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Eigenvalue bounds for the Orr–Sommerfeld equation. Part 2

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Rigorous estimates of amplification rates, wave speeds and sufficient conditions for linear stability are derived for the manifold of solutions of the Orr–Sommerfeld problem governing parallel motion in the boundary layer and in round pipes. The estimates for channel flow (part I) are improved and compared with numerical results for the neutral stability of Jeffery–Hamel flow.

1. Introduction

This paper continues an earlier investigation (Joseph 1968, hereafter called I) of the linear theory of parallel flow. In I, it was shown that the application of isoperimetric inequalities to the Orr–Sommerfeld problem in channels leads to rigorous estimates of the amplification rates and wave speeds over the entire manifold of solutions, as well as to improved sufficient conditions for linear stability. This paper extends and elaborates I in three ways. First, estimates of eigenvalues and regions of linear stability of the same generality and rigour as those given in I are constructed (see § 2) for a bounded domain approximation to boundary-layer flows. The estimates have the same form as those given in I but the isoperimetric inequalities themselves depend on the wave-number through the boundary conditions at the outer edge of the boundary layer. The channel estimates themselves are improved (§ 3) and compared (§ 4) to Eagles (1966) finite difference calculation of neutral Orr–Sommerfeld limits and wave speeds for Jeffery–Hamel flow in diverging channels. For these flows, the critical Reynolds number can be very low and the estimates given in I, and here, are respectable *a priori* estimates of the true situation. The paper concludes (§ 5) with estimates of amplification rates, wave speed and limits of linear stability for arbitrary parallel motions in round pipes.

2. Linear stability of the boundary layer

We consider the Orr–Sommerfeld problem

$$(U-C)(\phi''-\alpha^2\phi)-U''\phi = -(i/\alpha R)(\phi^{\text{iv}}-2\alpha^2\phi''+\alpha^4\phi), \tag{1a}$$

with boundary conditions $\phi(0)=\phi'(0)=\phi(\infty)=\phi'(\infty)=0$. Here $C=c_r+ic_i$ and α, R are real non-negative parameters. This problem is a dimensionless representation of the problem which is conventionally assumed to govern the linear stability of the boundary layer (thickness $Y = \delta$ at $y = 1$) in a stream $V(Y) = V(\infty)\, U(y)$ such that $U(y) = 1$ and $\phi = \tilde{\phi}$ for $y \geqslant 1$ with Reynolds number $R = V(\infty)\,\delta/\nu$.

In asymptotic analysis, this problem is approximated with an equivalent problem in the bounded domain $y \in [0, 1]$ (cf. Lin 1945; Reid 1965, p. 249). Outside the boundary layer ($y \geqslant 1$) we take $U \equiv 1$ and $U' \equiv 0$ and set

$$\tilde{\phi}(y) = A\, e^{-\alpha y} + B e^{-\beta y},$$

where $\beta^2 = i\alpha R(1-C) + \alpha^2$ and $\mathrm{Re}(\beta)$ is large like $(\alpha R)^{\frac{1}{2}}$. The viscous part of this solution, it is reasoned, decays much more rapidly than the inviscid part and it is conventional to seek solutions such that $\tilde{\phi} \sim e^{-\alpha y}$ for $y \geqslant 1$. Conversely, if the *single condition* $\tilde{\phi}'(1) + \alpha\tilde{\phi}(1)$ is set, then $B = 0$. The requirement that the velocities be continuous across the boundary layer ($\tilde{\phi}(1) = \phi(1)$, $\tilde{\phi}'(1) = \phi'(1)$) then implies that $\phi'(1) + \alpha\phi(1) = 0$. In asymptotic theory this single condition plus the requirement that the solution be inviscid at $y = 1$ suffices to determine a unique solution. Actually this inviscid requirement is equivalent to a second boundary condition on ϕ, namely $\tilde{\phi}''(1) = \phi''(1)$ and this suffices to guarantee not only the continuity of velocity but also of (zero) vorticity (since $\phi'' - \alpha^2\phi$ then must vanish). A second boundary condition at $y = 1$ is, of course, necessary for a well-posed problem in the bounded domain. It turns out that *a priori* estimates are not easily constructed for this 'best' bounded domain approximation to the linear stability of the boundary layer. If, however, one relaxes the condition of continuous vorticity but not of continuous velocity across the boundary layer then, relative to the problem for which $\tilde{\phi} = A\, e^{-\alpha y}$ for $y \geqslant 1$, $\tilde{\phi} = \phi$, $\tilde{\phi}' = \phi'$ and $\tilde{\phi}'''\phi'' = \phi'''\tilde{\phi}''$ at $y = 1$, that is, for (1 *a*) and

$$\phi(0) = \phi'(0) = 0, \quad \phi'(1) + \alpha\phi(1) = 0, \quad \phi'''(1) + \alpha\phi''(1) = 0, \tag{1b}$$

one may obtain exact, *a priori*, results. Asymptotic (large αR) solutions naturally satisfy (1 *a*, *b*), approximately, and do not lead to discontinuous vorticity. We shall find that in the worst possible circumstance (see figure 1 and equation (11)) no neutral or amplified solutions of (1 *a*, *b*) are possible when $\alpha R < 25/q$, where $q = \max U'(y)$ for $y \in [0, 1]$. Though this estimate does not establish that only large αR neutral solutions of (1 *a*, *b*) can exist, it does show that such solutions cannot exist for sufficiently small values of αR.

The analysis begins with relations for c_r and c_i which are obtained by multiplying (1 *a*) by the conjugate ($\bar{\phi}$) of ϕ and integrating over the range of y, using (1 *b*). In this way we find that

$$c_i = \{Q - \bar{Q} - (\alpha R)^{-1}(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)\}/(I_1^2 + \alpha^2 I_0^2) \tag{2}$$

and

$$c_r = \left\{\int_0^1 [U|\phi'|^2 + (\alpha^2 U + \tfrac{1}{2}U'')\,|\phi|^2]\,dy + \alpha\,|\phi(1)|^2\right\}\Big/(I_1^2 + \alpha^2 I_0^2), \tag{3}$$

where

$$\left.\begin{aligned} I_0^2 &= \int_0^1 |\phi^2|\,dy, \quad I_1^2 = \int_0^1 |\phi'|^2\,dy + \alpha|\phi(1)|^2, \\ I_2^2 &= \int_0^1 |\phi''|^2\,dy, \quad Q = \frac{i}{2}\int_0^1 U'\phi\bar{\phi}'\,dy. \end{aligned}\right\} \tag{4}$$

Equations (2) and (3) are formally identical (except for boundary terms) to equations of the same number given in I and governing channel flow. As in I, these equations and the relevant isoperimetric inequalities form a basis for estimating the allowed range of stability parameters over the manifold of solutions. We need only establish the relevant inequalities over a class of functions satisfying the boundary conditions which contain solutions as a subset.

