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Convective Instability in a Temperature and Concentration Field

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1. Introduction

In chemically homogeneous fluids, density differences induced by thermal gradients can drive fluid motions. If, in addition, there is a concentration gradient, *e.g.*, a salt gradient or a gradient of water vapor in air, then the density variations which drive the motion are controlled by two scalar fields. Such systems are frequently within the scope of the Boussinesq equations. It is possible to work out an energy theory for such motions not only for the chemically homogeneous fluid (see [2], [3] and [11]), but also for the fluid in which there is mass diffusion as well. Such a theory is developed in this paper. This theory allows us to establish results of a comprehensive character relative to the stability and uniqueness of such fluid motions. These results are conveniently grouped into three categories.

First, we establish a universal stability estimate for arbitrary nonlinear disturbances in bounded or periodic domains. This estimate guarantees asymptotic stability provided that a stability criterion holds. The criterion is a relation between the basic flow parameters and material parameters. It can be expressed as a relation between a Reynolds number (R), a Rayleigh number ($\mathscr{R}$), and a number ($\mathscr{C}$), which can be viewed as a measure of a typical concentration difference having the same form as $\mathscr{R}$ but with temperature parameters replaced with their concentration analogues. The estimate establishes the existence of an open region of the origin in a Cartesian $(R, \mathscr{R}, \mathscr{C})$ space in which the flow is stable (Theorem 1) and, if steady, unique (Theorem 2).

Second, an improved criterion for stability and uniqueness is achieved through the formulation of a variational maximum problem in Hilbert space and several auxiliary ordinary maximum problems. The variational problem generates a two-parameter family of stability boundaries as the smallest eigenvalues of the associated Euler equations. The two parameters are introduced as coupling constants in a linear combination of energy-like quantities associated with the nonlinear equations of momentum, temperature and concentration. The best choice of these coupling constants leads to a stability boundary, which we call "optimum". This boundary gives the largest region of guaranteed stability and uniqueness (Theorem 3) which can be achieved through a variation of the coupling constants. In the usual circumstances, the best coupling parameters may be found *uniquely* from an ordinary maximum problem (Theorems 4 and 5). The optimum stability boundary $R=\tilde{R}(\mathscr{R}, \mathscr{C})$ is monotone and convex in $\mathscr{R}$ and $\mathscr{C}$ (Theorem 6) and in certain boundary data as well (Theorem 7). Monotonicity and convexity results for the stability boundary are made possible by a convexity lemma for functions of several parameters.

Third, we obtain the exact representation for the stability-uniqueness boundary associated with motionless states supporting constant gradients of temperature and concentration (Theorem 8). In several situations, energy and linear limits coincide. The linear limit, when it has meaning, gives sufficient conditions for *instability*. It follows that when these two limits coincide (stability-instability boundary), one has the strongest possible result and a guarantee that finite disturbances will not grow in situations where tiny disturbances decay. (We say "subcritical" instabilities are excluded.) Subcritical instabilities can be excluded when the temperature and concentration gradients are destabilizing (Theorem 8). On the other hand, a region potentially open to subcritical instabilities is obtained (Theorem 9) when the temperature gradient is destabilizing (heated below) and the concentration gradient stabilizing (salty below). For plane-free layers in the limit of zero mass diffusion (infinite Schmidt number), this region open to subcritical instabilities collapses, and we have, once again, the strongest result. Our results here are in accord with the tentative, purely formal calculations of other investigators. These calculations lead to subcritical solutions which fill the entire region deemed open by energy theory as the Prandtl and Schmidt numbers take on all allowed values.

2. Boussinesq Equations for the Difference Motion

The Boussinesq equations are approximations to the compressible Navier-Stokes equations in which variations of density are neglected except in the body (buoyant) force term of the momentum equation (see references [1] and [6]).

Density variations may be induced by both concentration and thermal fields. It is usual that in the Boussinesq system, one relates these fields to the density with a linear equation of state

$$\frac{\rho}{\rho_0}=1-\alpha(T-T_0)+\mathfrak{s}\beta(C-C_0), \tag{2.1}$$

where

$$\alpha=-\left(\frac{\partial \ln \rho}{\partial T}\right)_C, \quad \beta=\left|\left(\frac{\partial \ln \rho}{\partial C}\right)_T\right|,$$

and T_0, C_0 and ρ_0 are the reference temperature, concentration and density, respectively, and $\mathfrak{s}$ is $+1$ or -1 according to whether the density of the species is greater or less than the density of the solvent.

As a consequence of these various approximations, one has for a mathematical description of the convective motion the "Boussinesq" system

$$\nabla\cdot V=0, \tag{2.2}$$

$$\frac{dV}{dt}=-\nabla\frac{p}{\rho_0}+[1-\alpha(T-T_0)+\mathfrak{s}\beta(C-C_0)]\,g+\nu\Delta V, \tag{2.3}$$

$$\frac{dT}{dt}=\kappa_t\Delta T+Q_t(\boldsymbol{r},t), \tag{2.4}$$

$$\frac{dC}{dt}=\kappa_c\Delta C+Q_c(\boldsymbol{r},t), \tag{2.5}$$

where $\boldsymbol{g}(\boldsymbol{r},t)$, $Q_t(\boldsymbol{r},t)$, $Q_c(\boldsymbol{r},t)$ are, respectively, a prescribed body force (typically gravity), heat source and mass source field. The constants ν, κ_t and κ_c are, respectively, the kinematic viscosity and thermal and species diffusivities. The field variables V, T and C represent velocity, temperature and species concentration, respectively.

It is convenient to consider bounded domains $\mathscr{V}$. Then, for conditions at the boundary $\partial\mathscr{V}$, we consider T_1, C_1, V_1, V_{1n} as preassigned.

$$\frac{\partial T}{\partial N}+\sigma_t T=T_1, \tag{2.6}$$

$$\frac{\partial C}{\partial N}+\sigma_c C=C_1, \tag{2.7}$$

and

$$V=V_1, \qquad \text{rigid surface}; \tag{2.8}$$

or

$$V\cdot N=V_{1n}, \qquad \text{free surface}; \tag{2.9}$$

$$\{[\nabla V+(\nabla V)^T]\cdot N\}\times N=\text{prescribed vector}. \tag{2.10}$$

Here σ_t and σ_c are positive piecewise continuous functions of position*, N is the normal to $\partial\mathscr{V}$ and $(\nabla V)^T$ is the transpose of the dyadic gradient of the velocity.

To treat the stability of the basic (V,T,C,p) motion, one considers an altered (V^*,T^*,C^*,p^*) motion which satisfies the same equations (2.2 – 2.5) and boundary

* The limit $\sigma_t\to\infty$ or $\sigma_c\to\infty$ corresponds to prescribed temperature or concentration fields on $\partial\mathscr{V}$. The limit $\sigma_t\to 0$ or $\sigma_c\to 0$ corresponds to prescribed heat flux or mass flux on $\partial\mathscr{V}$.

conditions (2.6−2.10) as the basic state, but which differs from this state initially. Difference variables

$$\hat{u}=V^*-V, \quad \hat{\theta}=T^*-T, \quad \text{and} \quad \hat{c}=C^*-C, \qquad (2.11\text{a, b, c})$$

are then introduced. The difference variables satisfy equations

$$\frac{\partial\hat{u}}{\partial t}+(V^*\cdot\nabla)\hat{u}+(\hat{u}\cdot\nabla)V=-\nabla\frac{(p^*-p)}{\rho_0}-(\alpha\hat{\theta}-\delta\beta\hat{c})g+\nu\Delta\hat{u}, \qquad (2.12)$$

$$\frac{\partial\hat{\theta}}{\partial t}+(V^*\cdot\nabla)\hat{\theta}+(\hat{u}\cdot\nabla)T=\kappa_t\Delta\hat{\theta}, \qquad (2.13)$$

$$\frac{\partial\hat{c}}{\partial t}+(V^*\cdot\nabla)\hat{c}+(\hat{u}\cdot\nabla)C=\kappa_c\Delta\hat{c}, \qquad (2.14)$$

$$\nabla\cdot\hat{u}=0, \qquad (2.15)$$

and boundary conditions

$$\frac{\partial\hat{\theta}}{\partial N}+\sigma_t\hat{\theta}=0, \qquad (2.16)$$

$$\frac{\partial\hat{c}}{\partial N}+\sigma_c\hat{c}=0, \qquad (2.17)$$

$\hat{u}=0$ (rigid surface, velocity V prescribed), (2.18)

$\hat{u}\cdot N=0$ (free surface, normal velocity $V\cdot N$ prescribed), (2.19)

$$(N\cdot\hat{d})\times N=0, \qquad (2.20)$$

where $\hat{d}$ is the strain-rate tensor of the difference motion, that is,

$$\hat{d}=D^*-D \quad \text{and} \quad D=\tfrac{1}{2}[\nabla V+(\nabla V)^T].$$

3. Energy Identities

We wish to determine conditions under which the altered flow will approach the basic flow asymptotically as $t\to\infty$. In treating this problem, we introduce the kinetic energy of the difference motion,

$$\tilde{K}=\int_{\mathscr{V}}\tfrac{1}{2}\hat{u}^2\,d\mathscr{V}, \qquad 3.1)$$

and the quantities

$$\tilde{\Theta}=\int_{\mathscr{V}}\tfrac{1}{2}\hat{\theta}^2\,d\mathscr{V}, \qquad (3.2)$$

$$\tilde{\Gamma}=\int_{\mathscr{V}}\tfrac{1}{2}\hat{c}^2\,d\mathscr{V}, \qquad (3.3)$$

where $\hat{\theta}$ and $\hat{c}$ are the temperature and concentration of the difference motion, respectively, and $\tilde{\Theta}$ and $\tilde{\Gamma}$ are the temperature modulus and concentration modulus.* If $\tilde{K}$, $\tilde{\Theta}$ and $\tilde{\Gamma}$ tend to zero as $t\to\infty$, then we say that the basic motion is asymptotically stable in the mean. The rates of change of $\tilde{K}$, $\tilde{\Theta}$ and $\tilde{\Gamma}$ are governed

* In writing integrals, we shall, in the sequel, omit infinitesimal volume elements; moreover, all integrals are understood to be extended over the entire region $\mathscr{V}$ except for integrals over $\partial\mathscr{V}$, the boundary of $\mathscr{V}$, which are indicated by a circle drawn through the integral sign.

by the fundamental formulae:

$$\frac{d\tilde{K}}{dt} = -\int [\hat{\boldsymbol{u}} \cdot \boldsymbol{D} \cdot \hat{\boldsymbol{u}} + (\alpha\hat{\theta} - s\beta\hat{c})\boldsymbol{g} \cdot \hat{\boldsymbol{u}} + 2\nu\hat{\boldsymbol{d}}:\hat{\boldsymbol{d}}], \tag{3.4}$$

$$\frac{d\tilde{\Theta}}{dt} = -\int (\hat{\theta}\hat{\boldsymbol{u}} \cdot \nabla T + \kappa_t \nabla\hat{\theta} \cdot \nabla\hat{\theta}) - \kappa_t \oint \sigma_t \hat{\theta}^2, \tag{3.5}$$

$$\frac{d\Gamma}{dt} = -\int (\hat{c}\,\hat{\boldsymbol{u}} \cdot \nabla C + \kappa_c \nabla\hat{c} \cdot \nabla\hat{c}) - \kappa_c \oint \sigma_c \hat{c}^2. \tag{3.6}$$

Equations (3.4)–(3.6) follow from the integration of the equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\hat{u}^2}{2}\right) + \boldsymbol{V} \cdot \nabla\left(\frac{\hat{u}^2}{2}\right) = -\hat{\boldsymbol{u}} \cdot \boldsymbol{D} \cdot \hat{\boldsymbol{u}} - (\alpha\hat{\theta} - s\beta\hat{c})\boldsymbol{g} \cdot \hat{\boldsymbol{u}} - \nu \nabla\hat{\boldsymbol{u}} : \nabla\hat{\boldsymbol{u}} + \operatorname{div} \boldsymbol{A}_0, \\
&\frac{\partial}{\partial t}\left(\frac{\hat{\theta}^2}{2}\right) + \boldsymbol{V} \cdot \nabla\left(\frac{\hat{\theta}^2}{2}\right) = -\hat{\theta}\hat{\boldsymbol{u}} \cdot \nabla T - \kappa_t \nabla\hat{\theta} \cdot \nabla\hat{\theta} + \operatorname{div} \boldsymbol{B}_t, \\
&\frac{\partial}{\partial t}\left(\frac{\hat{c}^2}{2}\right) + \boldsymbol{V} \cdot \nabla\left(\frac{\hat{c}^2}{2}\right) = -\hat{c}\,\hat{\boldsymbol{u}} \cdot \nabla C - \kappa_c \nabla\hat{c} \cdot \nabla\hat{c} + \operatorname{div} \boldsymbol{B}_c,
\end{aligned} \tag{3.7}$$

where

$$\boldsymbol{A}_0 = \nu \nabla\left(\frac{\hat{u}^2}{2}\right) - \hat{\boldsymbol{u}}\left(\frac{\hat{u}^2}{2} + \frac{p^* - p}{\rho_0}\right),$$

$$\boldsymbol{B}_t = \kappa_t \hat{\theta} \nabla\hat{\theta} - \hat{\boldsymbol{u}}\frac{\hat{\theta}^2}{2},$$

and

$$\boldsymbol{B}_c = \kappa_c \hat{c} \nabla\hat{c} - \hat{\boldsymbol{u}}\frac{\hat{c}^2}{2},$$

over $\mathscr{V}$ and the application of the divergence theorem and boundary conditions (2.16)–(2.20). Equations (3.7) follow in an obvious way from (2.12)–(2.15).

The energy integrals (3.4), (3.5) and (3.6) also hold for unbounded regions under suitable assumptions on the asymptotic behavior of the functions. Another justification of (3.4), (3.5) and (3.6) for infinite regions is available whenever the flow geometry is such that the disturbances can be assumed spatially periodic at each instant. For periodic disturbances, nonvanishing contributions on planes normal to symmetry axes cancel one another. Flows with free surfaces, $\partial\mathscr{V}_f$, are also described by (3.4), (3.5) and (3.6) (see JOSEPH [3] for complete discussion).

Subsequent deductions are made from the energy identities and kinematic constraints alone, and no further role is played by the local nonlinear conservation equations. It is convenient for the work which follows to emphasize the convective aspects of the stability problem. It is for this reason that we introduce new (dimensionless) variables defined for the basic state,

$$\begin{aligned}
\mathbf{H} &= g/\mathscr{g}, \\
\mathbf{H}_1 &= \nabla T/\alpha_m, \\
\mathbf{H}_2 &= \nabla C/\beta_m, \\
\mathbf{E} &= \boldsymbol{D}/m,
\end{aligned}$$

where

$$\mathscr{g} = \text{Max}\,|\,\boldsymbol{g}\,|,$$
$$\alpha_m = \text{Max}\,|\,\nabla T\,|,$$
$$\beta_m = \text{Max}\,|\,\nabla C\,|,$$
$$\mathscr{m} = \text{least characteristic value of } \boldsymbol{D},$$

over $\mathscr{V}$, and for the difference motion

$$\boldsymbol{v} = \hat{\boldsymbol{u}}/\mathring{u},$$
$$\theta = \hat{\theta}/\mathring{T},$$
$$c = \hat{c}/\mathring{C},$$
$$p = \hat{p}/\mathring{p},$$
$$\boldsymbol{d} = \hat{\boldsymbol{d}}\,l/\mathring{u},$$

where $\mathring{u}$ is a unit of velocity, l is a unit of length, and

$$\mathring{T} = \sqrt{\frac{P_r\,\alpha_m}{\mathscr{g}\,\alpha}}\,\mathring{u},$$
$$\mathring{C} = \sqrt{\frac{S_c\,\beta_m}{\mathscr{g}\,\beta}}\,\mathring{u},$$
$$\mathring{p} = \frac{\rho_0\,\nu}{l}\,\mathring{u}.$$

This change of variable generates the following dimensionless combinations:

$R = \dfrac{\mathscr{m}\,l^2}{\nu}$,	Reynolds number;
$\mathscr{R} = \sqrt{\dfrac{\alpha\,\alpha_m\,\mathscr{g}\,l^4}{\nu\,\kappa_t}}$,	Rayleigh number (for the temperature field);
$\mathscr{C} = \sqrt{\dfrac{\beta\,\beta_m\,\mathscr{g}\,l^4}{\nu\,\kappa_c}}$,	Rayleigh number (for the concentration field);
$Nu = l\,\sigma_t$,	Nusselt number;
$Sh = l\,\sigma_c$,	Sherwood number;
$Pr = \nu/\kappa_t$,	Prandtl number;
$Sc = \nu/\kappa_c$,	Schmidt number.

