sums to $-(\boldsymbol{v}, \boldsymbol{\Phi})_H + pR^{00} \int (v_4 \Phi_3 + v_3 \Phi_4)\, dP$, which is the left side of (3.5).

We now examine the convergence on the right side of (3.4). From (3.15) and the fact that $R^{mn} \varphi_3 + \hat{\Sigma}^{mn} R v_3 = \Sigma^{mn} R v_3 - R^{00} v_3^{mn}$, we have formally,

$$\begin{aligned} \Sigma \varepsilon^m \mu^n \boldsymbol{h}^{mn} = &-\delta_{i4}[\Sigma \varepsilon^m \mu^n (\Sigma^{mn} R v_3) - R^{00} \Sigma \varepsilon^m \mu^n v_3^{mn}] + \mu A_1 \Sigma \varepsilon^m \mu^n \boldsymbol{v}^{mn} \\ &+ \varepsilon\mu \boldsymbol{B}_1(\Sigma \varepsilon^m \mu^n \boldsymbol{v}^{mn}, \Sigma \varepsilon^m \mu^n \boldsymbol{v}^{mn}) + \varepsilon \boldsymbol{B}_0(\Sigma \varepsilon^m \mu^n \boldsymbol{v}^{mn}, \Sigma \varepsilon^m \mu^n \boldsymbol{v}^{mn}). \end{aligned} \tag{3.25}$$

The first three terms on the right (the linear terms) represent summations which converge in the sense of the norm of H to

$$-\delta_{i4}[R v_3 - R^{00} v_3] + \mu A_1 \boldsymbol{v} = -(\boldsymbol{D} - \boldsymbol{D}^{00})\boldsymbol{v} + \mu A_1 \boldsymbol{v}.$$

Furthermore since $\|\boldsymbol{B}_{0,1}(\boldsymbol{u}, \boldsymbol{v})\|_{L_2} \leqq K \|\boldsymbol{u}\|_H \|\boldsymbol{v}\|_H$, the last two terms on the right converge in the L_2 sense to $\mu\varepsilon \boldsymbol{B}_1(\boldsymbol{v}, \boldsymbol{v}) + \varepsilon \boldsymbol{B}_0(\boldsymbol{v}, \boldsymbol{v})$. Taking the $\langle .,. \rangle$ scalar product with an arbitrary element $\boldsymbol{\Phi}$ of H, we therefore have that the series $\Sigma \varepsilon^m \mu^n \langle \boldsymbol{h}^{mn}, \boldsymbol{\Phi} \rangle$ converges to the right side of (3.5).

It is thus verified that the limit $\boldsymbol{v}$ satisfies (3.5) for every $\Phi \in H$; in other words, $\boldsymbol{v}$ is a weak solution of our problem. Lemma 1 now states that it is a strict solution as well. This completes the proof of Theorem 2.

4. Uniform Convergence of the Taylor Series

In Section 3 it was shown that the series $\Sigma \varepsilon^m \mu^n \boldsymbol{v}^{mn}$ constructed as a formal solution of our problem actually converges to a solution in the mean square sense. In practice, we want to approximate the solution pointwise with a partial sum, and for this purpose we need a stronger result. In this section we obtain this stronger result and show that the convergence is uniform in subsets of Ω which exclude irregular points on the boundary. It may in fact also be shown that the derivatives of the formal series up to any desired order converge uniformly in compact subsets to the corresponding derivatives of the solution. The proof of this latter result is based on similar techniques and will not be given here in detail.

Theorem 3. *Let $\mathscr{D}$ be a subdomain of Ω such that $\overline{\mathscr{D}} \subset \Omega + \partial'\Omega$. The series* (2.8a) *constructed in Theorem 1 converges uniformly to $\boldsymbol{v}$ in $\overline{\mathscr{D}}$.*

We shall have need in the following of two more Hilbert spaces. The space H of 4-vector functions was defined in Section 2. We define the analogous space H' of 3-vectors $\boldsymbol{u}(P) = (u_1, u_2, u_3)$ as being the closure, in the norm

$$|\boldsymbol{u}|_{H'} = \Big(\int_\Omega |\nabla \boldsymbol{u}|^2 \, dP\Big)^{\frac{1}{2}}$$

of the set of smooth solenoidal vectors defined in Ω and vanishing near $\partial\Omega$. Finally we take H'' to be a space of scalar functions, namely, the closure in the Dirichlet norm:

$$|\psi|_{H''} = \Big(\int_\Omega |\nabla \psi|^2 \, dP\Big)^{\frac{1}{2}}$$

of the set of smooth functions vanishing near $\partial\Omega$. It is seen that H is equivalent to $H' \oplus H''$.