Let $\bar{H}(\alpha)$ be a Hilbert space containing complex-valued elements

$$\{\phi\colon \phi(0) = \phi'(0) = 0, \quad \phi' + \alpha\phi = 0\}$$

which appear as limit points of sequences of four times continuously differentiable functions in the completion under the norm $\sqrt{\int_0^1 |\phi''|^2\,dy}$. We assume that $U''(y)$ is continuous on $y \in [0, 1]$, so that regular solutions $(1a, b)$ are in $\bar{H}(\alpha)$.

Lemma 1. *Let* $\phi \in \bar{H}(\alpha)$. *Then*

$$I_1^2 \geqslant \lambda_1^2(\alpha)\, I_0^2, \quad I_2^2 \geqslant \lambda_2^2(\alpha)\, I_1^2 \quad \textit{and} \quad I_2^2 \geqslant \lambda_3^2(\alpha)\, I_0^2, \tag{5a–c}$$

where

$$\lambda_1^2(\alpha) \geqslant \tfrac{1}{4}\pi^2, \quad \lambda_2^2(\alpha) \geqslant \pi^2 \quad \textit{and} \quad \lambda_3^2(\alpha) \geqslant \left(\frac{4{\cdot}73}{2}\right)^4 \tag{5d}$$

with equality holding in (5*d*) *when* $\alpha = 0$.

Proof of Lemma 1

As in I, if the inequalities (5) hold for real-valued functions in $\bar{H}(\alpha)$, say $\phi \in H(\alpha)$, then they also hold for complex-valued functions. The values λ_1, λ_2 and λ_3 are found from minimum problems for $\phi \in H(\alpha)$. Thus,

$$\lambda_1^2(\alpha) = \min_{\phi} \frac{\int_0^1 (\phi')^2\,dy + \alpha[\phi(1)]^2}{\int_0^1 \phi^2\,dy}, \tag{6a}$$

$$\lambda_2^2(\alpha) = \min_{\phi} \frac{\int_0^1 (\phi'')^2\,dy}{\int_0^1 (\phi')^2\,dy + \alpha[\phi(1)]^2} \tag{6b}$$

and

$$\lambda_3^2(\alpha) = \min_{\phi} \frac{\int_0^1 (\phi'')^2\,dy}{\int_0^1 \phi^2\,dy}. \tag{6c}$$

Euler equations for (6*a–c*) are

$$\phi'' + \lambda^2\phi = 0, \quad \phi(0) = 0, \quad \phi'(1) + \alpha\phi(1) = 0, \tag{7a}$$

$$\left.\begin{aligned} \phi^{\text{iv}} + \lambda^2\phi'' = 0, \quad & \phi(0) = \phi'(0) = 0, \\ \phi'(1) + \alpha\phi(1) = 0, \quad & \phi'''(1) + \alpha\phi''(1) = 0 \end{aligned}\right\} \tag{7b}$$

and

$$\left.\begin{aligned} \phi^{\text{iv}} - \lambda^2\phi = 0, \quad & \phi(0) = \phi'(0) = 0, \\ \phi'(1) + \alpha\phi(1) = 0, \quad & \phi'''(1) + \alpha\phi''(1) = 0. \end{aligned}\right\} \tag{7c}$$

The boundary condition $\phi'''(1) + \alpha\phi''(1) = 0$ is not automatically satisfied by $\phi \in H(\alpha)$ but arises as a natural boundary condition in problems (7*b*) and (7*c*).

The eigenfunctions for problem (7*a*) have the form $\phi = A\sin\lambda y$. The principal eigenvalues $\lambda_1(\alpha)$ are the smallest positive root of the equation

$$\lambda\cos\lambda + \alpha\sin\lambda = 0.$$

The eigenfunctions for problems (7*b*) and (7*c*) are

$$\phi = A(\sin\lambda y - \lambda y) + B(\cos\lambda y - 1) \tag{8b}$$

and

$$\phi = A(\sinh\lambda y - \sin\lambda y) + B(\cosh\lambda y - \cos\lambda y), \tag{8c}$$

respectively. The principal eigenvalues are found as the smallest positive root of

$$\begin{vmatrix} \lambda_2(\cos\lambda_2 - 1) + \alpha\sin\lambda_2 - \alpha\lambda_2 & -\lambda_2\sin\lambda_2 + \alpha(\cos\lambda_2 - 1) \\ -\lambda_2\cos\lambda_2 - \alpha\sin\lambda_2 & \lambda_2\sin\lambda_2 - \alpha\cos\lambda_2 \end{vmatrix} = 0 \tag{9b}$$

and

$$\begin{vmatrix} \lambda_3(\cosh\lambda_3 - \cos\lambda_3) + \alpha(\sinh\lambda_3 - \sin\lambda_3) & \lambda_3(\sinh\lambda_3 + \sin\lambda_3) + \alpha(\cosh\lambda_3 - \cos\lambda_3) \\ \lambda_3(\cosh\lambda_3 + \cos\lambda_3) + \alpha(\sinh\lambda_3 + \sin\lambda_3) & \lambda_3(\sinh\lambda_3 - \sin\lambda_3) + \alpha(\cosh\lambda_3 + \cos\lambda_3) \end{vmatrix} = 0. \tag{9c}$$

respectively. Graphs of $\lambda_1(\alpha)$, $\lambda_2(\alpha)$ and $\lambda_3(\alpha)$ are given in figure 1.