The difference velocity and temperature then necessarily satisfy energy identities obtained from (3.4), (3.5) and (3.6).

$$\frac{dK}{d\tau} = \frac{d}{d\tau}\int \frac{v^2}{2} = -\int_{\mathscr{V}} [R\,\boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v} + 2\boldsymbol{d}:\boldsymbol{d} + (\mathscr{R}\,\theta - s\,\mathscr{C}\,c)\mathbf{H}\cdot\boldsymbol{v}], \tag{3.8}$$

$$Pr\,\frac{d\Theta}{d\tau} = Pr\,\frac{d}{d\tau}\int \frac{\theta^2}{2} = -\int_{\mathscr{V}} (\mathscr{R}\,\theta\,\boldsymbol{v}\cdot\mathbf{H}_1 + \nabla\theta\cdot\nabla\theta) - \oint Nu\,\theta^2, \tag{3.9}$$

$$Sh\,\frac{d\Gamma}{d\tau} = Sc\,\frac{d}{d\tau}\int \frac{c^2}{2} = -\int_{\mathscr{V}} (\mathscr{C}\,c\,\boldsymbol{v}\cdot\mathbf{H}_2 + \nabla c\cdot\nabla c) - \oint Sh\,c^2, \tag{3.10}$$

where ∇ and $\mathscr{V}$ have been made dimensionless with l along with

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in } \mathscr{V}, \tag{3.11}$$

and

$$\frac{\partial \theta}{\partial N} + Nu\,\theta = 0, \tag{3.12}$$

$$\frac{\partial c}{\partial N} + Sh\,c = 0, \tag{3.13}$$

and

$$\boldsymbol{v} = 0, \qquad \text{rigid surface;} \tag{3.14}$$

or

$$\boldsymbol{v} \cdot \boldsymbol{N} = 0 \quad \text{and} \quad (\boldsymbol{N} \cdot \boldsymbol{d}) \times \boldsymbol{N} = 0, \qquad \text{free surface} \tag{3.15}$$

on $\partial\mathscr{V}$.

Equations (3.8)–(3.15) are the basis for all subsequent mathematical deduction.

4. Universal Stability and Uniqueness of Steady Flow

We generalize the stability criteria developed by SERRIN [12] and JOSEPH [2] to convective motions with heat and mass transfer. A basic flow will be stable (stable in the mean) provided the kinetic energy, temperature and concentration modulus of the difference motion tends to zero as $t \to \infty$.

To apply the method, one considers the right-hand sides of (3.8), (3.9) and (3.10), and if they are all negative for arbitrary vectors $\boldsymbol{v}$ satisfying $\operatorname{div} \boldsymbol{v} = 0$ and scalars θ and c, then, as we shall see, there is asymptotic stability. Since the terms $-\int \boldsymbol{d}:\boldsymbol{d}$, $-\int \nabla\theta \cdot \nabla\theta$ and $-\int \nabla c \cdot \nabla c$ are always negative, it is clear that mechanical dissipation, heat conduction and mass diffusion tend to stabilize the flow. On the other hand, the integrals in which the basic state field variables $\mathbf{E}, \mathbf{H}, \mathbf{H}_1, \mathbf{H}_2$ are explicit are indefinite, and if the parameters R, $\mathscr{R}$ and $\mathscr{C}$ are sufficiently large, these can lead to instability. The surface terms $-\oint Nu\,\theta^2$ and $-\oint Sh\,c^2$ are more stabilizing as Nu and Sh grow larger. (See Theorem 7.)

Our first result establishes the existence of a neighborhood of the origin of the Cartesian space $(R, \mathscr{R}, \mathscr{C})$ in which the motion is stable.

Theorem 1. *Let $\mathscr{V} = \mathscr{V}(t)$ be a bounded region of space which can be included in a sphere of diameter l. Let V, T and C be the velocity vector, temperature and concentration satisfying prescribed conditions at the boundary of $\mathscr{V}$. Then the kinetic energy K of any disturbance motion, the modulus Θ of any temperature disturbance and the modulus Γ of any concentration disturbance satisfy the following inequalities:*

$$(\sqrt{K} + \gamma\,Pr\sqrt{\Theta} + \gamma\,Sc\sqrt{\Gamma})$$
$$\leqq (\sqrt{K_0} + \gamma\,Pr\sqrt{\Theta_0} + \gamma\,Sc\sqrt{\Gamma_0}) \exp\{-M(\sqrt{(\delta - R)\,\varepsilon\pi^2} - \mathscr{R} - \mathscr{C})\,\tau\}, \tag{4.1}$$

provided that

$$\varepsilon\pi^2\gamma = \sqrt{(\delta - R)\,\varepsilon\pi^2} \geqq \mathscr{R} + \mathscr{C} \geqq 0. \tag{4.2}$$

Here K_0, Θ_0 and Γ_0 are the initial disturbance values. $\delta \cong 80$ is the least positive root of $\operatorname{Tan}\sqrt{\delta/2} = \sqrt{\delta/2}$ *and $\varepsilon = 3$ for spherical regions. M is the minimum value* *of* γ, $1/\gamma Pr$ *and* $1/\gamma Sc$. *If* $\varepsilon\pi^2\gamma=\sqrt{(\delta-R)\,\varepsilon\pi^2}\geqq\mathscr{R}+\mathscr{C}>0$ *for all* τ, *then* $K\to 0$, $\Theta\to 0$ *and* $\Gamma\to 0$ *as* $\tau\to\infty$, *and the flow is asymptotically stable in the mean.**

Proof. We start with an auxiliary computation for the inequalities:

(1) By the Schwarz inequality,

$$|\textstyle\int \mathbf{H}\cdot \boldsymbol{v}\theta| \leqq \int |\mathbf{H}|\,|\boldsymbol{v}|\,|\theta| \leqq 2\sqrt{K\Theta}, \tag{4.3}$$

$$|\textstyle\int \mathbf{H}\cdot \boldsymbol{v}c| \leqq \int |\mathbf{H}|\,|\boldsymbol{v}|\,|c| \leqq 2\sqrt{K\Gamma}, \tag{4.4}$$

$$|\textstyle\int \mathbf{H}_1\cdot \boldsymbol{v}\theta| \leqq \int |\mathbf{H}_1|\,|\boldsymbol{v}|\,|\theta| \leqq 2\sqrt{K\Theta}, \tag{4.5}$$

$$|\textstyle\int \mathbf{H}_2\cdot \boldsymbol{v}c| \leqq \int |\mathbf{H}_2|\,|\boldsymbol{v}|\,|c| \leqq 2\sqrt{K\Gamma}, \tag{4.6}$$

and [12]

$$|\textstyle\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}| \leqq \int |\boldsymbol{v}|\,|\mathbf{E}|\,|\boldsymbol{v}| \leqq 2K. \tag{4.7}$$

(2) By an isoperimetric inequality, with $\theta=0$ and $c=0$ on $\partial\mathscr{V}$, and for a suitably chosen positive number ε, we have

$$\textstyle\int \nabla\theta\cdot\nabla\theta \geqq \varepsilon\pi^2\int\theta^2 = 2\varepsilon\pi^2\Theta, \tag{4.8}$$

$$\textstyle\int \nabla c\cdot\nabla c \geqq \varepsilon\pi^2\int c^2 = 2\varepsilon\pi^2\Gamma, \tag{4.9}$$

where $\varepsilon=3$ for the spherical region.

(3) With $\boldsymbol{v}=0$ on $\partial\mathscr{V}$ and $\operatorname{div}\boldsymbol{v}=0$ in $\mathscr{V}$, we have

$$\textstyle\int 2\boldsymbol{d}:\boldsymbol{d} = \int \nabla\boldsymbol{v}:\nabla\boldsymbol{v} \geqq \delta\int v^2 = 2\delta K, \tag{4.10}$$

where δ is the least positive root of $\operatorname{Tan}\sqrt{\frac{\delta}{2}}=\sqrt{\frac{\delta}{2}}$.

A combination of (3.8), (3.9), (3.10) and (4.3)–(4.10) leads to the estimates

$$\frac{dK}{d\tau}\leqq 2\big((R-\delta)K+\mathscr{R}\sqrt{K\Theta}+\mathscr{C}\sqrt{K\Gamma}\big), \tag{4.11}$$

$$Pr\frac{d\Theta}{d\tau}\leqq 2(-\varepsilon\pi^2\Theta+\mathscr{R}\sqrt{K\Theta}), \tag{4.12}$$

$$Sc\frac{d\Gamma}{d\tau}\leqq 2(-\varepsilon\pi^2\Gamma+\mathscr{C}\sqrt{K\Gamma}). \tag{4.13}$$

Let

$$\hat{K}^2=K,\quad \hat{\Theta}^2=\Theta,\quad \hat{\Gamma}^2=\Gamma,\quad \gamma^2=\frac{\delta-R}{\varepsilon\pi^2}\geqq 0,$$

and

$$N=\varepsilon\pi^2\gamma=\sqrt{(\delta-R)\,\varepsilon\pi^2}, \tag{4.14}$$

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* It is not hard to extend the theorem to nonspherical regions and to more general boundary conditions. The effect of these extensions is to change the values of δ and ε (*cf.* JOSEPH [2]). Exact values for δ and ε are known for spheres (*cf.* PAYNE & WEINBERGER [7]). SERRIN [12], VELTE [15] and, most recently, SORGER [13] give bounds for δ in other regions of simple geometry.

where K, $\hat{\Theta}$, $\hat{\Gamma}$, γ, and N are positive definite. In the new variables,

$$\frac{d\hat{K}}{d\tau}+\gamma N\hat{K}-(\mathscr{R}\hat{\Theta}+\mathscr{C}\hat{\Gamma})\leqq 0, \tag{4.15}$$

$$\gamma Pr\frac{d\hat{\Theta}}{d\tau}+N\hat{\Theta}-\gamma\mathscr{R}\hat{K}\leqq 0, \tag{4.16}$$

and

$$\gamma Sc\frac{d\hat{\Gamma}}{d\tau}+N\hat{\Gamma}-\gamma\mathscr{C}\hat{K}\leqq 0. \tag{4.17}$$

Since $-\hat{\Theta}\leqq 0$ and $-\hat{\Gamma}\leqq 0$, by addition of (4.15)–(4.17) it follows that

$$\frac{d}{d\tau}(\hat{K}+\gamma Pr\hat{\Theta}+\gamma Sc\hat{\Gamma})+(N-\mathscr{R}-\mathscr{C})(\gamma\hat{K}+\hat{\Theta}+\hat{\Gamma})\leqq 0. \tag{4.18}$$

We can find M such that

$$\gamma\hat{K}+\hat{\Theta}+\hat{\Gamma}\geqq M(\hat{K}+\gamma Pr\hat{\Theta}+\gamma Sc\hat{\Gamma})\geqq 0, \tag{4.19}$$

where $M=\gamma\,\mathrm{Min}(1,\,1/\gamma^2 Pr,\,1/\gamma^2 Sc)\geqq 0$. M can be found in these following cases:

$$\begin{array}{llll}
\text{(i)} & \gamma^2 Pr\leqq 1, \text{ and } \gamma^2 Sc\leqq 1; & M=\gamma, \\
\text{(ii)} & \gamma^2 Pr\geqq 1 \text{ and } \gamma^2 Sc\geqq 1; & M=\dfrac{1}{\gamma}\,\mathrm{Min}\left(\dfrac{1}{Pr},\,\dfrac{1}{Sc}\right), \\
\text{(iii)} & \gamma^2 Pr\leqq 1 \text{ and } \gamma^2 Sc\geqq 1; & M=\dfrac{1}{\gamma Sc}, \\
\text{(iv)} & \gamma^2 Pr\geqq 1 \text{ and } \gamma^2 Sc\leqq 1; & M=\dfrac{1}{\gamma Pr}.
\end{array} \tag{4.20}$$

Hence, from (4.19) and (4.20), we have

$$0\geqq\frac{d}{d\tau}(\hat{K}+\gamma Pr\hat{\Theta}+\gamma Sc\hat{\Gamma})+(N-\mathscr{R}-\mathscr{C})\,M(\hat{K}+\gamma Pr\hat{\Theta}+\gamma Sc\hat{\Gamma}) \tag{4.21}$$

provided $M\geqq 0$ and $N-\mathscr{R}-\mathscr{C}\geqq 0$. It follows that

$$(\hat{K}+\gamma Pr\hat{\Theta}+\gamma Sc\hat{\Gamma})\leqq(\hat{K}_0+\gamma Pr\hat{\Theta}_0+\gamma Sc\hat{\Gamma}_0)\,e^{-M(N-\mathscr{R}-\mathscr{C})\tau}. \tag{4.22}$$

The stability condition is

$$\sqrt{(\delta-R)\varepsilon\pi^2}>\mathscr{R}+\mathscr{C}. \tag{4.23}$$

For spherical regions, $\varepsilon=3$ and $\delta\cong 80$. For a shear motion, $\mathscr{R}=0$ and $\mathscr{C}=0$ is essentially SERRIN's result. We have universal stability when

$$R=\frac{m\,l^2}{\nu}<80 \quad\text{or}\quad \frac{V\,l}{\nu}<\sqrt{80}=8.98,$$

where

$$V=\mathrm{Max}\,|V|.$$

For a motionless fluid ($R=0$), we have

$$\sqrt{2368} > \mathcal{R} + \mathcal{C}.$$

The estimates also apply for two parallel rigid surfaces on which the temperature and concentration are prescribed. For this case, we have $\varepsilon=1, \delta=3.74\pi^2$ (see VELTE [15] and SORGER [13] for calculations leading to bounds on δ).

For a motionless fluid ($R=0$), we have universal stability when $\sqrt{368} > \mathcal{R} + \mathcal{C}$.

This compares with the result of linear theory for the case in which both basic temperature and concentration distribution are potentially unstable (SANI [8]) $1708 > \mathcal{R} + \mathcal{C}$.

As a second application, we have the following uniqueness theorem concerning steady flow in a fixed bounded region.

Theorem 2. *Let* V^*, T^*, C^*, p^* and V, T, C, p *be, respectively, the velocity, temperature and concentration of two steady flows in* $\mathcal{V}$ *subject to a prescribed velocity, temperature and concentration distribution on the boundary of* $\mathcal{V}$. *Then the two flows are identical, provided only that*

$$0 \leqq \mathcal{R} + \mathcal{C} < \sqrt{\varepsilon\pi^2(\delta - R)}. \tag{4.24}$$

Proof. Since the flow is steady, the kinetic energy K, the temperature modulus Θ and the concentration modulus Γ must be constant. On the other hand, it must satisfy (4.21). Since (4.24) is satisfied, this can happen if and only if $K_0 = \Theta_0 = \Gamma_0 = K = \Theta = \Gamma = 0$. This implies $V^* = V$, $T^* = T$ and $C^* = C$ (almost everywhere).