Lemma 4. *Let the vector $\boldsymbol{u} \in H'$ satisfy, for some 3-vector* $\mathrm{f} \in L_2(\Omega)$ *and $\boldsymbol{u}'$, $\boldsymbol{u}''$ $\in H'$, the identity*

$$(\boldsymbol{u}, \boldsymbol{\varphi})_{H'} = \int_\Omega [\boldsymbol{f} + \mathrm{u}_i' \, \partial_j \boldsymbol{u}''] \cdot \boldsymbol{\varphi} \, dP \tag{4.1}$$

for all $\boldsymbol{\varphi} \in H'$. Then $\boldsymbol{u}$ is bounded in $\mathscr{D}$ and

$$\sup_{\mathscr{D}} |\boldsymbol{u}| \leqq C_1 (|\boldsymbol{f}|_{L_2(\Omega)} + |\boldsymbol{u}'|_{H'} \, |\boldsymbol{u}''|_{H'} + |\boldsymbol{u}|_{H'}). \tag{4.2}$$

Let $\tau \in H''$ satisfy, for some scalar function $g \in L_2(\Omega)$, $\boldsymbol{u}' \in H'$, and $\tau' \in H''$, the identity

$$(\tau, \psi)_{H''} = \int_\Omega [g + u_i' \, \partial_j \tau'] \, \psi \, dP \tag{4.3}$$

for all $\psi \in H''$. Then τ is bounded in $\mathscr{D}$ and

$$\sup_{\mathscr{D}} |\tau| \leqq C_2 (|g|_{L_2(\Omega)} + |\boldsymbol{u}'|_{H'} \, |\tau'|_{H''} + |\tau|_{H''}). \tag{4.4}$$

Proof. The proof of (4.2) follows immediately from a part of a proof in FUJITA [20], namely, the proof of (*i*) on his page 82. The proof of (4.4) is achieved by the same type of argument, but is a well-known result.

Proof of Theorem 3. The details will be given for the case of an interior subdomain; i.e. $\bar{\mathscr{D}}\subset\Omega$. The analogous argument for the case when $\mathscr{D}$ is adjacent to the boundary can be constructed by referring to FUJITA [20, Sec. 5] and EDWARDS [21]. We define the three-vectors $\boldsymbol{u}^{mn}=(u_1^{mn}, u_2^{mn}, u_3^{mn})$ to be the first three components of $\boldsymbol{v}^{mn}$; and τ^{mn} to be the last, so that $\boldsymbol{v}^{mn}=(\boldsymbol{u}^{mn}, \tau^{nm})$. Then for the special case $\Phi_4\equiv 0$ (3.4) assumes the following form (here we define $\boldsymbol{\Phi}=(\Phi_1, \Phi_2, \Phi_3)\in H'$)

$$\begin{aligned}-(\boldsymbol{u}^{mn}, \boldsymbol{\Phi})_{H'}+\int_\Omega \tau^{mn}\Phi_3\,dP&=\langle \boldsymbol{h}^{mn}, \boldsymbol{\Phi}\rangle/R^{00}p\\ &=\int_\Omega [\Sigma^{m-1,n}\boldsymbol{B}_0(\boldsymbol{v},\boldsymbol{v})+\boldsymbol{A}_1\boldsymbol{v}^{m,n-1}+\Sigma^{m-1,n-1}\boldsymbol{B}_1(\boldsymbol{v},\boldsymbol{v})]\cdot\boldsymbol{\Phi}\,dP\end{aligned}\tag{4.5}$$

Also for the special case $\Phi_4=\psi$, $\Phi_i\equiv 0$, $i\leqq 3$, we have

$$\begin{aligned}-(\tau^{mn}, \psi)_{H''}+R^{00}\int_\Omega v_3^{mn}\psi\,dP=-\int_\Omega\{&\Sigma^{mn}Rv_3-R^{00}v_3^{mn}+[A_1v^{m,n-1}\\ &+\Sigma^{m,n-1}B_0(v,v)+\Sigma^{m-1,n-1}B_1(v,v)]_4\}\psi\,dP.\end{aligned}\tag{4.6}$$

An application of Lemma 4 to (4.5) and (4.6) now tells us the following (we use also the fact that $|A_1\boldsymbol{v}|_{L_2}\leqq C|\boldsymbol{v}|_H$):

$$\begin{aligned}\sup_{\mathscr{D}}|\boldsymbol{u}^{mn}|\leqq C_1\{&|\tau^{mn}|_{L_2}+|\boldsymbol{v}^{m,n-1}|_H+\Sigma^{m-1,n}|\boldsymbol{v}|_H|\boldsymbol{v}|_H\\ &+\Sigma^{m-1,n-1}|\boldsymbol{v}|_H|\boldsymbol{v}|_H+|\boldsymbol{u}^{mn}|_{H'}\},\\ \sup_{\mathscr{D}}|\tau^{mn}|\leqq C_2\{&|\boldsymbol{u}^{mn}|_{H'}+\Sigma^{mn}|R||\boldsymbol{u}|_{H'}+|\boldsymbol{v}^{m,n-1}|_H\\ &+\Sigma^{m-1,n-1}|\boldsymbol{v}|_H|\boldsymbol{v}|_H+\Sigma^{m-1,n}|\boldsymbol{v}|_H|\boldsymbol{v}|_H+|\tau^{mn}|_{H''}\}.\end{aligned}$$