The following two theorems generalize the results of I to boundary layers. As in I,

$$q = \max_{y\in[0,1]} |U'(y)| = U'_{\max}.$$

THEOREM 1. *Let $C(\alpha, R)$ be any eigenvalue of* (1*a*, *b*). *Then*

$$c_i \leqslant \frac{q}{2\alpha} - \left\{\frac{\lambda_1^2(\alpha)\,(\lambda_2^2(\alpha)+\alpha^2)}{\lambda_1^2(\alpha)+\alpha^2} + \alpha^2\right\}\Big/\alpha R. \tag{10}$$

Moreover, no amplified disturbances ($c_i > 0$) *of* (1*a*, *b*) *exist if*

$$\alpha Rq < f(\alpha) = \max\,[M_1, M_2], \tag{11}$$

where

$$M_1 = \lambda_3(\alpha)\,\lambda_2(\alpha) + 2^{\frac{3}{2}}\alpha^3$$

and

$$M_2 = \lambda_3(\alpha)\,\lambda_2(\alpha) + 2\alpha^2\lambda_1(\alpha).$$

The graph of the bound (11) is given in figure 2.

Proof of Theorem 1

Applying Schwarz's inequality to (2) we find that

$$c_i \leqslant \{qI_0I_1 - (\alpha R)^{-1}(I_2^2 + 2\alpha^2I_1^2 + \alpha^4I_0^2\}/(I_1^2 + \alpha^2I_0^2).$$

We use the estimates

$$2\alpha I_1I_0 \leqslant I_1^2 + \alpha^2I_0^2, \quad I_2^2/I_1I_0 \geqslant \lambda_2\lambda_3,$$

$$(I_2^2 + 2\alpha^2I_1^2 + \alpha^4I_0^2)/(I_1^2 + \alpha^2I_0^2) = \frac{I_2^2 + \alpha^2I_1^2}{I_1^2 + \alpha^2I_0^2} + \alpha^2 \geqslant \alpha^2 + \frac{\lambda_1^2(\lambda_2^2 + \alpha^2)}{\lambda_1^2 + \alpha^2}$$

and

$$\frac{I_2^2 + 2\alpha^2I_1^2 + \alpha^4I_0^2}{I_0I_1} \geqslant \left\{\begin{matrix} \lambda_3\lambda_2 + \dfrac{2\alpha^2}{I_0I_1}\left[\left(I_1 - \dfrac{\alpha}{\sqrt{2}}I_0\right)^2 + \dfrac{2\alpha}{\sqrt{2}}I_0I_1\right] \\ \lambda_3\lambda_2 + \dfrac{2\alpha^2I_1^2}{I_1I_0} \end{matrix}\right\} \geqslant \left\{\begin{matrix} M_1 \\ M_2 \end{matrix}\right\}.$$

as in I, to prove the theorem.

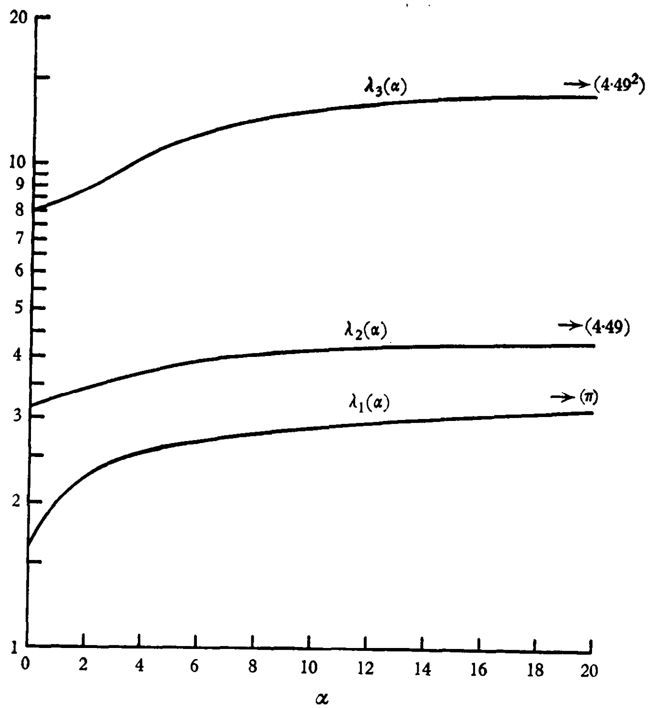

FIGURE 1. Principal eigenvalues for problem 6.

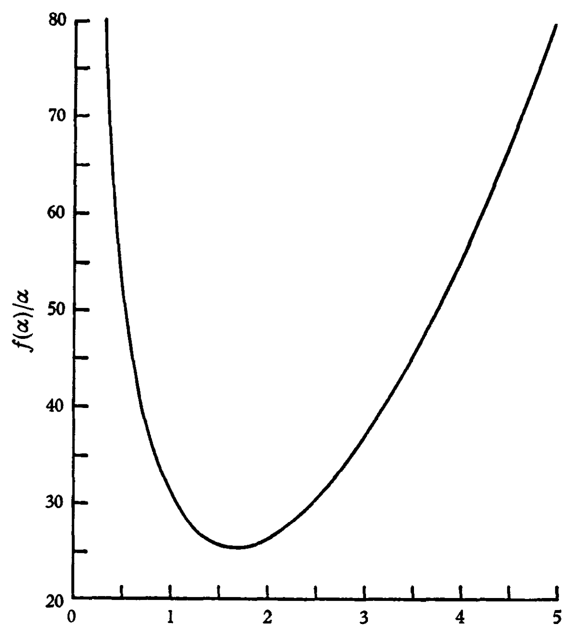

FIGURE 2. Graph of $f(\alpha)/\alpha$ as given by equation (11). In the boundary layer no linear disturbance can exist if qR is in the region below the graph.