Theorem 3 of the next chapter gives a greatly improved criterion for stability and uniqueness.

5. Optimum Stability Boundary

A value of the concept of universal stability (Theorem 1) is the generality with which it specifies the limits of stability. This generality is, however, attained at a cost of precision which could be attained by using all of the information about the known characteristics of the basic motion. It develops that the problem of finding the "optimum" limits of stability may be rigorously recast as a variational problem, and the limits obtained are the maximum eigenvalues of a bounded set generated by solutions to the appropriate Euler equations.

It is, in fact, possible to insure asymptotic stability by the introduction of a $\lambda_1 > 0$, $\lambda_2 > 0$ family of functionals (of a Lyaponouv type),

$$E_\lambda = K + \lambda_1 \, Pr \, \Theta + \lambda_2 \, Sc \, \Gamma,$$

which are loosely designated as "energy". Our goal is to determine the greatest values of R, $\mathcal{R}$ and $\mathcal{C}$ and also the best values of λ_1 and λ_2 such that as $t \to \infty$, $E_\lambda \to 0$.

We next simplify the problem by introducing two positive parameters μ_1, μ_2 ($0 \leqq \mu_1, \mu_2 \leqq 0$) defined by $\mu_1 = \mathcal{R}/R$ and $\mu_2 = \mathcal{C}/R$. We regard μ_1 and μ_2 as preassigned and use them to eliminate the explicit dependence on $\mathcal{R}$ and $\mathcal{C}$. In the same way, we could eliminate any two of the three stability parameters with two preassigned parameters. Then we introduce the notation

$$
\begin{aligned}
I_0(\boldsymbol{v},\theta,c)&=\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}+(\mu_1\theta-\mathscr{s}\mu_2 c)\mathbf{H}\cdot\boldsymbol{v},\\
I_1(\boldsymbol{v},\theta)&=\int \mu_1\mathbf{H}_1\cdot\boldsymbol{v}\theta,\\
I_2(\boldsymbol{v},c)&=\int \mu_2\mathbf{H}_2\cdot\boldsymbol{v}c,\\
\mathscr{D}_0(\boldsymbol{v},\boldsymbol{v})&=2\int \boldsymbol{d}:\boldsymbol{d},\\
\mathscr{D}_1(\theta,\theta)&=\int \nabla\theta\cdot\nabla\theta+\oint Nu\,\theta^2,\\
\mathscr{D}_2(c,c)&=\int \nabla c\cdot\nabla c+\oint Sh\,c^2,\\
I&=I_0+\lambda_1 I_1+\lambda_2 I_2,\\
\mathscr{D}&=\mathscr{D}_0+\lambda_1\mathscr{D}_1+\lambda_2\mathscr{D}_2.
\end{aligned}
\tag{5.1}
$$

From (3.8), (3.9), and (3.10) one finds that

$$
\begin{aligned}
\frac{dE_\lambda}{d\tau}&=\mathscr{D}\left[-1+R\left(-\frac{I}{\mathscr{D}}\right)\right]\\
&\leqq\mathscr{D}\left[-1+R\,\mathrm{Max}\left(-\frac{I}{\mathscr{D}}\right)\right]\\
&=\mathscr{D}[-1+R/\tilde{R}_\lambda],
\end{aligned}
\tag{5.2}
$$

where

$$
\tilde{R}_\lambda^{-1}=\tilde{R}_\lambda^{-1}(\lambda_1,\lambda_2,\mu_1,\mu_2,\tau)=\mathrm{Max}\left(-\frac{I}{\mathscr{D}}\right).
\tag{5.3}
$$

From the inequality (5.2), we have the following theorem:

Theorem 3*. *Let there exist* $a_0^2>0$, $a_1^2>0$ *and* $a_2^2>0$ *such that the inequalities*

$$
\begin{aligned}
\tfrac{1}{2}a_0^2\int v^2&\leqq\mathscr{D}_0(\boldsymbol{v},\boldsymbol{v}),\\
\tfrac{1}{2}Pr\,a_1^2\int\theta^2&\leqq\mathscr{D}_1(\theta,\theta),\\
\tfrac{1}{2}Sc\,a_2^2\int c^2&\leqq\mathscr{D}_2(c,c),
\end{aligned}
\tag{5.4}
$$

hold. Then, for any $\lambda_1>0$, $\lambda_2>0$ *and* $\mu_1\geqq 0$, $\mu_2\geqq 0$, *we have*

$$
E_\lambda(\tau)\leqq E_\lambda(0)\cdot\exp\left\{-\int_0^\tau(1-R/\tilde{R}_\lambda(\tau))\,\xi^2\,d\tau\right\}
\tag{5.5}
$$

provided $R<\tilde{R}_\lambda(\tau)$ *in the time interval* $[0,\tau]$, *where* $E_\lambda(0)$ *is the initial energy of the difference motion and* $\xi^2=\mathrm{Min}(a_0^2,a_1^2,a_2^2)$. *If* $R<\tilde{R}_\lambda(\tau)$ *for all* τ, *then* $E_\lambda\to 0$, *and the flow is asymptotically stable in the mean.*

Proof. Let the assumed inequalities hold. Then combine

$$
\begin{aligned}
E_\lambda&=\tfrac{1}{2}\int(v^2+\lambda_1 Pr\,\theta^2+\lambda_2 Sc\,c^2)\\
&\leqq a_0^{-2}\mathscr{D}_0(\boldsymbol{v},\boldsymbol{v})+a_1^{-2}\lambda_1\mathscr{D}_1(\theta,\theta)+a_2^{-2}\lambda_2\mathscr{D}_2(c,c)\\
&\leqq\xi^{-2}\mathscr{D},
\end{aligned}
$$

* It is clear that the analogue of uniqueness, Theorem 2, can be framed in terms of the value $\tilde{R}_\lambda$ when the basic motion is steady.

with (5.2) to produce

$$\frac{dE_\lambda}{d\tau} \leqq -\left(1-\frac{R}{\tilde{R}_\lambda}\right)\mathscr{D} \leqq -\left(1-\frac{R}{\tilde{R}_\lambda}\right)\xi^2 E_\lambda. \tag{5.6}$$

Integrate (5.6) over the time interval $[0,\tau]$, proving the theorem.

5A. Maximum Problem

The problem set here is conveniently divided into two parts. First, we regard positive parameters $\mu_1, \mu_2, \lambda_1, \lambda_2$ as given and seek the minimum value of $\tilde{R}_\lambda$ for which (3.11)–(3.15) hold, *i.e.*, stability is guaranteed provided $R<\tilde{R}_\lambda$. This leads to a maximum problem for $\frac{1}{\tilde{R}_\lambda}$:

$$\frac{1}{\tilde{R}_\lambda} = \operatorname*{Max}_{\mathbf{v},\theta,c} \frac{-I}{\mathscr{D}} = \operatorname*{Max}_{\mathbf{v},\theta,c}\left[-\frac{I_0(\mathbf{v},\theta,c)+\lambda_1 I_1(\mathbf{v},\theta)+\lambda_2 I_2(\mathbf{v},c)}{\mathscr{D}_0(\mathbf{v},\mathbf{v})+\lambda_1\mathscr{D}_1(\theta,\theta)+\lambda_2\mathscr{D}_2(c,c)}\right], \tag{5A.1}$$

over a field of twice continuously differentiable functions θ, c and $\mathbf{v}$ satisfying (3.11)–(3.15). Second, for fixed μ_1 and μ_2, we may select λ_1 and λ_2 to give the best possible limit for stability, *i.e.*, the largest values of $\tilde{R}_\lambda(\lambda_1,\lambda_2,\mu_1,\mu_2)$ over λ_1 and λ_2, and define

$$\tilde{R}(\mu_1,\mu_2) = \operatorname*{Max}_{\substack{\lambda_1>0\\ \lambda_2>0}} \tilde{R}_\lambda(\lambda_1,\lambda_2,\mu_1,\mu_2). \tag{5A.2}$$

The locus of values $\tilde{R}(\mu_1,\mu_2)$ gives the optimum stability boundary surface over the three-dimensional space with the coordinates $(R,\mathscr{R},\mathscr{C})$. The stability-boundary surface is determined as follows: We first fix $\frac{1}{\mu_1}=R/\mathscr{R}$. This determines a plane through the $\mathscr{C}$-axis. Then we fix $\frac{1}{\mu_2}=R/\mathscr{C}$ and determine a plane through the $\mathscr{R}$-axis. The intersection of these two planes determines a ray from the origin in $(R,\mathscr{R},\mathscr{C})$ space. A set of maximum values, $1/\tilde{R}_\lambda$, are then found for different λ_1 and λ_2 on the ray for the fixed values μ_1 and μ_2. The best λ_1 and λ_2 produces the maximum value of $\tilde{R}_\lambda$ on the given ray and determines the optimum value $\tilde{R}(\mu_1,\mu_2)$. The coordinates of the stability surface, $\tilde{\mathscr{R}}$, $\tilde{\mathscr{C}}$, are determined from given parameters $\tilde{\mathscr{R}}=\mu_1\tilde{R}$ and $\tilde{\mathscr{C}}=\mu_2\tilde{R}$. The stability-boundary surface is generated as μ_1,μ_2 take all positive values. Under the surface, the flow is stable (see Figure 1).

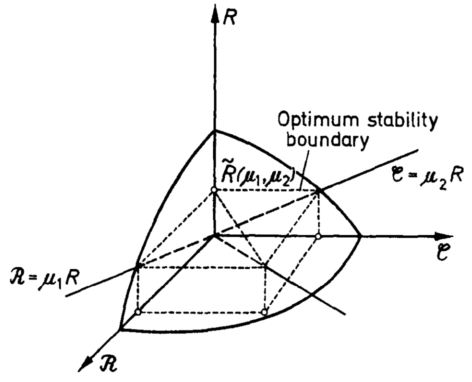

Fig. 1. Stability boundary and stability region. Points $(R,\mathscr{R},\mathscr{C})$ in the first octant and under the stability boundary are in the stability region. For such points the basic motion is stable and, if steady, then unique.

The maximum problem (5A.1) is easily formulated by the methods of variational calculus. Require that

$$\frac{1}{\tilde{R}_\lambda(\lambda_1, \lambda_2, \mu_1, \mu_2)} = \operatorname*{Max}_{\boldsymbol{v}, \theta, c} \{-[I_0(\boldsymbol{v}, \theta, c) + \lambda_1 I_1(\boldsymbol{v}, \theta) + \lambda_2 I_2(\boldsymbol{v}, c)]\}, \tag{5A.3}$$

hold for a class of twice-continuously differentiable functions $\boldsymbol{v}$, θ and c satisfying (3.11)−(3.15) and the normalizing condition

$$\mathscr{D} = \mathscr{D}_0(\boldsymbol{v}, \boldsymbol{v}) + \lambda_1 \mathscr{D}_1(\theta, \theta) + \lambda_2 \mathscr{D}_2(c, c) = 1. \tag{5A.4}$$

Lagrange multipliers R_λ and $p(\boldsymbol{x}, \tau)$ are introduced, and the problem of the system (3.11, 12, 13, 14, 15), (5A.1) and (5A.3) can be expressed as

$$\begin{aligned} \delta \Big\{ & I_0(\boldsymbol{v}, \theta, c) + \lambda_1 I_1(\boldsymbol{v}, \theta) + \lambda_2 I_2(\boldsymbol{v}, c) - 2 \int \frac{p}{R_\lambda} \nabla \cdot \boldsymbol{v} \\ & + \frac{1}{R_\lambda} [\mathscr{D}_0(\boldsymbol{v}, \boldsymbol{v}) + \lambda_1 \mathscr{D}_1(\theta, \theta) + \lambda_2 \mathscr{D}_2(c, c)] \Big\} = 0. \end{aligned} \tag{5A.5}$$

The Euler-Lagrange equations corresponding to (5A.5) are

$$\begin{aligned} R_\lambda \boldsymbol{v} \cdot \mathbf{E} + \tfrac{1}{2} \mu_1 R_\lambda (\mathbf{H} + \lambda_1 \mathbf{H}_1) \theta + \tfrac{1}{2} \mu_2 R_\lambda (-\mathscr{s} \mathbf{H} + \lambda_2 \mathbf{H}_2) c \\ = - \nabla p + \Delta \boldsymbol{v}, \end{aligned} \tag{5A.6}$$

$$\frac{\mu_1 R_\lambda}{2\lambda_1} (\mathbf{H} + \lambda_1 \mathbf{H}_1) \cdot \boldsymbol{v} = \Delta \theta, \tag{5A.7}$$

and

$$\frac{\mu_2 R_\lambda}{2\lambda_2} (-\mathscr{s} \mathbf{H} + \lambda_2 \mathbf{H}_2) \cdot \boldsymbol{v} = \Delta c, \tag{5A.8}$$

which are to be solved subject to the boundary conditions (3.12)−(3.15). Now, for any solution of equations (5A.6)−(5A.8), we have by suitable multiplications and the divergence theorem that

$$\begin{aligned} R_\lambda \int \boldsymbol{v} \cdot \mathbf{E} \cdot \boldsymbol{v} + \frac{\mu_1 R_\lambda}{2} \int (\mathbf{H} + \lambda_1 \mathbf{H}_1) \cdot \boldsymbol{v} \theta + \\ \frac{\mu_2 R_\lambda}{2} \int (-\mathscr{s} \mathbf{H} + \lambda_2 \mathbf{H}_2) \cdot \boldsymbol{v} c + \mathscr{D}_0(\boldsymbol{v}, \boldsymbol{v}) = 0, \end{aligned} \tag{5A.9}$$

$$\frac{\mu_1 R_\lambda}{2\lambda_1} \int (\mathbf{H} + \lambda_1 \mathbf{H}_2) \cdot \boldsymbol{v} \theta + \mathscr{D}_1(\theta, \theta) = 0, \tag{5A.10}$$

and

$$\frac{\mu_2 R_\lambda}{2\lambda_2} \int (-\mathscr{s} \mathbf{H} + \lambda_2 \mathbf{H}_2) \cdot \boldsymbol{v} c + \mathscr{D}_2(c, c) = 0. \tag{5A.11}$$

Add (5A.9)−(5A.11) and use the normalizing condition (5A.4) to produce

$$I = I_0(\boldsymbol{v}, \theta, c) + \lambda_1 I_1(\boldsymbol{v}, \theta) + \lambda_2 I_2(\boldsymbol{v}, c) = -\frac{1}{R_\lambda}.$$

The maximum of these eigenvalues $\frac{1}{R_\lambda}$ coincides with the solution of (5A.1)–(5A.4). [*cf.* remark following (5A.27)]. Hence, it follows that

$$\tilde{R}_\lambda(\lambda_1, \lambda_2, \mu_1, \mu_2) = \underset{v, \theta, c \in H_2}{\operatorname{Min}} R_\lambda(\lambda_1, \lambda_2, \mu_1, \mu_2)$$

for any of the positive set of eigenvalues R_λ. Then the eigenvalue problem and the optimum stability boundary are related by

$$\tilde{R}(\mu_1, \mu_2) = \underset{\lambda_1, \lambda_2 > 0}{\operatorname{Max}} \tilde{R}_\lambda(\lambda_1, \lambda_2, \mu_1, \mu_2) = \underset{\lambda_1, \lambda_2 > 0}{\operatorname{Max}} \left[\underset{v, \theta, c \in H_2}{\operatorname{Min}} R_\lambda(\lambda_1, \lambda_2, \mu_1, \mu_2) \right]. \tag{5A.12}$$