We now use the obvious relations $|\tau^{mn}|_{H''}+|\boldsymbol{u}^{mn}|_{H'}\leqq 2C_3|\boldsymbol{v}^{mn}|_H$, and $|\tau^{mn}|_{L_2}\leqq C_4|\tau^{mn}|_{H''}$ to obtain

$$\begin{aligned}\sup|\boldsymbol{v}^{mn}|\leqq C_5\{&|\boldsymbol{v}^{mn}|_H+|\boldsymbol{v}^{m,n-1}|_H+\Sigma^{m-1,n}|\boldsymbol{v}|_H|\boldsymbol{v}|_H\\ &+\Sigma^{m-1,n-1}|\boldsymbol{v}|_H|\boldsymbol{v}|_H+\Sigma^{mn}|R||\boldsymbol{v}|_H\}.\end{aligned}\tag{4.7}$$

From this and (3.24) we find that

$$\sup|\boldsymbol{v}^{mn}|\leqq C_6\{g^{mn}+g^{m,n-1}+\Sigma^{m-1,n}gg+\Sigma^{m-1,n-1}gg\}.$$

But since the g^{mn} satisfy (3.23), we have

$$\sup|v^{mn}|\leqq C_7(g^{mn}+g^{m,n-1}).$$

Thus for $\varepsilon\leqq\varepsilon_0$, $\mu\leqq\mu_0$,

$$\sup_{\mathscr{D}}\Big|\sum_{m,n=N}^{\infty}\varepsilon^m\mu^n\boldsymbol{v}^{mn}\Big|\leqq 2C_7\sum_{m,n=N-1}^{\infty}\varepsilon^m\mu^n g^{mn}\to 0\quad\text{as}\quad N\to\infty.$$

It follows that the series (2.8a) converges uniformly in $\bar{\mathscr{D}}$, which proves the theorem.

5. Various Extensions

In this section we discuss several variations on the problem treated in the preceding sections, and indicate that the main results attained there may be extended to the new situation. The first two extensions deal with an infinite layer of fluid heated from below. For $\mu=0$ this is the classical Bénard problem, and a constructive existence theorem was obtained by Rabinowitz. We show that the problem of seeking rectangular and hexagonal convection cells for the generalized problem fits within our framework. The third extension has to do with the case when the perturbation matrix A_1 contains derivatives of the solution of order two. This arises, for example, when one allows the viscosity to depend on temperature. The fourth and fifth extensions treat the situation when other boundary conditions are applied to θ and $\boldsymbol{u}$ on portions of the boundary.

(I). Infinite Fluid Layer. Rectangular Cells. Instead of a bounded domain we here take Ω to be the infinite layer $\{-h\leqq 2x_3\leqq h\}$. The boundary condition $U=0$ is applied at $x_3=\pm\frac{h}{2}$. Furthermore we require the solution to be periodic in x_1 with period a and in x_2 with period b. By symmetry considerations it is natural to seek solutions such that

$$v_1 \text{ is odd in } x_1 \text{ and even in } x_2$$
$$v_2 \text{ is even in } x_1 \text{ and odd in } x^2$$
$$v_3 \text{ is even in } x_1 \text{ and } x_2.$$

Let Ω_1 denote a period parallelepiped such as $\{-h<2x_3<h, 0<x_1<a, 0<x_2<b\}$. Then ε is defined as in (2.3), except that Ω is replaced by Ω_1.

This defines the first variation on our basic problem. It was shown by Judovich [11] that for almost all pairs (a, b), all eigenvalues of the linear version are simple. Independently, Rabinowitz proved this result for the lowest eigenvalue. Thus in most cases of importance, the first hypothesis in Theorem 1 is known to be satisfied.

We now show that the previous results hold in the present case. Following Judovich, we now take the Hilbert space H (see preceding (2.10)) to be the closure of the set of smooth solenoidal vector fields, with the above stated periodicity and evenness-oddness conditions, with respect to the norm (2.10), where again Ω is replaced by Ω_1. A weak solution is defined as before. Lemma 1 is again valid: Fujita's argument provides regularity in the interior, and Edward's argument provides it on the boundary. The rest of the proofs of Theorems 1, 2, and 3 are without change.

(II). Hexagonal Cells. Here Ω is the same layer as given above in (I); however we now seek flows with hexagonal symmetry. Specifically, we assume the (x, y)-plane to be covered with a network of hexagons of side a. The cylinders with height h and these hexagons as bases form the basic convective cells. The flow pattern in any one of these cells is to be the same as in all other cells. This requirement replaces the periodicity requirement of *(I)*. Furthermore we impose a natural symmetry requirement: the flow pattern and temperature distribution are to remain invariant under rotation of the (x_1, x_2)-plane by an angle $2\pi/3$.

Again, JUDOVICH [11] shows that for almost all values of a, all the eigenvalues of the linearized problem are simple. The Hilbert space H is defined to include only functions with the desired symmetry. The proofs of the theorems are obtained as before.