Just as in I, we find from (3) that

$$c_r(I_1^2+\alpha^2 I_0^2) = U(y_1)\int_0^1 |\phi'|^2\,dy+\alpha|\phi(1)|^2+(\alpha^2 U(y_2)+\tfrac{1}{2}U''(y_3))\,I_0^2,$$

where y_1, y_2, $y_3 \in (0,1)$ are mean values. From this follows:

THEOREM 2. *Let $C(\alpha, R)$ be any eigenvalue of $(1a,b)$. Then the following inequalities hold:*

$$\left.\begin{aligned} &U''_{\min} \geqslant 0,\\ &U_{\min} < c_r < U_{\max} + \frac{U''_{\max}}{2(\lambda_1^2+\alpha^2)};\end{aligned}\right\} \qquad (12a)$$

$$\left.\begin{aligned} &U''_{\min} \leqslant 0 \leqslant U''_{\max},\\ &U_{\min} + \frac{U''_{\max}}{2(\lambda_1^2+\alpha^2)} < c_r < U_{\max} + \frac{U''_{\max}}{2(\lambda_1^2+\alpha^2)};\end{aligned}\right\} \qquad (12b)$$

$$\left.\begin{aligned} &U''_{\max} \leqslant 0,\\ &U_{\min} + \frac{U''_{\min}}{2(\lambda_1^2+\alpha^2)} < c_r < U_{\max}.\end{aligned}\right\} \qquad (12c)$$

It should be noted that theorems 1 and 2 hold rigorously only relative to the problem $(1a,b)$. This problem is itself an approximation; in it we replace the infinite domain with a finite domain, we assume that viscous effects are negligible outside the boundary layer and the continuity of the vorticity across the boundary layer is not guaranteed. The boundary conditions at $y = 1$ are, however, compatible with known asymptotic (large αR) results. These give continuous (zero) vorticity in regions well within the main flow boundary layer. But for small αR (say damped solutions), the conditions at $y = 1$ may not faithfully approximate the true problem. For flows in bounded domains such approximations are not necessary and all the results apply without qualification. We turn to these bounded flows next.

3. Channel flow

Here we compare the channel flow estimates given in I with the result of a finite difference calculation for the Orr–Sommerfeld stability of Jeffery–Hamel flow (Eagles 1966). We first derive a sharper result than that given by theorem 1 of I.

THEOREM 3. *Let $C(\alpha, R)$ be any eigenvalue of $(1a)$ such that*

$$\phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0. \qquad (13)$$

Then theorem 1 holds with $\lambda_3^2 = (4{\cdot}73)^4$, $\lambda_2^2 = 4\pi^2$ and $\lambda_1^2 = \pi^2$.

Proof of Theorem 3

For the channel flow boundary conditions (13) the boundary terms in I_1^2 vanish. The values $\lambda_3^2 = (4{\cdot}73)^4$ and $\lambda_1^2 = \pi^2$ are as in I. The proof is constructed as for theorem 1 of this paper. It remains to establish that $\lambda_2^2 = 4\pi^2$.† The Euler equation for λ_2 is

$$\phi^{\text{iv}} + \lambda_2 \phi'' = 0 \quad \text{and} \quad \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0.$$

A fundamental solution of this problem is $\phi = A(\sin\lambda y - \lambda y) + B(\cos\lambda y - 1)$. For the minimizing solution we set $A = 0$, $\lambda = \lambda_2 = 2\pi$. This proves theorem 3.

The estimates

$$c_i \leqslant \frac{q}{2\alpha} - \left\{\frac{\pi^2(4\pi^2+\alpha^2)}{\pi^2+\alpha^2} + \alpha^2\right\}\Big/\alpha R,$$

and

$$M_1 = (4{\cdot}73)^2\, 2\pi + 2^{\frac{3}{2}}\alpha^3,$$

$$M_2 = (4{\cdot}73)^2\, 2\pi + 2\alpha^2\pi,$$

are an improvement over those given in I. In particular, by using the better estimate for λ_2^2, we raise the minimum critical Reynolds number by a factor of 2. These new values for M_1 and M_2 do not give the best *a priori* estimate of sufficient limits. The best estimates make use of the exact profile in a variational problem associated with (2) when $c_i = 0$ (Orr 1907). Such limits are, of course, tied to particular flows and require numerical integration.

4. Jeffery–Hamel flow

Eagles (1966) has examined the stability of a family of Jeffery–Hamel flows in diverging wedge-shaped channels on the basis of an Orr–Sommerfeld approximation. The relevant parameters are $\gamma = \theta M/\nu$, where θ is the semi-divergence angle of the wedge, M is one-half the volumetric flow rate and M/ν the Reynolds number. Under certain smoothness conditions, Fraenkel (1963) has shown that such motions approximate flow in diverging channels with curved walls. Presumably the result of Eagles' calculation has a relevance to the stability of such diverging channel flows and, for these, θ is the *local* semi-divergence angle. I refer the reader to Eagles' paper for a discussion of this point and for the justification and results of the Orr–Sommerfeld analysis.

For our purpose, we need only to set out the stability problem

$$D^4\phi - 2k^2D^2\phi + k^4\phi = ik(M/\nu)\{(W-C)(D^2\phi - k^2\phi) - D^2W\phi\} \tag{14a}$$

and

$$\phi(-1) = \phi'(-1) = \phi(1) = \phi'(1) = 0. \tag{14b}$$

Here $D = d/d\eta$ where η is the polar angle in the wedge and $W = W(\eta,\gamma)$, independent of radius.

The problem (1*a*) and (13) is equivalent to (14*a*, *b*) under the change of variables

$$2y = \eta + 1, \quad \alpha = 2k, \quad U(y) = W(\eta,\gamma), \quad U' = 2DW,$$

$$U'' = 4D^2W \quad \text{and} \quad R = 2M/\nu.$$

In these variables, stability is guaranteed by theorem 3 for k and M/ν, such that

$$k\,|DW|_{\max}(M/\nu) \leqslant \pi(4{\cdot}73)^2/4 + \max[2^{\frac{3}{2}}k^3,\ 8\pi k^2]. \tag{15}$$

† Let u be the minimizing function for λ_3; then

$$(4{\cdot}73)^4 = \lambda_3^2 = I_2^2/I_0^2 \geqslant 4\pi^2\, I_1^2/I_0^2 \geqslant 4\pi^4.$$

In particular, this shows that the inequality $I_1^2 \geqslant \pi^2 I_0^2$, which uses only two of four boundary conditions, is not a weak estimate.