The maximum problem (5A.1) over hydrodynamic solutions suffices for stability. This is true *a fortiori* over a wider class of functions. It is because of this that the maximum problem set in the standard way is equivalent to the problem of finding limits sufficient for stability. We seek a solution of (5A.1) in a vector-valued Hilbert space (H_2),

$$\begin{bmatrix} \boldsymbol{v} \\ \phi \\ \psi \end{bmatrix} = u \in H_2, \quad \phi = \theta \sqrt{\lambda_1}, \quad \psi = c \sqrt{\lambda_2},$$

completed under the norm corresponding to the scalar products

$$\mathscr{D}(u'; u'') = \mathscr{D}_0(\boldsymbol{v}'; \boldsymbol{v}'') + \mathscr{D}_1(\phi'; \phi'') + \mathscr{D}_2(\psi'; \psi''), \tag{5A.13}$$

where

$$\mathscr{D}_0(\boldsymbol{v}'; \boldsymbol{v}'') = \begin{cases} \int \nabla \boldsymbol{v}' : \nabla \boldsymbol{v}'', & \text{rigid surface}, \\ \int 2\boldsymbol{d}(\boldsymbol{v}') : \boldsymbol{d}(\boldsymbol{v}''), & \text{free surface}, \end{cases}$$

$$\mathscr{D}_1(\phi'; \phi'') = \int \nabla \phi' \cdot \nabla \phi'' + \oint Nu\, \phi' \phi'',$$

and

$$\mathscr{D}_2(\psi'; \psi'') = \int \nabla \psi' \cdot \nabla \psi'' + \oint Sh\, \psi' \psi''.$$

We then simplify the notations as follows: First, let

$$\hat{\boldsymbol{G}}_1 = \frac{\mu_1}{2\sqrt{\lambda_1}} (\mathbf{H} + \lambda_1 \mathbf{H}_1),$$

and

$$\hat{\boldsymbol{G}}_2 = \frac{\mu_2}{2\sqrt{\lambda_2}} (-\delta \mathbf{H} + \lambda_2 \mathbf{H}_2),$$

and write (5A.6, 7, 8) as

$$\tilde{R}_\lambda(\mathbf{E} \cdot \boldsymbol{v} + \hat{\boldsymbol{G}}_1 \phi + \hat{\boldsymbol{G}}_2 \psi) = -\nabla p + \Delta \boldsymbol{v}, \tag{5A.14}$$

$$\tilde{R}_\lambda \hat{\boldsymbol{G}}_1 \cdot \boldsymbol{v} = \Delta \phi, \tag{5A.15}$$

$$\tilde{R}_\lambda \hat{\boldsymbol{G}}_2 \cdot \boldsymbol{v} = \Delta \psi. \tag{5A.16}$$

Remark. *We wish to draw the readers' attention to our notational convention. Principal eigenvalues of Euler's equations will now be designated with a tilda overbar.*

No further reference is made to higher eigenvalues so that, in the sequel, tilda overbar quantities are eigenvalues, and parameters without overbars are preassigned. In (5A.14, 15, 16), we have the principal eigenvalue $R_\lambda = \tilde{R}_\lambda$. The value $\tilde{R}$ of (5A.12) is one of the values $\tilde{R}_\lambda$. In Theorems 4 and 5, $\tilde{R}(\mu_1, \mu_2)$ is an eigenvalue, and μ_1 and μ_2 are preassigned; in Theorem 6, $\tilde{R}e(\mathcal{R}, \mathcal{C})$ $[=\tilde{R}(\mu_1, \mu_2)]$ is the eigenvalue, and $\mathcal{R}, \mathcal{C}$ are preassigned. In Chapter 6, $R=0$ and $\tilde{\mathcal{R}}(\mathcal{C})$ is an eigenvalue, and $\mathcal{C}$ is preassigned, *etc.*

With the introduction of the 5×5 symmetric matrix

$$\hat{A} = \left[\begin{array}{c|cc} \mathbf{E} & \hat{G}_1 & \hat{G}_2 \\ \hline \hat{G}_1 & 0 & 0 \\ \hat{G}_2 & 0 & 0 \end{array}\right] \tag{5A.17}$$

and the operators

$$L u = \begin{bmatrix} \Delta \boldsymbol{v} \\ \Delta \phi \\ \Delta \psi \end{bmatrix}, \qquad \partial = \begin{bmatrix} \nabla \\ 0 \\ 0 \end{bmatrix}, \tag{5A.18}$$

one may write (5A.14, 15, 16) as

$$\tilde{R}_\lambda \begin{bmatrix} \mathbf{E} & \hat{G}_1 & \hat{G}_2 \\ \hat{G}_1 & 0 & 0 \\ \hat{G}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \Delta \boldsymbol{v} \\ \Delta \phi \\ \Delta \psi \end{bmatrix} - \begin{bmatrix} \nabla p \\ 0 \\ 0 \end{bmatrix}, \tag{5A.19}$$

or

$$\tilde{R}_\lambda \hat{A} u = L u - \partial p, \tag{5A.20}$$

or

$$M u \equiv (L - \tilde{R}_\lambda \hat{A}) u \equiv (L - A) u = \partial p \tag{5A.21}$$

where $A = R_\lambda A$. Replace boundary conditions with

$$B u = 0, \quad x \in \partial \mathscr{V} \tag{5A.22}$$

where

$$B = \begin{bmatrix} F & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dfrac{\partial}{\partial N} + Nu & 0 \\ \mathbf{0} & 0 & \dfrac{\partial}{\partial N} + Sh \end{bmatrix}, \tag{5A.23}$$

and

$$(F)_{ij} = \begin{cases} \delta_{ij}, & \text{rigid surface}, \\ (\delta_{ij} N_l + N_j \delta_{il} - 2 N_i N_j N_l) \dfrac{\partial}{\partial x_l}, & \text{free surface}, \end{cases} \tag{5A.24}$$

where $i, j = 1, 2, 3$ and sum over l. The matrix F arises out of the free-surface condition expressed as $\boldsymbol{d} \cdot \boldsymbol{N} = \tilde{\alpha} \boldsymbol{N}$ or, equivalently, as $(\boldsymbol{N} \cdot \boldsymbol{d} \cdot \boldsymbol{N}) \boldsymbol{N} = \tilde{\alpha} \boldsymbol{N} = \boldsymbol{d} \cdot \boldsymbol{N}$. It is easily verified that this last statement may be expressed as $\boldsymbol{F} \cdot \boldsymbol{v} = 0$. Moreover, we define the scalar product

$$\langle u'; u''\rangle = \int_{\mathscr{V}} [\boldsymbol{v}', \phi', \psi'] \begin{bmatrix} \boldsymbol{v}'' \\ \phi'' \\ \psi'' \end{bmatrix}$$

and

$$\langle u'; u''\rangle^* = \oint_{\partial\mathscr{V}} [\boldsymbol{v}', \phi', \psi'] \begin{bmatrix} \boldsymbol{v}'' \\ \phi'' \\ \psi'' \end{bmatrix}. \tag{5A.25}$$

Since

$$\begin{aligned} \langle u'; \hat{A} u''\rangle &= \langle u''; \hat{A} u'\rangle \\ &= \int [\boldsymbol{v}' \cdot \mathbf{E} \cdot \boldsymbol{v}'' + \hat{G}_1 \cdot (\boldsymbol{v}'\phi'' + \boldsymbol{v}''\phi') + \hat{G}_2 \cdot (\boldsymbol{v}'\psi'' + \boldsymbol{v}''\psi')] \\ &= I(u'; u''), \end{aligned} \tag{5A.26}$$

one may write

$$\frac{1}{\tilde{R}_\lambda} = \operatorname*{Max}_{u' \in H_2} \frac{-\langle u'; \hat{A} u'\rangle}{\mathscr{D}(u'; u')}. \tag{5A.27}$$

It can be shown that, under usual circumstances, $\langle u'; \hat{A} u'\rangle$ is completely continuous with respect to convergent sequences of elements in H_2, *i.e.*, with respect to $\mathscr{D}(u'; u')$. It follows from this that the maximum is attained for elements $u \in H_2$, which are also regular solutions of the Euler equations and natural boundary conditions for (5A.27), *i.e.*, (5A.22, 23). For this reason, it is permissible to regard H_2 as a space of smooth functions, as is done in the sequel. For further details see [11].

Now we enumerate properties of the inner product. First, note that for any vectors

$$u' = \begin{bmatrix} \boldsymbol{v}' \\ \phi' \\ \psi' \end{bmatrix}, \qquad u'' = \begin{bmatrix} \boldsymbol{v}'' \\ \phi'' \\ \psi'' \end{bmatrix},$$

where $\boldsymbol{v}'$ and $\boldsymbol{v}''$ are solenoidal, we have

$$\begin{aligned} \langle u'; Lu'' - \partial p\rangle &= \int [\boldsymbol{v}', \phi', \psi'] \begin{bmatrix} -\nabla p + \Delta \boldsymbol{v}'' \\ \Delta \phi'' \\ \Delta \psi'' \end{bmatrix} \\ = \langle u'; Lu''\rangle &= \int [\boldsymbol{v}', \phi', \psi''] \begin{bmatrix} \Delta \boldsymbol{v}'' \\ \Delta \phi'' \\ \Delta \psi'' \end{bmatrix} \\ &= \int (\boldsymbol{v}' \cdot \Delta \boldsymbol{v}'' + \phi' \Delta \phi'' + \psi' \Delta \psi'') \\ &= -\mathscr{D}(u'; u'') + \langle u'; \mathscr{B} u''\rangle^*, \end{aligned} \tag{5A.28}$$

where

$$\mathscr{B} u = \begin{bmatrix} \nabla \boldsymbol{v} \cdot N \\ \frac{\partial}{\partial N} \phi + Nu\, \phi \\ \frac{\partial}{\partial N} \psi + Sh\, \psi \end{bmatrix}, \qquad \text{rigid surface}, \tag{5A.29a}$$

$$\mathscr{B}u=\begin{bmatrix} 2\boldsymbol{d}(\boldsymbol{v})\cdot N \\ \dfrac{\partial}{\partial N}\phi+Nu\,\phi \\ \dfrac{\partial}{\partial N}\psi+Sh\,\psi \end{bmatrix}, \qquad \text{free surface.} \tag{5A.29b}$$

Equation (5A.28) is obtained from the relations

$$\boldsymbol{v}'\cdot\Delta\boldsymbol{v}''=\begin{cases}\operatorname{div}(\boldsymbol{v}'\cdot\nabla\boldsymbol{v}'')-(\nabla\boldsymbol{v}':\nabla\boldsymbol{v}''), & \text{rigid surface},\\ 2\operatorname{div}(\boldsymbol{v}'\cdot\boldsymbol{d}(\boldsymbol{v}''))-2\boldsymbol{d}(\boldsymbol{v}'):\boldsymbol{d}(\boldsymbol{v}''), & \text{free surface},\end{cases}$$

and

$$\begin{aligned}\langle u';Mu''\rangle&=\langle u';(L-A)u''\rangle\\ &=-\mathscr{D}(u';u'')+\langle u';\mathscr{B}u''\rangle^{*}-\langle u';Au''\rangle.\end{aligned} \tag{5A.30}$$

Let

$$\begin{aligned}H(u';u'')&=\mathscr{D}(u';u'')+\langle u';\tilde{R}_{\lambda}\hat{A}u''\rangle\\ &=\mathscr{D}(u';u'')+\tilde{R}_{\lambda}\langle u';\hat{A}u''\rangle.\end{aligned} \tag{5A.31}$$

If u' is an admissible function in competition for the maximum (5A.27), then

$$\frac{1}{\tilde{R}_{\lambda}}=\operatorname*{Max}_{u'}\frac{-\langle u';\hat{A}u'\rangle}{\mathscr{D}(u';u')}\geqq\frac{-\langle u';\hat{A}u'\rangle}{\mathscr{D}(u';u')}, \tag{5A.32}$$

and

$$H(u';u')\geqq 0. \tag{5A.33}$$

From (5A.28), (5A.30), we have

$$\begin{aligned}&\langle u';Mu''\rangle-\langle u'';Mu'\rangle\\ &=\langle u';Lu''\rangle-\langle u'';Lu'\rangle\\ &=\langle u';\mathscr{B}u''\rangle^{*}-\langle u'';\mathscr{B}u'\rangle^{*}.\end{aligned} \tag{5A.34}$$

If u' and u'' satisfy the boundary condition (5A.22), then the surface integrals in (5A.28, 30, 34) vanish, and this implies that M is a self-adjoint operator.

5B. Parametric Derivatives of the Boundary-Value Problem Associated with the Euler Equation and a Convexity Lemma

Let y^{α} ($\alpha=1, 2, \ldots$) stand for any scalar parameter and

$$u_{,\alpha}=\frac{\partial u}{\partial y^{\alpha}},$$

$$u_{,\alpha\beta}=\frac{\partial^2 u}{\partial y^{\alpha}\partial y^{\beta}},$$

$$u_{,\alpha^n}=\frac{\partial^n u}{\partial y^{\alpha^n}}.$$

We have from (5A.21, 22),

$$\begin{cases}Mu=(L-A)u=\partial p & \text{for } \boldsymbol{x}\in\mathscr{V}, \quad (5B.1)\\ Bu=0 & \text{for } \boldsymbol{x}\in\partial\mathscr{V}. \quad (5B.2)\end{cases}$$

Differentiate (5B.1, 2) n times with respect to any parameter y^α. This gives

$$M u_{,\alpha^n} + \sum_{k=1}^{n} \binom{n}{k} M_{,\alpha^k} u_{,\alpha^{n-k}} = \partial p_{,\alpha^n}, \tag{5B.3}$$

$$B u_{,\alpha^n} + \sum_{k=1}^{n} \binom{n}{k} B_{,\alpha^k} u_{,\alpha^{n-k}} = 0, \tag{5B.4}$$

and

$$\nabla \cdot v_{,\alpha^n} = 0. \tag{5B.5}$$

Multiply (5B.1, 2) by $u_{,\alpha^n}$, integrate over $\mathscr{V}$ and use (5A.34) to obtain

$$\begin{aligned} 0 = \langle u_{,\alpha^n}; M u\rangle &= \langle u; M u_{,\alpha^n}\rangle + \langle u_{,\alpha^n}; \mathscr{B} u\rangle^* - \langle u; \mathscr{B} u_{,\alpha^n}\rangle^* \\ &= \langle u; M u_{,\alpha^n}\rangle - \langle u; \mathscr{B} u_{,\alpha^n}\rangle^*. \\ &= \langle u; M u_{,\alpha^n}\rangle - \langle u; B u_{,\alpha^n}\rangle^*. \end{aligned} \tag{5B.6}$$

Note that ∇, Δ, N and $\frac{\partial}{\partial N}$ do not depend on parameters y^α.