(III). The Case when A_1 Involves Second Derivatives. The treatment in previous sections was restricted to linear matrices A_1 which involve derivatives of $\mathbf{v}$ of orders 0 and 1 only. However, if one allows the viscosity to depend on temperature, for example, then the problem is perturbed by a matrix A_1 containing terms linear in the second derivatives of the solution, i.e., terms of the form $a(x)D^2 v_i$. In this case the definition of a weak solution (see (3.4)) must be modified slightly. The term $\langle A_1 \mathbf{v}, \boldsymbol{\Phi}\rangle$ on the right of (3.5) formally contains integrals of the form $\int_\Omega a(x)D^2 v_i \Phi_j dx$, which integral has no meaning for arbitrary $\mathbf{v}$ in H, since only the first derivatives of $\mathbf{v}$ are required to be in L_2. However, the integrated-by-parts version,

$$-\int_\Omega D v_i D(a(x)\Phi_j)\,dx,$$

does exist for $\mathbf{v}$ and $\boldsymbol{\Phi}$ in H. In the fundamental identities (3.4), (3.5) defining weak solutions, we suppose such former integrals to be replaced by the latter. Of course the two integrals are the same if $\mathbf{v}$ can be proved to be regular.

As in the case treated in Sections 3 and 4, a weak solution can be constructed by power series; however the question of whether it is a classical solution is not so easily answered.

The answer can be given in the affirmative in the case in which $\partial\Omega$ is smooth (i.e., no irregular points exist on the boundary). This is accomplished by showing successively that each term $\mathbf{v}^{mn}$ is regular up to the boundary, to the extent of being in class $\mathscr{C}^{2+\alpha}(\Omega)$ for some $0<\alpha<1$. This is possible since each $\mathbf{v}^{mn}$ is the solution of a linear elliptic system with right side depending on the $\mathbf{v}^{\mu\nu}$ for $\mu+\nu<m+n$. Once the right side has been shown to be in $\mathscr{C}^\alpha(\Omega)$, FUJITA's existence and regularity theory up to the boundary, together with that of AGMON, DOUGLIS, and NIRENBERG [22] indicate that $\mathbf{v}^{mn}\in\mathscr{C}^{2+\alpha}$, with a concomitant estimate available in that norm. These estimates yield a set of inequalities of the type (3.19) relating the norms $|\mathbf{v}^{mn}|_{\mathscr{C}^{2+\alpha}}$. As before, such inequalities imply the convergence of the series in the norm of $\mathscr{C}^{2+\alpha}(\Omega)$ to a classical solution.

(IV). Other Boundary Conditions. We have treated only Dirichlet boundary conditions up to this point. However other boundary conditions of physical interest may be handled almost as easily, if we stay within the realm of weak solutions. This is done by simply changing the definitions of H and of weak solutions. However, regularity up to the boundary for the variant problems has not at present been established, so that we cannot prove that weak solutions are classical at the boundary.

First consider a radiative condition imposed on $\theta=U_4=\varepsilon v_4$:

$$\frac{\partial\theta}{\partial N}+\sigma(x)\,\theta=0$$

where $\sigma(x)$ is a positive smooth function defined on the boundary, and differentiation is in the normal direction. Suppose, in the classical form, that this condition is imposed on a portion $\partial_1\Omega$ of the boundary. Then to define the proper weak solution, one takes H to be the closure of a new set of smooth vector functions with respect to a new norm. The new set is the same as before, with the exception that v_4 is no longer required to vanish on $\partial_1\Omega$, only on $\partial\Omega-\partial_1\Omega$. The square of the new norm is the old one plus the extra term

$$p\int_{\partial_1\Omega}\sigma(s)\,v_4^2\,ds.$$

It is easy to see that regular weak solutions are classical. Furthermore the former construction of a weak solution by series can be carried through as easily in the present case.

(V). Free Surface Condition. Consider a free surface condition on the velocity $\boldsymbol{u}=(\varepsilon v_1, \varepsilon v_2, \varepsilon v_3)$:

$$\boldsymbol{u}\cdot N=0, \qquad (\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T)\cdot N\times N=0,$$

where N is the outward normal vector at a point on $\partial\Omega$, and $(\cdot)^T$ denotes the transpose. Suppose, in the classical form, that these conditions are imposed on a smooth section $\partial_2\Omega$ of the boundary. One obtains the correct definition of weak solution by observing that they arise as natural boundary conditions for a certain variational problem, namely the problem of minimizing the Dirichlet integral of $\boldsymbol{u}$ under the condition that $\boldsymbol{u}\cdot N=0$ on $\partial_2\Omega$. Thus one takes H this time to be the same as before except that on $\partial_2\Omega$, the first three components of the test functions are not required to vanish, but only to have normal component zero. The same norm for H is taken as before. The argument proceeds as before, again except for regularity at points of $\partial_2\Omega$.