Moreover, for those profiles in which DW changes sign and *any* solution (damped, neutral, amplified or higher mode),

$$W_{\min} + \frac{2(D^2W)_{\min}}{\pi^2+4k^2} < c_r < W_{\max} + \frac{2(D^2W)_{\max}}{\pi^2+4k^2}. \tag{16}$$

In figure 3 the estimate (15) is superposed onto Eagles' neutral curve for the profile $\gamma = 4{\cdot}71$ ($|DW|_{\max} \simeq 3{\cdot}34$). Figure 4 gives a similar result for a profile with backflow ($\gamma = 5{\cdot}45$, $|DW|_{\max} \simeq 6$). The comparison between the *a priori* estimate and the true Orr–Sommerfeld result is certainly no disappointment. The numerical precision of (15) relative to these very unstable profiles is, of course, lost for the more stable profiles, e.g. Poiseuille flow. If, however, one replaces the factor $|DW|_{\max}$ in (15) with a mean value of $|DW|$, say

$$\overline{|DW|}\int_{-1}^{1}|\phi|\,|\overline{\phi}'|\,d\eta = \int_{-1}^{1}|DW|\,|\phi|\,|\overline{\phi}'|\,d\eta,$$

then the altered (15) is a numerically better estimate. Unfortunately, I see no way to obtain the more exact estimate without a relaxation of mathematical rigour.

For the profiles with back flow Eagles found that along the lower branch of the neutral curve, kM/ν tends to a finite value and the wave speeds are more negative than the smallest value of W. The estimate (16) shows clearly that c_r dominates $W_{\min}$, the difference being due to an effect of the curvature of the profile. For $\gamma = 5{\cdot}45$ Eagles finds that

$$(c_r)_{\min} = -0{\cdot}737 \leqslant c_r \leqslant 1{\cdot}72 = (c_r)_{\max},$$

where the value on the left is the limiting value on the lower branch ($R \to \infty$, $k \to 0$) and the value on the right is the limiting value on the upper branch ($R \to \infty$, $k \to 4{\cdot}29$). For

$$\gamma = 5{\cdot}45, \quad W_{\min} = -0{\cdot}245, \quad W_{\max} = 3{\cdot}089, \quad (D^2W)_{\min} = -17{\cdot}98$$

and $$(D^2W)_{\max} = 7{\cdot}87.$$

Since $$(c_r)_{\min} < W_{\min},$$

it is clear from (16) that the negative wave speed is somehow associated with an effect of the large negative value of D^2W. For $k = 0$, (16) gives

$$-3{\cdot}88 < c_r$$

and for $k = 4{\cdot}29$, $$c_r < 3{\cdot}278.$$

We note that the estimate (16) can be written as an equality (see equation (12) of part I)

$$\overline{W} + \frac{2(\overline{D^2W})}{\pi^2+4k^2} = c_r,$$

where the overbar quantities are (different) mean values on the range of W and D^2W for $\eta \in [-1, 1]$.

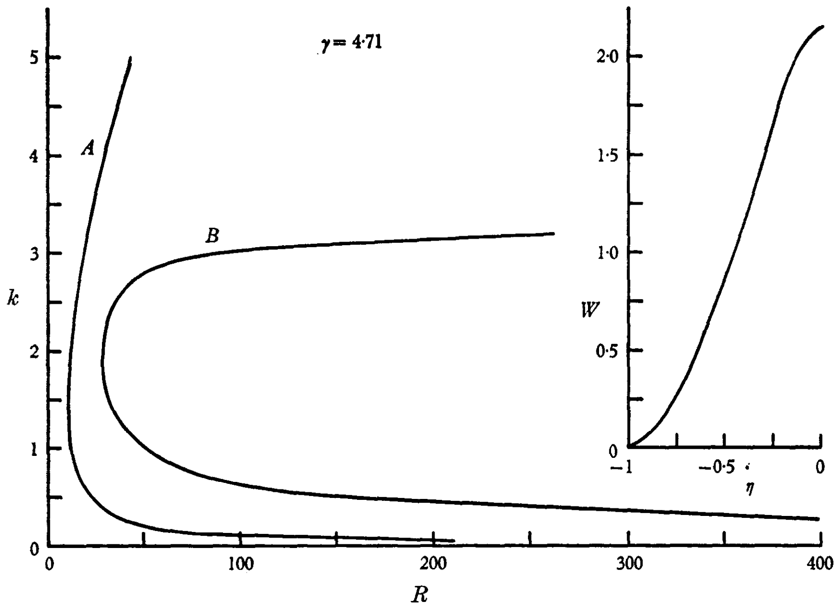

FIGURE 3. Stability of Jeffery–Hamel flow for $\gamma = 4{\cdot}71$, $(DW)_{max} \cong 3{\cdot}34$. The neutral curve B is taken from the paper by Eagles. In the region to the left of A there can be no amplified linear disturbance of any bounded $(-1 \leqslant \eta \leqslant 1)$ parallel motion with a maximum velocity less than $(DW)_{max}$. The graph of the velocity profile for $\gamma = 4{\cdot}71$ is given in the upper right-hand corner.

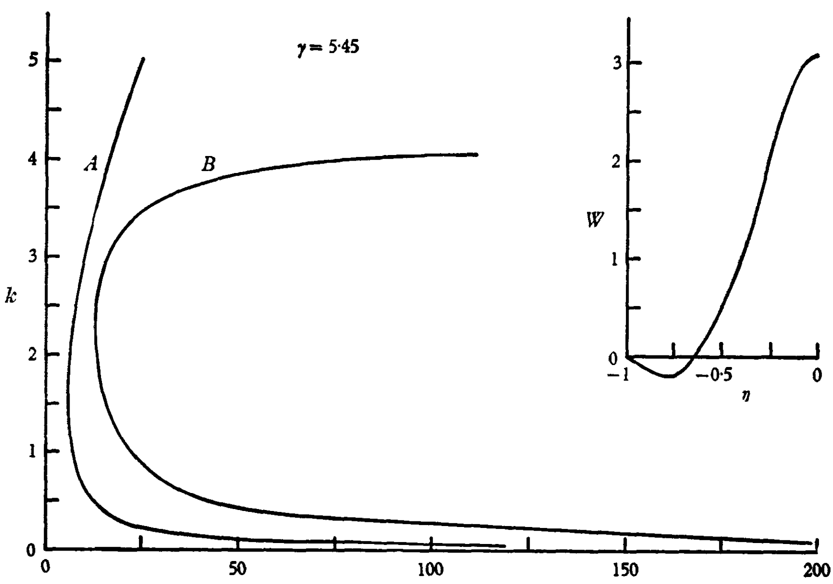

FIGURE 4. Stability of Jeffery–Hamel flow for $\gamma = 5{\cdot}45$ and $(DW)_{max} \cong 6$. The curves A and B are as in figure 3.