From (5B.1), (5A.29a, 29b), we have

$$M_{,\alpha^n} = -A_{,\alpha^n}, \qquad B_{,\alpha^n} = \mathscr{B}_{,\alpha^n}, \qquad (n = 1, 2, \ldots). \tag{5B.7}$$

From (5B.3, 4, 6) and (5B.7), it follows that

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \binom{n}{k} \left[\langle u; B_{,\alpha^k} u_{,\alpha^{n-k}}\rangle^* - \langle u; M_{,\alpha^k} u_{,\alpha^{n-k}}\rangle\right] \\ &= \sum_{k=1}^{n} \binom{n}{k} \left[\langle u; \mathscr{B}_{,\alpha^k} u_{,\alpha^{n-k}}\rangle^* + \langle u; A_{,\alpha^k} u_{,\alpha^{n-k}}\rangle\right]. \end{aligned} \tag{5B.8}$$

Moreover, boundary-value problems for $u_{,\alpha}$ and $u_{,\alpha\beta}$ are formed by direct differentiation. Thus,

$$M_{,\alpha} u + M u_{,\alpha} = \partial p_{,\alpha}, \tag{5B.9}$$

$$M_{,\alpha\beta} u + M_{,\alpha} u_{,\beta} + M_{,\beta} u_{,\alpha} + M u_{,\alpha\beta} = \partial p_{,\alpha\beta}, \tag{5B.10}$$

$$\operatorname{div} v = \operatorname{div} v_{,\alpha} = \operatorname{div} v_{,\alpha\beta} = 0, \tag{5B.11}$$

for $x \in \mathscr{V}$ and

$$B_{,\alpha} u + B u_{,\alpha} = 0, \tag{5B.12}$$

$$B_{,\alpha\beta} u + B_{,\alpha} u_{,\beta} + B_{,\beta} u_{,\alpha} + B u_{,\alpha\beta} = 0, \tag{5B.13}$$

$$v \cdot N = v_{,\alpha} \cdot N = v_{,\alpha\beta} \cdot N = 0, \tag{5B.14}$$

for $x \in \partial\mathscr{V}$. Also

$$\langle u; B u_{,\alpha\beta}\rangle^* = \langle u; \mathscr{B} u_{,\alpha\beta}\rangle^*, \tag{5B.15}$$

$$\langle u_{,\alpha}; B u_{,\beta}\rangle = \langle u_{,\alpha}; \mathscr{B} u_{,\beta}\rangle^* \tag{5B.16}$$

Lemma 1. *Let M, B and the u which solves the boundary-value problem* (5B.9 – 14) *be twice-continuously differentiable functions of the parameters y^α, y^β for each $x \in \overline{\mathscr{V}}$ $(\overline{\mathscr{V}} = \mathscr{V} + \partial\mathscr{V})$. Then*

a) $$\langle u; A_{,\alpha} u\rangle + \langle u; B_{,\alpha} u\rangle^* = 0, \tag{5B.17}$$

b) $$\langle u; A_{,\alpha\beta} u\rangle + \langle u; B_{,\alpha\beta} u\rangle^* = 2H(u_{,\alpha}; u_{,\beta}). \tag{5B.18}$$

Proof.

a) (5B.17) follows directly from (5B.8) by taking $n=1$.

b) Multiply (5B.10, 13) by u and integrate over $\mathscr{V}$ and $\partial\mathscr{V}$, respectively, to obtain

$$\langle u; A_{,\alpha\beta} u\rangle - \langle u; M_{,\alpha} u_{,\beta}\rangle - \langle u; M_{,\beta} u_{,\alpha}\rangle - \langle u; M u_{,\alpha\beta}\rangle = 0, \quad (5\text{B}.19)$$

and

$$\langle u; B_{,\alpha\beta} u\rangle^* + \langle u; B_{,\alpha} u_{,\beta}\rangle^* + \langle u; B_{,\beta} u_{,\alpha}\rangle^* + \langle u; \mathscr{B} u_{,\alpha\beta}\rangle^* = 0. \quad (5\text{B}.20)$$

Note that $M_{,\alpha}$ and $B_{,\alpha}$ are symmetric. Then

$$\langle u_{,\beta}; M_{,\alpha} u\rangle + \langle u_{,\beta}; M u_{,\alpha}\rangle = 0 \quad (5\text{B}.21)$$

and

$$\langle u_{,\beta}; B_{,\alpha} u\rangle^* + \langle u_{,\beta}; \mathscr{B} u_{,\alpha}\rangle^* = 0 \quad (5\text{B}.22)$$

follow from (5B.9, 12). Thus we have the relations

$$\begin{aligned}\langle u; M_{,\alpha} u_{,\beta}\rangle &= \langle u_{,\beta}; M_{,\alpha} u\rangle \\ &= -\langle u_{,\beta}; M u_{,\alpha}\rangle \\ &= H(u_{,\beta}; u_{,\alpha}) - \langle u_{,\beta}; \mathscr{B} u_{,\alpha}\rangle^*.\end{aligned} \quad (5\text{B}.23)$$

Similarly,

$$\langle u; M_{,\beta} u_{,\alpha}\rangle = H(u_{,\alpha}; u_{,\beta}) - \langle u_{,\alpha}; \mathscr{B} u_{,\beta}\rangle^*, \quad (5\text{B}.24)$$

$$\begin{aligned}\langle u; B_{,\alpha} u_{,\beta}\rangle^* &= \langle u_{,\beta}; B_{,\alpha} u\rangle^* \\ &= -\langle u_{,\beta}; \mathscr{B} u_{,\alpha}\rangle^*,\end{aligned} \quad (5\text{B}.25)$$

and

$$\langle u; B_{,\beta} u_{,\alpha}\rangle^* = -\langle u_{,\alpha}; \mathscr{B} u_{,\beta}\rangle^*. \quad (5\text{B}.26)$$

The relation

$$\begin{aligned}\langle u; M u_{,\alpha\beta}\rangle &= \langle u_{,\alpha\beta}; M u\rangle + \langle u; \mathscr{B} u_{,\alpha\beta}\rangle^* \\ &\quad - \langle u_{,\alpha\beta}; \mathscr{B} u\rangle^* = \langle u; \mathscr{B} u_{,\alpha\beta}\rangle^*\end{aligned} \quad (5\text{B}.27)$$

is a consequence of $Mu = Bu = 0$ and (5A.34). Now, substitute (5B.23, 24, 25, 26, 27) into (5B.19, 20) and sum (5B.19) and (5B.20) to obtain (5B.18). This proves Lemma 1.

Lemma 2 (A Convexity Lemma). *Let the hypotheses of Lemma 1 prevail. Then*

$$\sum_{\alpha,\beta=1}^{N} dy^\alpha dy^\beta \left(\langle u; B_{,\alpha\beta} u\rangle^* + \langle u; A_{,\alpha\beta} u\rangle\right) = 2 \sum_{\alpha,\beta=1}^{N} dy^\alpha dy^\beta H(u_{,\alpha}; u_{,\beta}) \geqq 0. \quad (5\text{B}.28)$$

Proof. The inequality (5B.28) follows from the fact that

$$U = \sum_{k=1}^{N} dy^k u_{,k} \in H_2$$

and is an admissible function for which (5A.33) is true:

$$0 \leqq H(U; U) = 2 \sum_{\alpha,\beta=1}^{N} dy^\alpha dy^\beta H(u_{,\alpha}; u_{,\beta}). \quad (5\text{B}.29)$$

To prove the first equality of (5B.28), we multiply $dy^\alpha dy^\beta$ on (5B.18) and sum over repeated indices. This proves the lemma.

5C. The Best Coupling Parameter and the Geometry of the Stability Boundary

The optimum stability boundary is defined by (5A.12). A partial resolution of this problem is given by

Theorem 4. *Let* $\tilde{R}_\lambda(\lambda_1,\lambda_2,\mu_1,\mu_2)$ *and* $u(\mathbf{x},\lambda_1,\lambda_2,\mu_1,\mu_2)$, *which solve the maximum problem* (5A.27) *and the equivalent eigenvalue problem for the Euler equation* (5A.21), *be continuously differentiable functions of their arguments. Let the best values of the coupling parameters* $(\tilde{\lambda}_1,\tilde{\lambda}_2)$ *which solve* (5A.2) *be finite and not zero. Then these best values satisfy the relations*

$$\tilde{\lambda}_1=\frac{\int \mathbf{H}\cdot \mathbf{v}\phi}{\int \mathbf{H}_1\cdot \mathbf{v}\phi}, \tag{5C.1}$$

and

$$\tilde{\lambda}_2=\frac{-s\int \mathbf{H}\cdot \mathbf{v}\psi}{\int \mathbf{H}_2\cdot \mathbf{v}\psi}, \tag{5C.2}$$

where $u=\begin{bmatrix}\mathbf{v}\\ \phi\\ \psi\end{bmatrix}$ *is the maximizing function for* (5A.27). *Moreover,*

$$\tilde{\mathscr{R}}=\mu_1\tilde{R}(\mu_1,\mu_2)=-\frac{\mathscr{D}_1(\phi,\phi)}{\sqrt{\tilde{\lambda}_1}\int \mathbf{H}_1\cdot \mathbf{v}\phi}\,. \tag{5C.3}$$

and

$$\tilde{\mathscr{C}}=\mu_2\tilde{R}(\mu_1,\mu_2)=-\frac{\mathscr{D}_2(\psi,\psi)}{\sqrt{\tilde{\lambda}_2}\int \mathbf{H}_2\cdot \mathbf{v}\psi}, \tag{5C.4}$$

Proof. Under the hypotheses of the theorem, the best $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ must appear as roots of the equations

$$\left.\frac{\partial\tilde{R}_\lambda}{\partial\lambda_1}\right|_{\lambda_2}=0 \quad\text{and}\quad \left.\frac{\partial\tilde{R}_\lambda}{\partial\lambda_2}\right|_{\lambda_1}=0, \tag{5C.5}$$

as is necessarily true for solutions of (5A.2). The proof is an immediate consequence of the relation

$$0=\langle u; A_{,\alpha}u\rangle, \tag{5C.6}$$

which follows from (5B.17), written for $\alpha=1, y^1=\lambda_1$, $\alpha=2, y^2=\lambda_2$ and $B_{,1}=B_{,2}=0$. When $\alpha=1$,

$$A_{,1}=\left[\begin{array}{c|cc} \dfrac{\partial\tilde{R}_\lambda}{\partial\lambda_1}\mathbf{E} & \tilde{R}_\lambda\dfrac{\partial\hat{G}_1}{\partial\lambda_1}+\hat{G}_1\dfrac{\partial\tilde{R}_\lambda}{\partial\lambda_1} & \mathbf{0}\\ \hline \tilde{R}_\lambda\dfrac{\partial\hat{G}_1}{\partial\lambda_1}+\hat{G}_1\dfrac{\partial\tilde{R}_\lambda}{\partial\lambda_1} & 0 & 0\\ \mathbf{0} & 0 & 0 \end{array}\right] =\tilde{R}_\lambda\left[\begin{array}{c|cc} \mathbf{0} & \dfrac{\partial\hat{G}_1}{\partial\lambda_1} & \mathbf{0}\\ \hline \dfrac{\partial\hat{G}_1}{\partial\lambda_1} & 0 & 0\\ \mathbf{0} & 0 & 0 \end{array}\right]. \tag{5C.7}$$

This leads through (5C.6, 7) to the relation

$$0=2\int \tilde{R}_\lambda \frac{\partial \hat{G}_1}{\partial \lambda_1}\cdot v\phi = -\frac{\mu_1 \tilde{R}_\lambda}{4\tilde{\lambda}_1^{\frac{3}{2}}}\int (\mathbf{H}-\tilde{\lambda}_1 \mathbf{H}_1)\cdot v\phi$$

and proves (5C.1). Similarly, for $\alpha=2$, we have

$$-\frac{\mu_2 \tilde{R}_\lambda}{4\tilde{\lambda}_2^{\frac{3}{2}}}\int (-s\mathbf{H}-\tilde{\lambda}_2 \mathbf{H}_2)\cdot v\psi = 0,$$

which implies (5C.2). Multiply (5A.15, 16) by ϕ, ψ and integrate over $\mathscr{V}$ to obtain

$$\tilde{R}_\lambda \int \hat{G}_1 \cdot v\phi = -\mathscr{D}_1(\phi,\phi), \tag{5C.8}$$

and

$$\tilde{R}_\lambda \int \hat{G}_2 \cdot v\psi = -\mathscr{D}_2(\psi,\psi). \tag{5C.9}$$

Then use the values λ_1 and $\tilde{\lambda}_2$ given by (5C.1) and (5C.2) in (5C.8, 9) to obtain (5C.3, 4). This completes the proof.

Equations (5C.1, 2) are valuable for estimating the best values of λ_1, λ_2 (see [5] for an application). Obviously, when $\mathbf{H}_1=\mathbf{H}$ or $-s\mathbf{H}_2=\mathbf{H}$, the best values are $\tilde{\lambda}_1=1$ or $\tilde{\lambda}_2=1$, respectively.

The differentiability requirement is clearly necessary, as the proof shows. In physics there are cases in which this requirement is not satisfied, and then Theorem 4 leads to a false result. This is the case for a plane motionless fluid layer heated from above. For this flow, $\mathbf{H}=-\mathbf{H}_1=-e$. From (5C.1), there results the unacceptable value $\tilde{\lambda}_1=-1$. (5C.1) fails because $\tilde{R}_\lambda(\lambda_1)$ is not continuously differentiable. See Section 6A for details.

The conditions under which one may determine uniquely the best values of λ_1 and λ_2 (5A.12) are the subject of the following:

Theorem 5. *Let the hypotheses of Theorem 4 prevail. Then there exists one and only one pair of values* $(\tilde{\lambda}_1, \tilde{\lambda}_2)$

$$0<\lambda_1, \lambda_2<\infty$$

satisfying the pair of equations

$$\frac{\partial \tilde{R}_\lambda}{\partial \lambda_1}=\frac{\partial \tilde{R}_\lambda}{\partial \lambda_2}=0. \tag{5C.10, 11}$$

Moreover, $\tilde{R}_\lambda(\lambda_1,\lambda_2)$ *attains its maximum value when* $\lambda_1=\lambda_1$ *and* $\lambda_2=\lambda_2$.

Proof. To prove this result, we establish that every pair of values (λ_1,λ_2) for which (5C.10, 11) holds makes $\tilde{R}_\lambda$ a local maximum. $\tilde{R}=\tilde{R}_\lambda$ is by hypothesis the absolute maximum. Every path between two such maximum points passes through a minimum. It must therefore be possible to find a path crossing the valley on an arc in the plane $\lambda_1=$const. or $\lambda_2=$const. At a point of such a path one of the equations (5C.10, 11) holds. We show that this point cannot be a minimum. Hence, there cannot be two local maximum points, and the maximum is unique.

To establish that each stationary point is locally a maximum, we let (λ_1, λ_2) denote the stationary point and then show that the sufficient condition

$$\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_1^2}<0, \quad \frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_2^2}<0,$$

and

$$\left(\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_1^2}\right)\left(\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_2^2}\right)-\left(\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_1 \partial \lambda_2}\right)^2>0,$$

holds when $\lambda_1=\bar{\lambda}_1$ and $\lambda_2=\bar{\lambda}_2$.