6. Applications

We have given a proof of existence and analyticity for a construction which enjoys wide popularity in applications. The essential feature of this construction is the expansion in the norm. For the standard Bénard problem this expansion is readily carried out and can be used to obtain many interesting results relative to the convection [13]. Of course, the standard problem is a restricted model of the physical situation and more complex models are not so easily handled. For this purpose multiparameter expansions have much to offer. One may, by such expansions, demonstrate the existence of convection at values of R below the critical values of linear theory. The stability of such solutions (which bifurcate downward and are called "subcritical") may also be reached through such expansions. BUSSE [16], for example, has formally calculated stable, subcritical, convective motion in hexagons between rigid or free, conducting, planes in a number of situations which generalize the Bénard problem. He allows material properties to depend linearly on the temperature, and the density (Equation 1.1) to have a quadratic polynomial dependence on the temperature (e.g., water near its freezing point). A similar result prevails for the standard Boussinesq equations generalized to accomodate a uniform distribution of heat sources of intensity S

(Equation 1.5). Here subcritical hexagons are predicted through the expansions and are observed experimentally (KRISHNAMURTI [17]).

For definiteness, and because much is known about this heat source problem, we shall in this chapter confine our attention to it. Our main aim here is to show how a two parameter construction can be used to obtain not only bifurcation results but the existence of subcritical motion in hexagons. Given the correctness of a certain explicit calculation of KRISHNAMURTI the existence of such a motion, analytic in ε and μ, follows rigorously from our theorems.

Before undertaking an examination of this example it is appropriate to make a few remarks about the potential of the expansions for revealing observable motions. Linear theory, of course, has not this potential and this lack may apply to some of our results here, or those of [16] and [17]. We can, using the expansions, guarantee existence and analyticity of a nonlinear convective motion. The motion however, may (or may not) persist as an analytic solution for values of the parameters sufficiently large to guarantee ease of observation. In the best of circumstances one is usually confined, as a practical matter, to the first few terms of the power series and we certainly have no guarantee that these suffice for calculations of the observable motion. For the heat source problem there are, however, independent *global* results of energy theory [23]. These results are in accord with the results of the perturbation expansions and are consistent with the idea that *strongly* subcritical motion can be induced by intense internal heating.

In the postulated situation of uniform heat sources one finds that $\eta s(z)=x_3/\kappa\beta=(h/\kappa\beta)z=\eta z$. Then from (1.6) we have

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\nabla w+p\Delta\boldsymbol{u}+p\boldsymbol{i}\theta, \tag{6.1}$$

$$(\boldsymbol{u}\cdot\nabla)\theta=Ru_3+\mu z u_3+\Delta\theta, \tag{6.2}$$

$$\nabla\cdot\boldsymbol{u}=0 \tag{6.3}$$

and

$$\theta=0, \boldsymbol{u}=0 \quad \text{or} \quad u_3=\frac{\partial u_1}{\partial z}=\frac{\partial u_2}{\partial z}=0 \quad \text{at} \quad z=\pm\frac{1}{2} \tag{6.4}$$

where $\mu=\eta R$. One seeks a periodic solution of this problem as a double Taylor series in ε and μ where ε is the preassigned L_2 norm (2.3)

$$\varepsilon^2=\int(|\boldsymbol{u}|^2+|\theta|^2)\,dP. \tag{6.5}$$

The extensions of Theorems 1 and 2 given in Section 5 guarantee that such Taylor series exist.

In [15], [16], [17] and elsewhere a parameter $\hat{\varepsilon}$, called an amplitude, is used rather than ε. We show that this parameter may also be identified with a preassigned norm for the nonlinear problem and that ε and $\hat{\varepsilon}$ are analytic functions one of the other, and are invertible. For example, in [17] formal solutions for (6.1 − 4) are sought as series

$$\begin{aligned}\begin{bmatrix}\boldsymbol{u}\\ \theta\end{bmatrix}&=\sum_{j=0}\sum_{i=1}\begin{bmatrix}\boldsymbol{u}^{(ij)}\\ \theta^{(ij)}\end{bmatrix}\hat{\varepsilon}^i\eta^j;\\ R(\hat{\varepsilon},\eta)&=\sum_{j=0}\sum_{i=0}R^{(ij)}\hat{\varepsilon}^i\eta^j\end{aligned} \tag{6.6}$$

where $\boldsymbol{u}^{(10)}$ and $\theta^{(10)}$ are eigenfunctions for the linear homogeneous problem. The inhomogeneous problems for the higher order coefficients admit the addition of arbitrary constant multiples of solutions of the linear homogeneous problem as solutions. An additional condition is needed to determine a unique solution, and for this purpose it is required that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u_3^{(ij)} \theta^{(10)} dz = 0, \qquad i+j \geqq 2 \tag{6.7}$$

where $u_3 = \boldsymbol{i} \cdot \boldsymbol{u}$.