5. Stability of parallel flow in round pipes

The strict mathematical problem for parallel flow in round pipes applies only to the Hagen–Poiseuille motion. As is true of channel and boundary-layer flows, it is customary to regard the linearized parallel flow problem as an approximation to nearly parallel motions and, on that account, to consider the Orr–Sommerfeld problem for arbitrary axial velocity components $U(r)$. This problem does not lend itself to exact analysis, and nearly all of the linear work for the viscous flow is approximate. It is in this context that we set the theorems of this section.

Unlike the plane problem, the Orr–Sommerfeld problem for pipe flow is not known to satisfy an analogue to Squires theorem. This leaves open the possibility that instability can here be induced first by a non-axisymmetric disturbance. For the Hagen–Poiseuille flow the linear theory gives absolute stability (as far as its result is known) against disturbances which are axisymmetric or to non-axisymmetric disturbances (Salwen & Grosch 1968) which are small enough. Evidently, the motion has strong stability properties against axisymmetric disturbances (Leite 1959). Parallel profiles which are suitably deviated from the parabolic can, however, be unstable to axisymmetric disturbances (Gill 1965). Though Leite reports that non-axisymmetric disturbances were more rapidly damped than axisymmetric disturbances, the experiments of Fox, Lessen & Bhat (1968) indicate that the Hagen–Poiseuille motion is unstable to an induced spiral mode with a first mode azimuthal variation. This aspect of the result of Fox *et al.* (spiral mode, first mode azimuthal wave-number, finite amplitude) is consistent with the outcome of the non-linear energy analysis of Joseph & Carmi (1969), though the experimental critical Reynolds number (≈ 2100) is an order of magnitude larger than the value (81·49) which guarantees certain non-linear stability.

The results given below apply only to the linear situation. They do, however, cover arbitrary velocity profiles and are mathematically rigorous. Though some of the estimates apply to the three-dimensional problem, the strongest results are stated relative to axisymmetric disturbances.

To obtain the governing boundary-value problem, we linearize the Navier–Stokes equations (in cylindrical co-ordinates) around the parallel flow

$$\mathbf{U} = (0, 0, U(r)).$$

For the disturbance velocity $\mathbf{V} = (V_r, V_\theta, V_z)$ we introduce normal modes

$$\mathbf{V}(r, \theta, z, t, R) \sim \mathbf{u}(r, \alpha, N, C, R)\, e^{i(\alpha z + N\theta - \alpha C t)},$$

where $\mathbf{u} = (w, v, u)$ and obtain

$$i\alpha R(U-C)\, w = -Dp + \mathscr{L}_N w - (2iNv/r^2), \tag{17a}$$

$$i\alpha R(U-C)\, v = -(iN/r)\, p + \mathscr{L}_N v + (2iNw/r^2), \tag{17b}$$

$$i\alpha R(U-C)\, u + RwDU = -i\alpha p + \mathscr{L}_N u \tag{17c}$$

and

$$(1/r)\, D(rw) + (iN/r)\, v + i\alpha u = 0, \tag{17d}$$

where $D = \dfrac{d}{dr}$, $\mathscr{L}_N = \dfrac{1}{r} D(rD) - \dfrac{(N^2+1)}{r^2} - \alpha^2$ and $L_N = \mathscr{L}_N + \dfrac{1}{r^2}$.

For our initial results we require only that u, v, w and p be bounded in the Dirichlet norm and

$$u(1) = v(1) = w(1) = 0. \tag{18}$$

The analysis starts with the identities

$$c_i = -\{\langle DU(u\bar{w}+w\bar{u})\rangle + 2R^{-1}D_{\alpha N}\}/2\alpha\langle|\mathbf{u}|^2\rangle \tag{19}$$

and

$$c_r = \{2\alpha\langle U\,|\mathbf{u}|^2\rangle + i\langle DU(u\bar{w}-w\bar{u})\rangle\}/2\alpha\langle|\mathbf{u}|^2\rangle, \tag{20}$$

where

$$\begin{aligned} D_{\alpha N}(\mathbf{u}) &= -\langle w\mathscr{L}_N\bar{w}\rangle - \langle v\mathscr{L}_N\bar{v}\rangle - \langle uL_N\bar{u}\rangle - 2iN\left(\left\langle\frac{\bar{v}}{r}\frac{w}{r}\right\rangle - \left\langle\frac{\bar{w}}{r}\frac{v}{r}\right\rangle\right) \\ &= \langle|Dw|^2+|Dv|^2+|Du|^2+\alpha^2(|w|^2+|v|^2+|u|^2)\rangle \\ &\quad + N^2\left\langle\left|\frac{u}{r}\right|^2\right\rangle + \left\langle\frac{|Nw+iv|^2}{r^2} + \frac{|Nv-iw|^2}{r^2}\right\rangle \end{aligned} \tag{21}$$

and

$$\langle a\rangle = \int_0^1 ra\,dr.$$

To obtain (19) and (20) we eliminate p from (17) and find

$$iR(NG-\alpha rB) = iNL_N u - i\alpha r\mathscr{L}_N v + 2\alpha N(w/r) \tag{22a}$$

and

$$R(-iNA + D(rB)) = -iN\mathscr{L}_N w - 2N^2\left(\frac{v}{r^2}\right) + D(r\mathscr{L}_N v) + 2iND(w/r), \tag{22b}$$

where

$$A = i\alpha w(U-C), \quad B = i\alpha(U-C)v, \quad G = i\alpha(U-C)u + wDU. \tag{23}$$