From (5B.18),

$$\langle u; A_{,\alpha\beta} u\rangle=2H(u_{,\alpha}; u_{,\beta})$$

where

$$A_{,\alpha\beta}=\tilde{R}_{\lambda,\alpha\beta}\hat{A}+\tilde{R}_{\lambda,\alpha}\hat{A}_{,\beta}+\tilde{R}_{\lambda,\beta}\hat{A}_{,\alpha}+\tilde{R}_\lambda \hat{A}_{,\alpha\beta}.$$

Then, with $\tilde{R}_\lambda\langle u; \hat{A} u\rangle=-1$, one finds that

$$\tilde{R}_{\lambda,\alpha\beta}=-\tilde{R}_\lambda[2H(u_{,\alpha}; u_{,\beta})-\tilde{R}_\lambda\langle u; \hat{A}_{,\alpha\beta} u\rangle - \tilde{R}_{\lambda,\alpha}\langle u; \hat{A}_{,\beta} u\rangle-\tilde{R}_{\lambda,\beta}\langle u; \hat{A}_{,\alpha} u\rangle],$$

where α, $\beta=1, 2$, and

$$\hat{A}_{,\alpha\beta}(\lambda_1, \lambda_2)=\left[\begin{array}{c|cc} 0 & \hat{G}_{1,\alpha\beta} & \hat{G}_{2,\alpha\beta} \\ \hline \hat{G}_{1,\alpha\beta} & 0 & 0 \\ \hat{G}_{2,\alpha\beta} & 0 & 0 \end{array}\right].$$

The derivatives $\tilde{R}_{\lambda,\alpha\beta}$ are evaluated on the optimum stability boundary where (5C.1–4) hold. This yields

(i) $\alpha=1, \quad \beta=1, \quad y^1=\lambda_1$

$$\hat{G}_{1,11}=\frac{\mu_1[\lambda_1 \mathbf{H}_1+\frac{3}{2}(\mathbf{H}-\lambda_1 \mathbf{H}_1)]}{4\lambda_1^{\frac{5}{2}}}, \quad \hat{G}_{2,11}=0, \quad \tilde{R}_{\lambda,1}=0, \tag{5C.12}$$

$$\tilde{R}_{\lambda,11}\Big|_{(\bar{\lambda}_1,\bar{\lambda}_2)}=-\tilde{R}_\lambda\left[2H(u_{,1}; u_{,1})+\frac{\mathscr{D}_1}{2\bar{\lambda}_1^2}\right].$$

(ii) $\alpha=2, \quad \beta=2, \quad y^2=\lambda_2, \quad \tilde{R}_{\lambda,2}=0,$

$$\tilde{R}_{\lambda,22}\Big|_{(\bar{\lambda}_1,\bar{\lambda}_2)}=-\tilde{R}_\lambda\left[2H(u_{,2}; u_{,2})+\frac{\mathscr{D}_2}{2\bar{\lambda}_2^2}\right]. \tag{5C.13}$$

(iii) $\alpha=1, \quad \beta=2, \quad y^1=\lambda_1, \quad y^2=\lambda_2, \quad \tilde{R}_{\lambda,1}=0, \quad \tilde{R}_{\lambda,2}=0,$

$$\tilde{R}_{\lambda,12}\Big|_{(\bar{\lambda}_1,\bar{\lambda}_2)}=-\tilde{R}_\lambda\cdot 2H(u_{,1}; u_{,2}). \tag{5.14}$$

Note that $H(u_{,1}; u_{,1})$, $H(u_{,2}; u_{,2})$, $\mathscr{D}_1$ and $\mathscr{D}_2$ are positive definite. Then

$$\tilde{R}_{\lambda,11}\Big|_{(\bar{\lambda}_1,\bar{\lambda}_2)}=\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_1^2}<0, \quad \hat{R}_{\lambda,22}\Big|_{(\bar{\lambda}_1,\bar{\lambda}_2)}=\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_2^2}<0. \tag{5C.15, 16}$$

Equations (5C.12, 13, 14) are now combined to form the estimate

$$\left(\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_1^2}\right)\left(\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_2^2}\right)-\left(\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_1 \partial \lambda_2}\right)^2$$

$$=\tilde{R}_\lambda^2\left[4(H(u_{,1};u_{,1})H(u_{,2};u_{,2})-H(u_{,1};u_{,2})^2)\right. \tag{5C.17}$$

$$\left.+\frac{H(u_{,1};u_{,1})\mathcal{D}_2}{\tilde{\lambda}_2^2}+\frac{H(u_{,2};u_{,2})\mathcal{D}_1}{\tilde{\lambda}_1^2}+\frac{\mathcal{D}_1\mathcal{D}_2}{4\tilde{\lambda}_1^2\tilde{\lambda}_2^2}\right]>0.$$

Equations (5C.15, 16, 17) imply that $\tilde{R}_\lambda$ is a relative maximum at every stationary point $(\tilde{\lambda}_1,\tilde{\lambda}_2)$.

To show that there can be only one such maximum, we assume the contrary and note that on *every* path between the two maximum points, there is a minimum point. In particular, we can choose a path through the valley on which either (5C.10) or (5C.11) (say (5C.10)) holds at a point. Let the coordinates of this point be (λ_1^*,λ_2). It is easily verified that (5C.1), (5C.3) and (5C.15) hold when $(\tilde{\lambda}_1,\tilde{\lambda}_2)$ are replaced with (λ_1^*,λ_2). But here

$$\frac{\partial^2 \tilde{R}_\lambda}{\partial \lambda_1^2}<0,$$

and (λ_1^*,λ_2) cannot locate a minimum. Hence, there are not two local maxima, and the uniqueness of $(\tilde{\lambda}_1,\tilde{\lambda}_2)$ is established.

With these preliminary results aside, we can now examine the geometry of the stability boundary. It is convenient here to change variables and regard $\mathcal{R}=\mu_1 R_\lambda$ and $\mathcal{C}=\mu_2 R_\lambda$ as preassigned. We examine the properties of the stability boundary $\tilde{Re}(\mathcal{R},\mathcal{C})(=\tilde{R}(\mu_1,\mu_2))$.

Theorem 6. *Let the hypotheses of Theorem 4 and Lemmas 1 and 2 prevail. Then on the stability boundary* $\tilde{Re}(\mathcal{R},\mathcal{C})$

$$\frac{\partial \tilde{Re}}{\partial \mathcal{R}}=-\frac{2\int \mathbf{H}\cdot \boldsymbol{v}\phi}{\sqrt{\tilde{\lambda}_1}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}}=\frac{2\mathcal{D}_1(\phi,\phi)}{\mathcal{R}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}}\leqq 0, \tag{5C.18}$$

where the equality holds when $\mathcal{R}=0$. *Similarly,*

$$\frac{\partial \tilde{Re}}{\partial \mathcal{C}}=\frac{2\int s\mathbf{H}\cdot \boldsymbol{v}\psi}{\sqrt{\tilde{\lambda}_2}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}}=\frac{2\mathcal{D}_2(\psi,\psi)}{\mathcal{C}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}}\leqq 0, \tag{5C.19}$$

where the equality holds when $\mathcal{C}=0$. *Moreover,*

$$\frac{\partial^2 \tilde{Re}}{\partial \mathcal{R}^2}d\mathcal{R}^2+2\frac{\partial^2 \tilde{Re}}{\partial \mathcal{R}\partial \mathcal{C}}d\mathcal{R}\,d\mathcal{C}+\frac{\partial^2 \tilde{Re}}{\partial \mathcal{C}^2}d\mathcal{C}^2<0, \tag{5C.20}$$

and

$$\frac{\partial^2 \tilde{Re}}{\partial \mathcal{R}^2}<0, \quad \frac{\partial^2 \tilde{Re}}{\partial \mathcal{C}^2}<0.$$

That is, $\tilde{Re}(\mathcal{R},\mathcal{C})$ *is a monotone decreasing function of* $\mathcal{R}$ *and* $\mathcal{C}$ *and convex downward.*

Proof. Let $\mathscr{R}=R_1$ and $\mathscr{C}=R_2$. We regard R_1 and R_2, rather than μ_1 and μ_2, as preassigned and write

$$G_1=\frac{R_1}{2\sqrt{\lambda_1}}(\mathbf{H}+\lambda_1\mathbf{H}_1) \quad \text{and} \quad G_2=\frac{R_2}{2\sqrt{\lambda_2}}(-\mathfrak{s}\mathbf{H}+\lambda_2\mathbf{H}_2).$$

Let $\alpha=1, y^1=R_1, \alpha=2, y^2=R_2$. Obviously $B_{,1}=B_{,2}=0$. Then (5B.17) gives

$$0=\left\langle u; \frac{\partial A}{\partial R_1}u\right\rangle=\left\langle u; \frac{\partial A}{\partial R_2}u\right\rangle, \tag{5C.21}$$

where

$$\frac{\partial A}{\partial R_1}=\left[\begin{array}{c|cc} \frac{\partial \tilde{R}_\lambda}{\partial R_1}\mathbf{E} & \frac{\partial \boldsymbol{G}_1}{\partial R_1} & \mathbf{0} \\ \hline \frac{\partial \boldsymbol{G}_1}{\partial R_1} & 0 & 0 \\ \mathbf{0} & 0 & 0 \end{array}\right].$$

When unfolded, (5C.21) shows that

$$\frac{\partial \tilde{R}_\lambda}{\partial R_1}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}+2\frac{1}{2\sqrt{\lambda_1}}\int(\mathbf{H}+\lambda_1\mathbf{H}_1)\cdot\boldsymbol{v}\phi=0. \tag{5C.22}$$

On the optimum stability boundary, (5C.1) and (5C.3) hold and

$$\begin{aligned}\frac{1}{2\sqrt{\tilde{\lambda}_1}}\int(\mathbf{H}+\tilde{\lambda}_1\mathbf{H}_1)\cdot\boldsymbol{v}\phi&=\sqrt{\tilde{\lambda}_1}\int\mathbf{H}_1\cdot\boldsymbol{v}\phi\\ &=\frac{1}{\sqrt{\tilde{\lambda}_1}}\int\mathbf{H}\cdot\boldsymbol{v}\phi=\frac{-\mathscr{D}_1(\phi,\phi)}{R_1}.\end{aligned} \tag{5C.23}$$

Consider

$$\tilde{R}e=\tilde{R}(R_1,R_2)=\max_{\lambda_1,\lambda_2>0}\tilde{R}_\lambda=\tilde{R}_\lambda(\tilde{\lambda}_1(R_1,R_2),\tilde{\lambda}_2(R_1,R_2),R_1,R_2), \tag{5C.24}$$

and

$$\frac{\partial\tilde{R}e}{\partial R_i}=\frac{\partial\tilde{R}_\lambda}{\partial R_i}+\frac{\partial\tilde{R}_\lambda}{\partial\tilde{\lambda}_1}\frac{\partial\tilde{\lambda}_1}{\partial R_i}+\frac{\partial\tilde{R}_\lambda}{\partial\tilde{\lambda}_2}\frac{\partial\tilde{\lambda}_2}{\partial R_i}, \qquad i=1,2.$$

Since

$$\frac{\partial\tilde{R}_\lambda}{\partial\tilde{\lambda}_1}=\frac{\partial\tilde{R}_\lambda}{\partial\tilde{\lambda}_2}=0,$$

we have

$$\frac{\partial\tilde{R}e}{\partial R_1}=\frac{\partial\tilde{R}_\lambda}{\partial R_1} \quad \text{and} \quad \frac{\partial\tilde{R}e}{\partial R_2}=\frac{\partial\tilde{R}_\lambda}{\partial R_2},$$

which combined with (5C.22, 23) gives the left equality of (5C.18). Similarly, we prove (5C.19).

Note that $\phi=0$, $\psi=0$ are the only solutions of (5A.21, 22) with $R_1=0$ and $R_2=0$, respectively. This and the first equality of (5C.18) and (5C.19) prove that

$$\frac{\partial \tilde{R}e}{\partial R_1}=0 \qquad \text{when } R_1=0,$$

and

$$\frac{\partial \tilde{R}e}{\partial R_2}=0 \qquad \text{when } R_2=0.$$

To complete the proof of (5C.18) and (5C.19), we need to establish the inequalities. This is accomplished in two steps. We first demonstrate that the inequalities hold when R_1 and R_2 are small; then we remove the restriction. Let us suppose that the maximum problem (5A.27) has been solved and the best values of the coupling constants [(5C.1) and (5C.2)] obtained. Then with $\mathscr{D}(u;u)=1$, we may define

$$\tilde{R}e \int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}=-J(R_1,R_2), \tag{5C.25}$$

where

$$J(R_1,R_2)=1+\frac{2R_1}{\sqrt{\tilde{\lambda}_1}}\int \mathbf{H}\cdot\boldsymbol{v}\phi+\frac{2R_2}{\sqrt{\tilde{\lambda}_2}}\int(-\mathfrak{s}\mathbf{H}\cdot\boldsymbol{v}\psi).$$

Substitute (5C.25) into (5C.18, 19) to obtain

$$\frac{\partial Re}{\partial R_1}=-\frac{2\tilde{R}e\,\mathscr{D}_1(\phi,\phi)}{R_1\,J(R_1,R_2)}$$

and

$$\frac{\partial \tilde{R}e}{\partial R_2}=-\frac{2\tilde{R}e\,\mathscr{D}_2(\psi,\psi)}{R_2\,J(R_1,R_2)}.$$

Since $\mathscr{D}_1$, $\mathscr{D}_2$, $\tilde{R}e$, R_1 and R_2 are positive and $J(R_1,R_2)>0$ for sufficiently small values of R_1 and R_2 (say, $0<R_1\leqq R_1^*$), it follows that $\frac{\partial \tilde{R}e}{\partial R_1}$ and $\frac{\partial \tilde{R}e}{\partial R_1}$ are negative for $R_1\leqq R_1^*$ and $R_2\leqq R_2^*$.

To prove that $\frac{\partial \tilde{R}e}{\partial R_1}<0$, $\frac{\partial \tilde{R}e}{\partial R_2}<0$ for $R_1>R_1^*$ and $R_2>R_2^*$ is more difficult and requires the convexity lemma.