The identity

$$\int_{\Omega} \boldsymbol{u} \cdot (\Delta \boldsymbol{u}^{(10)} + \boldsymbol{i}\, \theta^{(10)})\, dP = 0$$

holds for any $\boldsymbol{u}$. Let $\boldsymbol{u}(P, \varepsilon)$, $\theta(P, \varepsilon)$ and $R(\varepsilon)$ be the solutions (for a fixed value of $\mu = R\eta$) of (6.1 – 5) whose existence and analyticity in ε is guaranteed and define

$$\hat{\varepsilon} = \frac{\int_{\Omega} u_3\, \theta^{(10)}\, dP}{\int_{\Omega} u_3^{(10)} \theta^{(10)}} = \frac{\int_{\Omega} \boldsymbol{u} \cdot \Delta \boldsymbol{u}^{(10)}\, dP}{\int_{\Omega} \boldsymbol{u}^{(10)} \Delta \boldsymbol{u}^{(10)}}. \tag{6.8}$$

Then define the analytic function

$$G(\varepsilon, \hat{\varepsilon}) = a\,\hat{\varepsilon} - \varepsilon \int_{\Omega} v_3(P, \varepsilon, \mu)\, \theta^{(10}, dP = 0$$

where $a = \int_{\Omega} u_3^{(10)}\, \theta^{(10)} dP$. Also, $v_3(P, 0, 0) = v_3^{00}(P) = b u_3^{(10)}$ $(b \neq 0)$ when $\mu = 0$. For sufficiently small μ

$$\frac{\partial G}{\partial \hat{\varepsilon}}(\varepsilon, 0) = a \neq 0$$

and

$$\frac{\partial G}{\partial \varepsilon}(0, \hat{\varepsilon}) = -\int_{\Omega} v_3(P, 0, \mu)\, \theta^{(10)}\, dP \neq 0.$$

The implicit function theorem therefore guarantees that $\hat{\varepsilon}$ and ε are analytic functions of each other and are invertible. Then $\boldsymbol{u}$, θ and R are also analytic functions of $\hat{\varepsilon}$ and may be developed in the series (6.6). The orthogonality condition (6.7) then follows from the series representation (6.6) and from (6.8).

We pursue the analysis in terms of the ε and μ parameters. Let $R_c = R(\varepsilon = 0, \mu)$ be the smallest eigenvalue of the linear problem associated with (6.1 – 5). This linear problem is expressed in the matrix notation of Section 2 as

$$\boldsymbol{D}\boldsymbol{v} - \partial w + \mu \boldsymbol{Q} \boldsymbol{v} = 0 \tag{6.9}$$

along with (2.5), (2.6) and boundary conditions (6.4) written for the vector $\boldsymbol{v}$. Here

$$Q_{ij} = \delta_{i4}\, \delta_{j3}\, z\,.$$

In general the linear problem with $\mu \neq 0$ is not self-adjoint with respect to $\langle \cdot, \cdot \rangle$ and the boundary conditions. Moreover (6.9) has variable coefficients. It is possible to use (6.9) as a pivot problem for a Taylor series expansion of (2.4) in powers of ε alone. Our result guarantees the existence of such a series for sufficiently small μ. To carry out the analysis in this way, however, we should be obliged to introduce the adjoint problem to satisfy the requirements of existence of the Fredholm theory. In addition we should have to face up to the problem of solving inhomogeneous linear equations with variable coefficients. It is for these reasons that we turn to the two parameter construction of Section 2.

The Taylor coefficients for the two parameter expansions are determined sequentially as the unique solutions of (2.9) and (6.4). The pivot problem ($m=n=0$) is the linear Bénard convection. This problem is self-adjoint and has constant coefficients. For the rigid surface boundary conditions there is a smallest simple eigenvalue $R^{00}=1708$ for a wave number $k=3.12$. The eigenfunctions v_3^{00} and v_4^{00} are even functions of $z\in[-\frac{1}{2},\frac{1}{2}]$ and all eigenfunctions separate into solutions proportional to $f(x, y)$ where

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+k^2 f=0. \tag{6.10}$$

For free surfaces the same situation prevails except $R^{00}=\dfrac{27}{4}\pi^4$ and $k=\pi/2$.

The boundary value problems for $n=0$ determine the Taylor coefficients for classical Bénard convection. Having once calculated these $n=0$ coefficients one may use them in multiparameter expansions for any of a variety of generalizations of the classical problem [16, 17]. In every situation for which the conduction solution is unique [3] when $R<R^{00}$ it is necessarily true that $R^{10}=0$ and $R^{20}\geqq 0$. For the classical problem no downward bifurcation is possible.

To determine the nature of the bifurcation at $R_c=R(0,\mu)$ we need to know the sign of the lower order derivatives

$$R^{mn}=\frac{1}{m!\,n!}\frac{\partial^{m+n}R^{(0,0)}}{\partial\varepsilon^m\partial\mu^n}.$$

These derivatives are related (equation 3.18) to solution derivatives

$$\boldsymbol{v}^{\nu l}(P)=\frac{1}{\nu!\,l!}\frac{\partial^{\nu+l}\boldsymbol{v}(P)}{\partial\varepsilon^\nu\partial\mu^l}$$

of lower order $\nu\leqq m$, $l\leqq n$, $\nu+l<m+n$. In the simplified notation of Chapter 2 we have

$$\begin{aligned}\langle\boldsymbol{v}^{00},\boldsymbol{D}^{mn}\boldsymbol{v}^{00}\rangle=&-\langle\boldsymbol{v}^{00},\hat{\Sigma}^{mn}\boldsymbol{D}\boldsymbol{v}\rangle+\langle\boldsymbol{v}^{00},\Sigma^{m-1,n}(\boldsymbol{v}\cdot\partial)\boldsymbol{v}\rangle\\&-\langle\boldsymbol{v}^{00},\boldsymbol{Q}\boldsymbol{v}^{m,n-1}\rangle,\end{aligned} \tag{6.11}$$

where

$$\langle\boldsymbol{v}^{00},\boldsymbol{D}^{mn}\boldsymbol{v}^{00}\rangle=pR^{mn}\int v_3^{00}v_4^{00}\,dP=-p\frac{R^{mn}}{R^{00}}\int|\nabla v_4|^2\,dP \tag{6.12}$$

and $A^{\nu l}=0$ if $\nu=-1$ or $l=-1$ for any A.