We integrate ur multiplied by the complex conjugate of (22a) and wr multiplied by the complex conjugate of (22b) over $[0, 1]$ and subtract the result. Now integrating by parts and using (17d) we find that

$$-i\alpha\langle ru\bar{B}\rangle + \langle wD(r\bar{B})\rangle = iN\langle v\bar{B}\rangle,$$

$$-i\alpha\langle ur\mathscr{L}_N\bar{v})\rangle + \langle wD(r\mathscr{L}_N\bar{v})\rangle = iN\langle v, \mathscr{L}_N\bar{v}\rangle$$

and

$$\alpha\langle u, (\bar{w}/r)\rangle + i\langle wD(\bar{w}/r)\rangle = -N\langle(\bar{w}/r^2)\,v\rangle.$$

In this way we arrive at the relation

$$-R(\langle w\bar{A}\rangle + \langle v\bar{B}\rangle + \langle u\bar{G}\rangle) = D_{\alpha N}(\mathbf{u}). \tag{24}$$

Equations (19) and (20) are found as the real and imaginary parts of (24), using (23).

The next two theorems establish estimates of amplification rates and waves speeds over the manifold of solution of (17) and (18) which are bounded in the Dirichlet norm.

THEOREM 4. *Let $C(\alpha, N, R)$ be any eigenvalue of* (17) *and* (18). *Then*

$$c_i \leqslant \frac{q}{2\alpha} - \frac{\eta_0^2+\alpha^2+(N-1)^2}{\alpha R}, \tag{25}$$

where $\eta_0 = 2{\cdot}405$ is the first zero of $J_0(\eta)$ and $q = \max|DU|$ for $r\in[0, 1]$.

THEOREM 5. *Let $C(\alpha, N, R)$ be any eigenvalue of* (17) *and* (18). *Then*

$$U_{\min} - \frac{q}{2\alpha} < c_r < U_{\max} + \frac{q}{2\alpha}. \tag{26}$$

Theorem 5 follows from (20) in an obvious way. It should be noted that (26) does not depend on N and is not uniform in α. A much stronger result holds for the axisymmetric case (theorem 6).

Proof of Theorem 4

By the arithmetic-geometric mean inequality we have

$$-\frac{\langle DU(u\bar{w}+w\bar{u})\rangle}{2\alpha\langle|\mathbf{u}|^2\rangle} \leqslant \frac{q(\langle|u|^2\rangle+\langle|w|^2\rangle)}{2\alpha\langle|\mathbf{u}|^2\rangle} \leqslant \frac{q}{2\alpha}$$

and from (19)
$$c_i \leqslant \frac{q}{2\alpha} - D_{\alpha N}(\mathbf{u})/\alpha R\langle|\mathbf{u}|^2\rangle. \tag{27}$$

We bound $D_{\alpha N}(\mathbf{u})$ from below. Using the isoperimetric inequality

$$\langle D\phi^2\rangle \geqslant \eta_0^2\langle\phi^2\rangle,$$

which holds for $\phi(1) = 0$, we find that

$$\begin{aligned}
D_{\alpha N} &= \langle|Dw|^2+|Dv|^2+|Du|^2+\alpha^2(|w|^2+|v|^2+|u|^2)+N^2|u/r|^2\rangle \\
&\quad + \left\langle\frac{|Nw+iv|^2}{r^2}+\frac{|Nv-iw|^2}{r^2}\right\rangle \\
&\geqslant (\eta_0^2+\alpha^2)\{\langle|w|^2\rangle+\langle|v|^2\rangle\}+(\eta_0^2+\alpha^2+N^2)\langle|u|^2\rangle \\
&\quad + \langle|Nw+iv|^2+|Nv-iw|^2\rangle \\
&\geqslant (\eta_0^2+\alpha^2+N^2+1)\{\langle|w|^2\rangle+\langle|v|^2\rangle\}+(\eta_0^2+\alpha^2+N^2)\langle|u|^2\rangle \\
&\quad - 2N|\langle\bar{v}w\rangle-\langle\bar{w}v\rangle|
\end{aligned} \tag{28}$$

and
$$(\langle v\bar{w}\rangle-\langle\bar{v}w\rangle) \leqslant \langle|v|^2\rangle+\langle|w|^2\rangle. \tag{29}$$

Equations (28) and (29) are combined to form the estimate

$$D_{\alpha N} \geqslant (\eta_0^2+\alpha^2+(N-1)^2)\langle|\mathbf{u}|^2\rangle,$$

proving theorem 4.

For the axisymmetric case, a stronger result holds. Setting $v = 0$ and $N = 0$ in ($17d$) and ($22a, b$) we obtain

$$i\alpha R\{(U-C)\mathscr{L}_0 w - \psi w\} = \mathscr{L}_0^2 w, \tag{30a}$$

where $\psi = rD\{(1/r)DU\}$ and $\mathscr{L}_0 = \mathscr{L}_N$ for $N = 0$ and $\tilde{\mathscr{L}}_0 = \mathscr{L}_0 + \alpha^2$. A Frobenius analysis at the origin shows that we may obtain two non-singular solutions of ($30a$) which vanish at the origin like r and r^3, respectively. Actually our results hold for functions w convergent in the norm $\langle|\tilde{\mathscr{L}}_0 w|^2\rangle$ which, by Sobelev's inequality, implies the boundedness of w. Hence, we shall state our results relative to $w \in \hat{H}$ where $\hat{H}$ is the complex-valued Hilbert space associated with the above-mentioned norm and the stable boundary conditions

$$w(1) = Dw(1) = 0, \quad w(0)\,\text{bounded}. \tag{30b}$$

We designate the same Hilbert space for real valued functions as H.