We next write the convexity lemma for

$$y^1=\lambda_1,\quad y^2=\lambda_2,\quad y^3=R_1,\quad y^4=R_2.$$

Let

$$U_3=dy^1 u_{,1}+dy^2 u_{,2}+dy^3 u_{,3},$$

$$U_4=dy^1 u_{,1}+dy^2 u_{,2}+dy^4 u_{,4},$$

and

$$\lambda_{1,3}=\frac{\partial\lambda_1}{\partial R_1},\quad \lambda_{1,4}=\frac{\partial\lambda_1}{\partial R_2},\quad \lambda_{2,3}=\frac{\partial\lambda_2}{\partial R_1},\quad \lambda_{2,4}=\frac{\partial\lambda_2}{\partial R_2}.$$

For fixed values of Nu and Sh, we have $B_{,\alpha}=B_{,\alpha\beta}=0$ and by (5B.28),

$$
\begin{aligned}
H(U_3;U_3)=&\sum_{\alpha,\beta=1}^{3} dy^\alpha dy^\beta \langle u; A_{,\alpha\beta}u\rangle \\
&=[d\tilde{\lambda}_1^2\langle u;A_{,11}u\rangle+d\tilde{\lambda}_2^2\langle u;A_{,22}u\rangle \\
&\quad +dR_1^2\langle u;A_{,33}u\rangle+2d\tilde{\lambda}_1\, d\tilde{\lambda}_2\langle u;A_{,12}u\rangle \\
&\quad +2d\tilde{\lambda}_1\, dR_1\langle u;A_{,13}u\rangle+2d\tilde{\lambda}_2\, dR_1\langle u;A_{,23}u\rangle] \\
&\geqq 0.
\end{aligned}
\tag{5C.26}
$$

This inequality implies the desired result. To show it, we must first express the $A_{,\alpha\beta}$ in terms of the derivatives of R_λ. Toward this end write

$$
A_{,\alpha\beta}=\left[\begin{array}{c|cc} \tilde{R}_{\lambda,\alpha\beta}\mathbf{E} & G_{1,\alpha\beta} & G_{2,\alpha\beta} \\ \hline G_{1,\alpha\beta} & 0 & 0 \\ G_{2,\alpha\beta} & 0 & 0 \end{array}\right],
$$

and

$$
\langle u; A_{,\alpha\beta}u\rangle=\tilde{R}_{\lambda,\alpha\beta}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}+2\int G_{1,\alpha\beta}\cdot\boldsymbol{v}\phi+2\int G_{2,\alpha\beta}\boldsymbol{v}\psi .
$$

These derivatives are to be evaluated on the optimum stability boundary. We can then use (5C.23) to calculate

(i) $$\langle u;A_{,11}u\rangle=\tilde{R}_{\lambda,11}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}-\frac{1}{2\tilde{\lambda}_1^2}\mathscr{D}_1(\phi,\phi), \tag{5C.27}$$

(ii) $$\langle u;A_{,22}u\rangle=\tilde{R}_{\lambda,22}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}-\frac{1}{2\tilde{\lambda}_2^2}\mathscr{D}_2(\psi,\psi), \tag{5C.28}$$

(iii) $$\langle u;A_{,33}u\rangle=\tilde{R}_{\lambda,33}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}, \tag{5C.29}$$

(iv) $$\langle u;A_{,12}u\rangle=\tilde{R}_{\lambda,12}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}, \tag{5C.30}$$

(v) $$\langle u;A_{,13}u\rangle=\tilde{R}_{\lambda,13}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}, \tag{5C.31}$$

(vi) $$\langle u;A_{,23}u\rangle=\tilde{R}_{\lambda,23}\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}. \tag{5C.32}$$

An easy combination of (5C.26−32) now leads to

$$
\begin{aligned}
0\leqq H(U_3;U_3)=\hat{I}_0\, dR_1^2[&\tilde{R}_{\lambda,11}\tilde{\lambda}_{1,3}^2+\tilde{R}_{\lambda,22}\tilde{\lambda}_{2,3}^2+\tilde{R}_{\lambda,33} \\
&+2\tilde{R}_{\lambda,12}\tilde{\lambda}_{1,3}\tilde{\lambda}_{2,3}+2\tilde{\lambda}_{1,3}\tilde{R}_{\lambda,13}+2\tilde{\lambda}_{2,3}\tilde{R}_{\lambda,23}] \\
&-d\tilde{\lambda}_1^2\frac{\mathscr{D}_1(\phi,\phi)}{2\tilde{\lambda}_1^2}-d\tilde{\lambda}_2^2\frac{\mathscr{D}_2(\psi,\psi)}{2\tilde{\lambda}_2^2},
\end{aligned}
\tag{5C.33}
$$

where

$$
\hat{I}_0=\int \boldsymbol{v}\cdot\mathbf{E}\cdot\boldsymbol{v}.
$$

By the chain rule,

$$
\begin{aligned}
\frac{\partial^2\tilde{Re}}{\partial R_1^2}=\tilde{Re}_{,33}=&\tilde{R}_{\lambda,11}\tilde{\lambda}_{1,3}^2+\tilde{R}_{\lambda,22}\tilde{\lambda}_{2,3}^2+\tilde{R}_{\lambda,33} \\
&+2\tilde{R}_{\lambda,12}\tilde{\lambda}_{1,3}\tilde{\lambda}_{2,3}+2\tilde{R}_{\lambda,13}\tilde{\lambda}_{1,3}+2\tilde{R}_{\lambda,23}\tilde{\lambda}_{2,3} \\
&+\tilde{R}_{\lambda,1}\tilde{\lambda}_{1,33}+\tilde{R}_{\lambda,2}\tilde{\lambda}_{2,33}.
\end{aligned}
\tag{5C.34}
$$

Since $\tilde{R}_{\lambda,1}=\tilde{R}_{\lambda,2}=0$ on the optimum stability boundary, (5C.33, 34) can be combined to produce the inequality

$$\hat{I}_0\, dR_1^2\,\tilde{R}e_{,33} \geqq \mathscr{D}_{12}>0,$$

where

$$\mathscr{D}_{12}=d\tilde{\lambda}_1^2\,\mathscr{D}_1(\phi,\phi)/2\tilde{\lambda}_1^2+d\tilde{\lambda}_2^2\,\mathscr{D}_2(\psi,\psi)/2\tilde{\lambda}_2^2>0.$$

Hence,

$$\hat{I}_0\,\tilde{R}e_{,33}>0. \tag{5C.35}$$

Next, combine (5C.18) and (5C.35) to obtain

$$\frac{2\mathscr{D}_1(\phi,\phi)}{R_1}\left(\frac{\partial^2\tilde{R}e}{\partial R_1^2}\bigg/\frac{\partial\tilde{R}e}{\partial R_1}\right)>0. \tag{5C.36}$$

Since $\mathscr{D}_1$, R_1 are positive, we have

$$\frac{\partial^2\tilde{R}e}{\partial R_1^2}\bigg/\frac{\partial\tilde{R}e}{\partial R_1}>0. \tag{5C.37}$$

The last inequality is then integrated over R_1 ($R_1^*<R_1$) and fixed $R_2=R_2^*$ to produce

$$\ln\left[\left|\frac{\partial\tilde{R}e}{\partial R_1}(R_1,R_2^*)\right|\bigg/\left|\frac{\partial\tilde{R}e}{\partial R_1}(R_1^*,R_2^*)\right|\right]>0,$$

and

$$\left|\frac{\partial\tilde{R}e}{\partial R_1}(R_1,R_2^*)\right|\bigg/\left|\frac{\partial\tilde{R}e}{\partial R_1}(R_1^*,R_2^*)\right|>1.$$

Since

$$\frac{\partial\tilde{R}e}{\partial R_1}(R_1^*,R_2^*)<0,$$

we have that

$$\frac{\partial\tilde{R}e}{\partial R_1}(R_1,R_2^*)<\frac{\partial\tilde{R}e}{\partial R_1}(R_1^*,R_2^*)<0, \tag{5C.38}$$

for $R_1>R_1^*$. Similarly,

$$\frac{\partial\tilde{R}e}{\partial R_2}(R_1^*,R_2)<\frac{\partial\tilde{R}e}{\partial R_2}(R_1^*,R_2^*)<0, \tag{5C.39}$$

for $R_2>R_2^*$.

We want to show that $\partial\tilde{R}e/\partial R_1$, $\partial\tilde{R}e/\partial R_2$ are negative for all R_1 and R_2 on the domain of $\tilde{R}e$. To do this, we examine (5C.18, 19). First, note that in (5C.19), $\partial\tilde{R}e\,(R_1,R_2)/\partial R_2<0$ for $R_2 \gtreqqless R_2^*$ but only for $R_1\leqq R_1^*$. This implies that $J(R_1,R_2)>0$ for $R_2\gtreqqless R_2^*$ but only for $R_1\leqq R_1^*$. Then, through (5C.18), it follows that

$$\partial\tilde{R}e/\partial R_1\,(R_1^*,R_2)<0$$

for $R_2\gtreqqless R_2^*$ and not just $R_2\leqq R_2^*$. From (5C.38), it follows that

$$\frac{\partial\tilde{R}e}{\partial R_1}(R_1,R_2)<\frac{\partial\tilde{R}e}{\partial R_1}(R_1^*,R_2)<0,$$

for $R_1 > R_1^*$. Then,

$$\frac{\partial \tilde{R}e}{\partial R_1}(R_1, R_2) < 0, \qquad \text{for all } R_1 \text{ and } R_2 \text{ on the domain of } \tilde{R}e.$$

Similarly, we prove that

$$\frac{\partial \tilde{R}e}{\partial R_2}(R_1, R_2) < 0, \qquad \text{for all } R_1 \text{ and } R_2 \text{ on the domain of } \tilde{R}e.$$

Thus, $\tilde{R}e(R_1, R_2)$ is a monotone decreasing function of R_1 and R_2. Also, by (5C.18), $\int \boldsymbol{v} \cdot \mathbf{E} \cdot \boldsymbol{v}$ is always negative. This implies through (5C.35) that

$$\frac{\partial^2 \tilde{R}e}{\partial R_1^2} < 0. \tag{5C.40}$$

Similarly, we prove that

$$\frac{\partial^2 \tilde{R}e}{\partial R_2^2} < 0. \tag{5C.41}$$

This follows from (5C.26) and the relation

$$\begin{aligned} H(U_4; U_4) = \hat{I}_0\, dR_2^2 [\tilde{R}_{\lambda,11}\, \tilde{\lambda}_{1,4}^2 + \tilde{R}_{\lambda,22}^2\, \tilde{\lambda}_{2,4}^2 + \tilde{R}_{\lambda,44} \\ + 2\tilde{R}_{\lambda,12}\, \tilde{\lambda}_{1,4}\, \tilde{\lambda}_{2,4} + 2\tilde{R}_{\lambda,14}\, \tilde{\lambda}_{1,4} + 2\tilde{R}_{\lambda,24}\, \tilde{\lambda}_{2,4}] - \mathscr{D}_{12} \\ = \hat{I}_0\, dR_2^2 \cdot \tilde{R}e_{,44} - \mathscr{D}_{12} \geqq 0. \end{aligned} \tag{5C.42}$$

I_0 is negative and $\mathscr{D}_{12}$, dR_2^2 are positive. This proves (5C.41).

To complete our proof, we note that

$$\begin{aligned} H(U_3; U_4) = \hat{I}_0\, dR_1\, dR_2 [\tilde{R}_{\lambda,11}\, \tilde{\lambda}_{1,3}\, \tilde{\lambda}_{1,4} \\ + \tilde{R}_{\lambda,22}\, \tilde{\lambda}_{2,3}\, \tilde{\lambda}_{2,4} + \tilde{R}_{\lambda,34} + \tilde{R}_{\lambda,12}(\tilde{\lambda}_{1,3}\, \tilde{\lambda}_{2,4} + \tilde{\lambda}_{1,4}\, \tilde{\lambda}_{2,3}) \\ + \tilde{R}_{\lambda,13}\, \tilde{\lambda}_{1,4} + \tilde{R}_{\lambda,14}\, \tilde{\lambda}_{1,3} + \tilde{R}_{\lambda,23}\, \tilde{\lambda}_{2,4} + \tilde{R}_{\lambda,24}\, \tilde{\lambda}_{2,3}] - \mathscr{D}_{12} \\ = \hat{I}_0\, dR_1\, dR_2\, \tilde{R}e_{,34} - \mathscr{D}_{12} \end{aligned} \tag{5C.43}$$

and then combine (5C.33, 42, and 43) to find that

$$\begin{aligned} 0 \leqq H(U_3; U_3) + H(U_4; U_4) + 2H(U_3; U_4) \\ = \hat{I}_0 [dR_1^2\, \tilde{R}e_{,33} + dR_2^2\, \tilde{R}e_{,44} + 2\, dR_1\, dR_2\, \tilde{R}e_{,34}] - 4\mathscr{D}_{12}, \end{aligned} \tag{5C.44}$$

where the first inequality follows from the convexity lemma. Thus,

$$dR_1^2\, \tilde{R}e_{,33} + dR_2^2\, \tilde{R}e_{,44} + 2\, dR_1\, dR_2\, \tilde{R}e_{,34} < 0,$$

or

$$\frac{\partial^2 \tilde{R}e}{\partial R_1^2}\, dR_1^2 + 2\frac{\partial^2 \tilde{R}e}{\partial R_1\, \partial R_2}\, dR_1\, dR_2 + \frac{\partial^2 \tilde{R}e}{\partial R_2}\, dR_2^2 < 0. \tag{5C.45}$$

Hence, $\tilde{R}e(R_1, R_2) = \tilde{R}e(\mathscr{R}, \mathscr{C})$ is convex downward. This completes the proof.

The theorem is a universal stability result of a kind. Thus, if coordinates $(\tilde{R}e(\mathscr{R}, \mathscr{C}), \mathscr{R}, \mathscr{C})$ are known at two points, say P and Q, on the optimum stability boundary, the points in the vertical plane below PQ are in the stability region. For example (Joseph [3]),

(a) $\tilde{R}e(0, 0)=\sqrt{1708}$ for plane Couette flow.

(b) $\tilde{R}e(\mathscr{R}=\sqrt{1708}, 0)=0$ for a motionless fluid heated from below. Hence, by Theorem 6, plane Couette flow heated from below is stable provided that

$$R<\sqrt{1708}-\mathscr{R}. \tag{5C.46}$$

The estimate here is not the best possible. The optimum stability boundary is not the arc of the straight line (5C.46) but, rather, an arc of the circle [3]

$$R<\sqrt{1708-\mathscr{R}^2}.$$

Similarly, points below the plane containing any three points of the stability boundary are in the stability region. We know, for example (Theorem 8, Chapter 6), that

(c) $\tilde{R}e(0,\mathscr{C}=\sqrt{1708})=0$ for a motionless fluid concentrated from above. Hence, by theorem 6 *plane Couette flow heated from below and concentrated from above is certainly stable provided that*

$$R<\sqrt{1708}-\mathscr{R}-\mathscr{C}.$$

5D. Effect of Boundary Conditions on Stability

We consider that all basic state parameters, except *Nu* and *Sh*, are fixed and determine the effect of *Nu*, *Sh* on the stability limit $\tilde{R}(Nu,Sh)$. A review of the literature on the stability of a fluid layer contained by rigid boundaries and heated from below reveals marked differences of the values of the critical Rayleigh numbers (1700 ± 51 for a linear temperature distribution). The main source of these differences seems to be the difficulty in constructing experimental apparatus approximating the theoretical boundary condition of prescribed temperature. The thermal capacity of the boundary ignored in the mathematical statement has a resistance to heat transfer and this is measured by the Nusselt number. Similar remarks tie the mass transfer to the Sherwood number. The effect of these resistance measures of boundary heat and mass transfer is the subject of Theorem 7.

It has been shown (JOSEPH & SHIR [5]) that when $Re=0$ the stability boundary $\tilde{R}h(Nu)$ is nondecreasing. This result can be extended. Consider μ_1,μ_2 as preassigned, and let

$$\begin{aligned}\tilde{R}(Nu, Sh)&=\tilde{R}_\lambda(Nu, Sh, \tilde{\lambda}_1(Nu, Sh), \tilde{\lambda}_2(Nu, Sh))\\ &=\underset{\lambda_1,\lambda_2>0}{\operatorname{Max}}\ \tilde{R}_\lambda(Nu, Sh, \lambda_1, \lambda_2)\end{aligned} \tag{5D.1}$$

define an optimum stability boundary. The best λ_1,λ_2 are found as roots of (5C.5), and with certain obvious changes Theorem 4 holds also.