We next establish properties of the derivatives:

(a) $R^{01}=R^{10}=0$, $R^{20}\geqq 0$.

(b) *Let* $m>0, n>0$ *or* $m>2, n\geqq 0$. *To compute* R^{mn} *we need* $(m+1)\times(n+1)-2$ *elements of the mn matrix of elements* $\boldsymbol{v}^{\nu l}$. *The two deleted elements are* $\boldsymbol{v}^{m-1,n}$ *and* $\boldsymbol{v}^{mn}$. Property (b) has not been noted elsewhere.

To establish these properties we first observe that for any solenoidal $\boldsymbol{v}(\partial\cdot\boldsymbol{v}=0)$ such that $\boldsymbol{v}\in H$ and any vector $\boldsymbol{\Phi}$

$$\langle\boldsymbol{\Phi},(\boldsymbol{v}\cdot\partial)\boldsymbol{\Phi}\rangle=0. \tag{6.13}$$

Hence

$$\langle\boldsymbol{v}^{00},\Sigma^{m-1,n}(\boldsymbol{v}\cdot\partial)\boldsymbol{v}\rangle=\langle\boldsymbol{v}^{00},(\boldsymbol{v}^{00}\cdot\partial)\boldsymbol{v}^{m-1,n}\rangle+\langle\boldsymbol{v}^{00}\hat{\Sigma}^{m-1,n}(\boldsymbol{v}\cdot\partial)\boldsymbol{v}\rangle. \tag{6.14}$$

It is also true that

$$p^{-1}\langle\boldsymbol{v}^{00},\boldsymbol{Q}\,\boldsymbol{v}^{00}\rangle=\int v_4^{00}\,v_3^{00}\,z\,dP=\iint dx\,dy\int_{-\frac{1}{2}}^{\frac{1}{2}}z\,v_4^{00}\,v_3^{00}\,dz=0 \tag{6.15}$$

because v_3^{00} and v_4^{00} are even functions. This relation holds whenever $Q_{ij}=\delta_{14}\,\delta_{j3}\,S(z)$ is an odd function of z.

Put $m=1$, $n=0$ into (6.11) and use (6.12) and (6.13) to find that $R^{10}=0$. Put $m=0$, $n=1$ into (6.11) and use (6.12) and (6.13) to find that $R^{01}=0$. Of course, $\boldsymbol{D}^{10}=\boldsymbol{D}^{01}=0$.

We prove property (b) next and then show that $R^{20}\geqq 0$*. Property (b) follows from the formula

$$\begin{aligned}\langle\boldsymbol{v}^{00},\boldsymbol{D}^{mn}\,\boldsymbol{v}^{00}\rangle=&\langle\boldsymbol{v}^{10},\boldsymbol{Q}\,\boldsymbol{v}^{m-1,n-1}\rangle-\langle\boldsymbol{v}^{00},\boldsymbol{Q}\,\boldsymbol{v}^{m,n-1}\rangle\\&+\langle\boldsymbol{v}^{10},\boldsymbol{D}^{m-1,n}\,\boldsymbol{v}^{00}\rangle+\langle\boldsymbol{v}^{10},\hat{\Sigma}^{m-1,n}\boldsymbol{D}\,\boldsymbol{v}\rangle\\&-\langle\boldsymbol{v}^{00},\hat{\Sigma}^{mn}\boldsymbol{D}\,\boldsymbol{v}\rangle-\langle\boldsymbol{v}^{10},\Sigma^{m-2,n}(\boldsymbol{v}\cdot\partial)\boldsymbol{v}\rangle\\&+\langle\boldsymbol{v}^{00},\hat{\Sigma}^{m-1,n}(\boldsymbol{v}\cdot\partial)\boldsymbol{v}\rangle.\end{aligned} \tag{6.16}$$

It is obvious that (6.16) contains no derivative of $\boldsymbol{v}$ of order mn. There is also no derivative of $\boldsymbol{v}$ of order $m-1,n$. The only place where such a derivative could appear is as a term of

$$\langle\boldsymbol{v}^{00},\hat{\Sigma}^{mn}\boldsymbol{D}\,\boldsymbol{v}\rangle=\langle\boldsymbol{v}^{00},\boldsymbol{D}^{10}\,\boldsymbol{v}^{m-1,n}\rangle+\cdots$$

and $\boldsymbol{D}^{10}=0$.