Equations ($30a, b$) define the axisymmetric Orr–Sommerfeld problem in round pipes. As in the plane problem, we define

$$\mathscr{I}_0^2 = \langle |w|^2 \rangle,$$

$$\mathscr{I}_1^2 = \langle |Dw|^2 + |w/r|^2 \rangle = -\langle \overline{w}, \tilde{\mathscr{L}}_0 w \rangle$$

and $$\mathscr{I}_2^2 = \langle |D^2 w|^2 \rangle + 3(\langle |Dw/r|^2 \rangle - \langle |w/r^2|^2 \rangle) = \langle |\tilde{\mathscr{L}}_0 w|^2 \rangle.$$

With $N = 0$, ($17d$) is written as

$$\frac{D(wr)}{r} = -i\alpha u, \tag{31a}$$

$$\alpha^2 \langle |\mathbf{u}|^2 \rangle = \alpha^2 \langle |u|^2 + |w|^2 \rangle = \mathscr{I}_1^2 + \alpha^2 \mathscr{I}_0^2 \tag{31b}$$

and $$\alpha^2 D_{\alpha 0} = \alpha^2 \langle |Dw|^2 + |Du|^2 + \alpha^2 (|w|^2 + |u|^2) + |w/r|^2 \rangle$$
$$= \mathscr{I}_2^2 + 2\alpha^2 \mathscr{I}_1^2 + \alpha^4 \mathscr{I}_0^2. \tag{31c}$$

$$\alpha^2 \langle U \mathbf{u}^2 \rangle = \langle U(|Dw|^2 + |w/r|^2 + \alpha^2 |w|^2) - (DU/r)\, |w|^2 \rangle.$$

Now we use (31) to rewrite (27) and (20) as

$$c_i \leqslant \frac{q}{2\alpha} - (\mathscr{I}_2^2 + 2\alpha^2 \mathscr{I}_1^2 + \alpha^4 \mathscr{I}_0^2)/\alpha R(\mathscr{I}_1^2 + \alpha^2 \mathscr{I}_0^2) \tag{32a}$$

and $$c_r = \int_0^1 \{U(|Dw|^2 + |w/r|^2 + \alpha^2 |w|^2) + \tfrac{1}{2}\Lambda |w|^2\} r\, dr / (\mathscr{I}_1^2 + \alpha^2 \mathscr{I}_0^2), \tag{32b}$$

where $\Lambda = r^3 D\{(1/r)\, DU\}$. Of course, equations (32) can be derived directly from (30).

To prove the cylindrical analogue of theorems 1 and 2 of I, we need only to establish the appropriate analogue for lemma 1.

LEMMA 2. *Let* $w \in H$. *Then*

$$\mathscr{I}_1^2 \geqslant \eta_1^2 \mathscr{I}_0^2, \quad \mathscr{I}_2^2 \geqslant \eta_2^2 \mathscr{I}_1^2 \quad \text{and} \quad \mathscr{I}_2^2 \geqslant \eta_3^2 \mathscr{I}_0^2, \tag{33a, b, c}$$

where $$\eta_1^2 = (3{\cdot}83)^2, \quad \eta_2^2 = (5{\cdot}13)^2 \quad \text{and} \quad \eta_3^2 = (4{\cdot}61)^4.$$

Proof of Lemma 2

The values of η_1^2, η_2^2 and η_3^2 are established as minimum values for the functionals ($w \in H$)

$$\frac{\mathscr{I}_1^2}{\mathscr{I}_0^2} = \frac{-\langle w, \tilde{\mathscr{L}}_0 w \rangle}{\langle w^2 \rangle}, \quad \frac{\mathscr{I}_2^2}{\mathscr{I}_1^2} = \frac{\langle (\tilde{\mathscr{L}}_0 w)^2 \rangle}{\langle -w, \tilde{\mathscr{L}}_0 w \rangle} \quad \text{and} \quad \frac{\mathscr{I}_2^2}{\mathscr{I}_0^2} = \frac{\langle (\tilde{\mathscr{L}}_0 w)^2 \rangle}{\langle w^2 \rangle}, \tag{34a–c}$$

respectively. It is well known that the minimum value of ($34a$) defines the principal eigenvalue for the Bessel equation with eigenfunction $J_1(\eta_1 r)$ where $\eta_1^2 = (3{\cdot}83)^2$ is the first zero of $J_1(\eta_1) = 0$.

The Euler problem for ($34b$) is

$$\left.\begin{aligned} &\tilde{\mathscr{L}}_0^2 w + \eta^2 \tilde{\mathscr{L}}_0 w = 0, \quad w(1) = Dw(1) = 0, \\ &w \quad \text{bounded at } r = 0. \end{aligned}\right\} \tag{35b}$$

Two linearly independent solutions of this problem are r and $J_1(\eta r)$. A linear combination of these satisfies the boundary conditions, provided that

$$J_1(\eta) - \eta J_1'(\eta) = 0, \tag{36b}$$

where primes denote differentiation with respect to the argument. The principal eigenvalue η_2 is found as the smallest positive root of (36*b*).

The Euler problem for (34*c*) is

$$\left.\begin{array}{l}\tilde{\mathscr{L}}_0^2 w - \eta^2 w = 0, \quad w(1) = Dw(1) = 0, \\ w \quad \text{bounded at } r = 0.\end{array}\right\} \tag{35c}$$

Two linearly independent solutions of (35*c*) are $J_1(i\sqrt{(\eta)}\,r)$ and $J_1(\sqrt{(\eta)}\,r)$. A linear combination of these satisfies the boundary conditions, provided that

$$J_1(i\sqrt{\eta})\,J_1'(\sqrt{\eta}) - iJ_1(\sqrt{\eta})\,J_1'(i\sqrt{\eta}) = 0. \tag{36c}$$

The principal eigenvalue is found as the smallest positive root η_3 of (36*c*).

Lemma 2 allows us to reduce the estimation problem associated with equations (32) to the problem treated in I and under § 2 above.

THEOREM 6. *Let* $C(\alpha, R)$ *be any eigenvalue of* (30*a*, *b*). *Then theorems* 1 *and* 2 *apply, word for word, provided that* λ_1, λ_2 *and* λ_3 *are replaced by* η_1, η_2 *and* η_3 *and* U'' *is replaced with* Λ.

The proof of theorem 6 can be constructed by following the details of the proofs of theorems 1 and 2.

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Note added in proof. It is obvious enough, but deserves mention, that various minor improvements in the estimates of this paper are possible at a cost in simplicity of the mathematical constructions. Where the numerical improvement possible seemed slight, I have chosen the simplest possible construction.