To obtain (6.16) set (6.14) into (6.11). Then find that

$$\begin{aligned}\langle\boldsymbol{v}^{00},(\boldsymbol{v}^{00}\cdot\partial)\boldsymbol{v}^{m-1,n}\rangle&=-\langle\boldsymbol{v}^{m-1,n},(\boldsymbol{v}^{00}\cdot\partial)\boldsymbol{v}^{00}\rangle\\&=-\langle\boldsymbol{v}^{m-1,n},\boldsymbol{D}^{00}\,\boldsymbol{v}^{10}\rangle=-\langle\boldsymbol{v}^{10},\boldsymbol{D}^{00}\,\boldsymbol{v}^{m-1,n}\rangle\\&=\langle\boldsymbol{v}^{10},\boldsymbol{D}^{m-1,n}\boldsymbol{v}^{00}\rangle+\langle\boldsymbol{v}^{10},\hat{\Sigma}^{m-1,n}\boldsymbol{D}\,\boldsymbol{v}\rangle\\&\quad-\langle\boldsymbol{v}^{10},\Sigma^{m-2,n}(\boldsymbol{v}\cdot\partial)\boldsymbol{v}\rangle+\langle\boldsymbol{v}^{10},\boldsymbol{Q}\,\boldsymbol{v}^{m-1,n-1}\rangle\end{aligned}$$

where the last equality follows from (2.9a). This proves (6.16)

Let $m=2$, $n=0$. Then evaluate (6.16) and find that

$$\langle\boldsymbol{v}^{00},\boldsymbol{D}^{20}\,\boldsymbol{v}^{00}\rangle=\langle\boldsymbol{v}^{10},\boldsymbol{D}^{00}\,\boldsymbol{v}^{10}\rangle. \tag{6.17}$$

* Property (b) follows whenever $R^{10}=R^{10}=0$. The relation $R^{01}=0$ holds generally and one can ordinarily choose a temperature reference T_0 so as to force $R^{01}=0$ (cf. Busse [16]). Hence properties (a) and (b) hold for most generalized Bénard problems and not just for constant heat sources.

Since $\langle \boldsymbol{v}^{00}, \boldsymbol{D}^{00}\boldsymbol{v}^{00}\rangle = 0$ we have

$$R^{00} = \frac{-\int v_4 \Delta v_4 \, dP}{2\int v_3 v_4 \, dP + \sum_{i=1}^{3} \int v_i \Delta v_i \, dP} \tag{6.18}$$

when $\boldsymbol{v} = \boldsymbol{v}^{00}$. R^{00} can be obtained as a minimum of the functional of L.H.S. of (6.18) for solenoidal $\boldsymbol{v}$ defined on a suitable Hilbert space. For these $\boldsymbol{v}$

$$\langle \boldsymbol{v}, \boldsymbol{D}^{00}\boldsymbol{v}\rangle \leqq 0 .$$

$\boldsymbol{v}^{10}$ is such a $\boldsymbol{v}$ and by (6.11)

$$\frac{-p R^{20}}{R^{00}} \int |\nabla \boldsymbol{v}_4|^2 \, dP \leqq 0$$

or $R^{20} \geqq 0$. This proves properties (a) and (b).

Finally let us consider the possible existence of subcritical solutions. A subcritical solution will exist if

$$R(\varepsilon, \mu) - R(0, \mu) < 0 .$$

Since R is analytic and

$$\frac{\partial R(0,0)}{\partial \varepsilon} = \frac{\partial R(0,0)}{\partial \mu} = 0,$$

subcritical solutions will exist if

$$\frac{\partial^2 R(0,0)}{\partial \varepsilon \, \partial \mu} \varepsilon \mu < -\frac{1}{2} \frac{\partial^2 R(0,0)}{\partial \varepsilon^2} \varepsilon^2 + O(\varepsilon^l \mu^\nu), \qquad l + \nu \geqq 3 . \tag{6.19}$$

Since $\varepsilon\mu$ can be chosen as either positive or negative we can always satisfy (6.19) when $R^{11} \neq 0$, $|\varepsilon| < |\mu|$ and $|\mu|$ is sufficiently small. Hence if $R^{11} \neq 0$, *then there exists a subcritical solution of* (6.1-5).

This solution is analytic in ε and μ and is subcritical for norms $|\varepsilon| < |\mu|$ where $|\mu|$ is sufficiently small. The value of $R^{11} = \partial^2 R / \partial \varepsilon \partial \mu$ may be obtained from (4.15) as

$$\frac{-R^{11}}{R^{00}} \int |\nabla v_4^{00}|^2 \, dP = -\int z (v_4^{00} v_3^{10} - v_4^{10} v_3^{00}) \, dP$$

KRISHNAMURTI [17] finds that $R^{11} \neq 0$ for hexagons, but $= 0$ for all other solutions of (6.9).*

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* In [17] it is asserted that LORTZ [24] has proved the convergence of ε series (when $\mu = 0$). LORTZ's proof presumably applies to periodic disturbances between free-conducting surfaces. The proof, however, is not correct even in this restricted context as the vertical vorticity has been set to zero. The proof may, however, apply to periodic rolls between free conducting layers.

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