Theorem 7. *Let Theorem 4 hold for fixed μ_1,μ_2, variable Nu, Sh and λ_1,λ_2 and eigenvalue $\tilde{R}_\lambda(Nu,Sh,\lambda_1,\lambda_2)$. Then*

$$\frac{\partial\tilde{R}}{\partial Nu}=\tilde{R}\oint\phi^2>0 \tag{5D.2}$$

and

$$\frac{\partial\tilde{R}}{\partial Sh}=\tilde{R}\oint\psi^2>0, \tag{5D.3}$$

so that $\tilde{R}(Nu, Sh)$ *increases tending, as* $Nu \to \infty$, $Sh \to \infty$, *to a stationary value*

$$\left.\frac{\partial \tilde{R}}{\partial Nu}\right|_{Nu\to\infty} \to 0, \qquad \left.\frac{\partial \tilde{R}}{\partial Sh}\right|_{Sh\to\infty} \to 0. \tag{5D.4}$$

Moreover,

$$\frac{\partial^2 \tilde{R}}{\partial Nu^2} < 0, \qquad \frac{\partial^2 \tilde{R}}{\partial Sh^2} < 0, \tag{5D.5}$$

$$d Nu^2 \frac{\partial^2 \tilde{R}}{\partial Nu^2} + 2\, dNu\, dSh \frac{\partial^2 \tilde{R}}{\partial Nu\, \partial Sh} + dSh^2 \frac{\partial^2 \tilde{R}}{\partial Sh^2} < 0, \tag{5D.6}$$

and $\tilde{R}(Nu, Sh)$ *is convex downward.*

Proof. By normalization we have

$$\langle u; A u\rangle = -1, \tag{5D.7}$$

and

$$\langle u; A_{,\alpha} u\rangle + \langle u; B_{,\alpha} u\rangle^* = 0. \tag{5D.8}$$

Let $A = \tilde{R}_\lambda \hat{A}$. Equations (5D.7, 8) give

$$\tilde{R}_{\lambda,\alpha} = \tilde{R}_\lambda^2 \langle u; \hat{A}_{,\alpha} u\rangle + \tilde{R}_\lambda \langle u; B_{,\alpha} u\rangle^*, \tag{5D.9}$$

where

$$\hat{A}(\lambda_1, \lambda_2)_{,\alpha} = \left[\begin{array}{c|cc} \mathbf{0} & \hat{\boldsymbol{G}}_{1,\alpha} & \hat{G}_{2,\alpha} \\ \hline \hat{\boldsymbol{G}}_{1,\alpha} & 0 & 0 \\ \hat{\boldsymbol{G}}_{2,\alpha} & 0 & 0 \end{array}\right], \qquad B_{,\alpha} = \left[\begin{array}{c|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & Nu_{,\alpha} & 0 \\ \mathbf{0} & 0 & Sh_{,\alpha} \end{array}\right]. \tag{5D.10, 11}$$

Let $y^1 = \lambda_1$, $y^2 = \lambda_2$, $y^3 = Nu$, $y^4 = Sh$. Then,

$$\hat{A}_{,3} = 0, \quad B_{,3} = \left[\begin{array}{c|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 0 \end{array}\right], \qquad \hat{A}_{,4} = 0, \quad B_{,4} = \left[\begin{array}{c|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1 \end{array}\right],$$

and

$$\tilde{R}_{\lambda,3} = \tilde{R}_\lambda \oint \phi^2,$$
$$\tilde{R}_{\lambda,4} = \tilde{R}_\lambda \oint \psi^2.$$

On the optimum stability boundary $\tilde{R}_{\lambda,1} = \tilde{R}_{\lambda,2} = 0$, so that $\tilde{R}_{,3} = \tilde{R}_{\lambda,3}$ and $\tilde{R}_{,4} = \tilde{R}_{\lambda,4}$ hold. This gives (5D.2, 3). In the limit $Nu \to \infty$, $\phi \to 0$ on $\partial\mathscr{V}$ and $Sh \to \infty$, $\psi \to 0$ on $\partial\mathscr{V}$. This proves (5D.4).

The second part of Theorem 7 is proved as follows: First, let

$$U_3 = d y^1 u_{,1} + d y^2 u_{,2} + d y^3 u_{,3},$$
$$U_4 = d y^1 u_{,1} + d y^2 u_{,2} + d y^4 u_{,4},$$

and

$$\lambda_{1,3} = \frac{\partial \lambda_1}{\partial Nu}, \qquad \lambda_{1,4} = \frac{\partial \lambda_1}{\partial Sh},$$

$$\lambda_{2,3} = \frac{\partial \lambda_2}{\partial Nu}, \qquad \lambda_{2,4} = \frac{\partial \lambda_2}{\partial Sh}.$$

Since B depends linearly on Nu and Sh, $B_{,\alpha\beta}=0$, for α, $\beta=1, 2, 3, 4$. Then (5B.28) may be evaluated as

$$H(U_3;U_3)=\sum_{\alpha,\beta=1}^{3} dy^\alpha dy^\beta \langle u; A_{,\alpha\beta} u\rangle \geqq 0, \tag{5D.12}$$

where

$$\begin{aligned}\langle u; A_{,\alpha\beta} u\rangle &= \tilde{R}_{\lambda,\alpha\beta}\langle u; \hat{A} u\rangle + \tilde{R}_{\lambda,\alpha}\langle u; \hat{A}_{,\beta} u\rangle \\ &\quad + \tilde{R}_{\lambda,\beta}\langle u; \hat{A}_{,\alpha} u\rangle + \tilde{R}_\lambda \langle u; \hat{A}_{,\alpha\beta} u\rangle .\end{aligned} \tag{5D.13}$$

In (5D.13), obviously, $\hat{A}_{,3}=\hat{A}_{,4}=0$. On the optimum stability boundary, $\tilde{R}_{\lambda,1}=\tilde{R}_{\lambda,2}=0$, and

$$\langle u; \hat{A}_{,1} u\rangle = 0, \qquad \langle u; \hat{A}_{,2} u\rangle = 0 .$$

Thus, (5D.13) can be written as

$$\langle u; A_{,\alpha\beta} u\rangle = -\frac{\tilde{R}_{\lambda,\alpha\beta}}{\tilde{R}_\lambda} + \tilde{R}_\lambda \langle u; \hat{A}_{,\alpha\beta} u\rangle . \tag{5D.14}$$

Form the derivatives $\hat{A}_{,\alpha\beta}$ from (5D.10) and calculate

(i) $$\langle u; A_{,11} u\rangle = -\frac{1}{\tilde{R}_\lambda}\left[\tilde{R}_{\lambda,11} + \tilde{R}_\lambda \frac{\mathscr{D}_1}{2\tilde{\lambda}_1^2}\right], \tag{5D.15}$$

(ii) $$\langle u; A_{,22} u\rangle = -\frac{1}{\tilde{R}_\lambda}\left[\tilde{R}_{\lambda,22} + \tilde{R}_\lambda \frac{\mathscr{D}_2}{2\tilde{\lambda}_2^2}\right], \tag{5D.16}$$

(iii) $$\langle u; A_{,33} u\rangle = -\frac{1}{\tilde{R}_\lambda}\tilde{R}_{\lambda,33}, \tag{5D.17}$$

(iv) $$\langle u; A_{,12} u\rangle = \frac{-1}{\tilde{R}_\lambda}\tilde{R}_{\lambda,12}, \tag{5D.18}$$

(v) $$\langle u; A_{,13} u\rangle = \frac{-1}{\tilde{R}_\lambda}\tilde{R}_{\lambda,13}, \tag{5D.19}$$

(vi) $$\langle u; A_{,23} u\rangle = -\frac{1}{\tilde{R}_\lambda}\tilde{R}_{\lambda,23}. \tag{5D.20}$$

Equations (5D.12–20) are combined to produce

$$\begin{aligned}H(U_3;U_3) &= -\frac{1}{\tilde{R}_\lambda} dNu^2 [\tilde{R}_{\lambda,11}\tilde{\lambda}_{1,3}^2 + \tilde{R}_{\lambda,22}\tilde{\lambda}_{2,3}^2 + \tilde{R}_{\lambda,33} \\ &\quad + 2\tilde{R}_{\lambda,12}\tilde{\lambda}_{1,3}\tilde{\lambda}_{2,3} + 2\tilde{R}_{\lambda,13}\tilde{\lambda}_{1,3} + 2\tilde{R}_{\lambda,23}\tilde{\lambda}_{2,3}] \\ &\quad - d\tilde{\lambda}_1^2 \frac{\mathscr{D}_1}{2\tilde{\lambda}_1^2} - d\tilde{\lambda}_2^2 \frac{\mathscr{D}_2}{2\tilde{\lambda}_2^2}\end{aligned} \tag{5D.21}$$

$$= -\frac{1}{\tilde{R}} dNu^2 \frac{\partial^2 \tilde{R}}{\partial Nu^2} - \mathscr{D}_{12} \geqq 0, \tag{5D.22}$$

where the last equality follows from the definition of $\tilde{R}$ and the chain rule. This implies $\frac{\partial^2 \tilde{R}}{\partial Nu^2}<0$. Similarly, we have

$$H(U_4; U_4) = -\frac{dSh^2}{\tilde{R}} \frac{\partial^2 \tilde{R}}{\partial Sh^2} - \mathscr{D}_{12} \geqq 0, \tag{5D.23}$$

which implies $\frac{\partial^2 \tilde{R}}{\partial Sh^2}<0$. In the same way

$$H(U_3; U_4) = -\frac{dNu\, dSh}{\tilde{R}} \frac{\partial^2 \tilde{R}}{\partial Nu\, \partial Sh} \geqq \mathscr{D}_{12}. \tag{5D.24}$$

Equations (5D.22, 23, 24) and the convexity lemma are combined to produce

$$\frac{\partial^2 R^2}{\partial Nu^2} dNu^2 + 2\frac{\partial^2 \tilde{R}}{\partial Nu\, \partial Sh} dNu\, dSh + \frac{\partial^2 \tilde{R}}{\partial Sh^2} dSh^2 < 0. \tag{5D.25}$$

This implies the downward convexity of $\tilde{R}(Nu, Sh)$ and completes the proof.

Sani [9] has obtained these boundary condition convexity results relative to the linearized Boussinesq equation for the motionless, constant gradient layer. He employs the transplantation method of Pólya & Schiffer [8]. This method could, we believe, also be used to produce the convexity results of Theorems 6 and 7, provided that the method is supplemented with the appropriate analogue of the convexity lemma.

6. Subcritical Thermohaline Convective Instability in Fluid Layers

In this chapter, we examine the stability of a basic motionless state to imposed linear temperature and concentration gradients.* The basic state is potentially open to subcritical instabilities when the temperature gradient is destabilizing (heated from below), and the concentration gradient is stabilizing (salted from below). For this situation, energy theory makes possible the specification of ranges of stability parameters (subcritical) for which large amplitude disturbances may grow though infinitesimal disturbances decay. Though there is, of course, no *a priori* guarantee that the local conservation equations will actually admit subcritical solutions, this is, in fact, indicated by approximate calculations of Veronis [16] and of Sani [10]. The consistency of these calculations with the rigorous energy result does suggest that the local nonlinear conservation equations do have subcritical solutions and that they cover (as the ratio of the Prandtl number to

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* This configuration models a "solar Pond". The "Pond" is a contained fluid layer which is both heated and salted below (*cf.* Tabor [14] and Weinberger [17]), so that the upper fluid layers thermally insulate the lower. Like the Dead Sea, the pond is washed by fresh water at its free surface and salted at its bottom, ensuring the existence of a stabilizing salt gradient in the vertical. The dark bottom of the pond is an effective absorber of radiative energy of the sun, which has the effect of heating the pond from below. Without the stable salt gradient, the limit of heating that could be achieved in this way is determined by the stability condition for the onset of convective motions. The stabilizing salt gradient enables significantly larger temperature differences to develop before the fluid turns over. Weinberger [17] reports that "a 25 by 25 meter experimental pond, with a blackened bottom, constructed under adverse conditions near the shores of the Dead Sea, in which a salt concentration was artificially created, reached a temperature of over 90 deg. *C* at a depth of 80 centimeters before being destroyed by leakage".

the Schmidt number takes on all positive values) the entire region deemed open by energy theory. Moreover, when the above mentioned ratio has its zero value, the energy and linear limits coincide, excluding possible subcritical solutions and showing that the energy bound is optimal. The calculations from energy and linear theory thus lead to limits sufficient for stability and instability as well as to the subcritical regions bounded by these two limits.

The energy limit for heated and salted below is not obtained within the context of the general resolution of the problem for the best λ (equations (5C.1) and (5C.2)). In this case, the hypothesis that the relevant eigenfunctions are continuously differentiable in the parameters fails. It is nonetheless quite easy to find these best values, as we demonstrate in the subsection below.

6A. Heated from Above

Consider the stability of a homogeneous motionless fluid heated from above. It is to be expected on physical grounds that this configuration is absolutely stable. A mathematically rigorous proof that this is indeed the case can be constructed directly from the energy identities (equations (3.8) and (3.9)) which in the present context may be written as

$$\frac{dK}{d\tau} = -\int \mathcal{R}\mathbf{H}\cdot \boldsymbol{v}\,\theta - 2\int \boldsymbol{d}:\boldsymbol{d},$$

$$Pr\,\frac{d\Theta}{d\tau} = -\int \mathcal{R}\mathbf{H}_1\cdot \boldsymbol{v}\,\theta - \int \nabla\theta\cdot\nabla\theta - \oint Nu\,\theta^2,$$

with $\mathbf{H} = -\mathbf{H}_1 = -\boldsymbol{e}$ where $\boldsymbol{e}$ points vertically above. By addition, one produces

$$\frac{d}{d\tau}(K + Pr\,\Theta) = -\int(2\boldsymbol{d}:\boldsymbol{d} + \nabla\theta\cdot\nabla\theta) - \oint Nu\,\theta^2 < 0.$$

One then bounds the right-hand side from above with $-(K + Pr\,\Theta)$ and integrates the resulting inequality to obtain an exponential decay of the form

$$(K + Pr\,\Theta) \leqq (K_0 + Pr\,\Theta_0)\, e^{-a^2\tau},$$

where a^2 is clearly independent of $\mathcal{R}$. It is clear that for this case one should choose $\lambda_1 = 1$. On the other hand, if one evaluates equation (5C.1) for the condition $\mathbf{H} = -\mathbf{H}_1$, there results the unacceptable value $\tilde{\lambda}_1 = -1$. Why does the theory of the best λ_1 appear to fail?

Consider the Euler-Lagrange equations for this special case:

$$\frac{\mathcal{R}_\lambda}{2\sqrt{\lambda_1}}(\lambda_1 - 1)(\boldsymbol{e}\cdot\boldsymbol{v}) = \Delta\phi, \tag{6A.1}$$

$$\frac{\mathcal{R}_\lambda}{2\sqrt{\lambda_1}}(\lambda_1 - 1)\,\phi\,\boldsymbol{e} = -\nabla p + \Delta\boldsymbol{v}, \tag{6A.2}$$

where $\mathscr{R}_\lambda = \mu_1 R_\lambda$. It is well known that the minimum positive value for the parameter

$$\frac{\mathscr{R}_\lambda}{2\sqrt{\lambda_1}}(\lambda_1 - 1)$$

is the critical value $\mathscr{R}_L$ of linear theory. It follows that

$$\tilde{\mathscr{R}}_\lambda = \min_{\phi, v} \mathscr{R}_\lambda = \frac{2\sqrt{\lambda_1}}{\lambda_1 - 1} \mathscr{R}_L$$

has a simple pole for $\lambda_1 = 1$ and is not continuously differentiable on $0 < \lambda_1 < \infty$. The problem of the best λ_1 cannot here be resolved by parameter differentiation, but, *a fortiori*, by inspection, *i.e.*,

$$\max_{\lambda_1 > 0} \tilde{\mathscr{R}}_\lambda \to \infty$$

for $\tilde{\lambda}_1 = 1$, and we obtain anew the proof of the absolute stability of the conductive state.

6B. Thermohaline Convection

In the absence of sources of heat and mass, the nonlinear Boussinesq equations admit as solutions linear variations (in the vertical) of temperature and concentration. These solutions are independent of the domain provided that the values assumed by the linear distributions are in accord with preassigned boundary values. We shall consider situations in which the temperature gradient is destabilizing (heated from below), but the concentration gradient may be stabilizing (salty below) or destabilizing (salty above). It is convenient to speak of transversally infinite fluid layers, though some of our results apply to bounded regions as well.

The temperature and concentration distribution corresponding to circumstances specified above are

$$\frac{T - T_1}{T_0 - T_1} = 1 - z, \tag{6B.1}$$

$$\frac{C - C_1}{C_0 - C_1} = 1 - z, \tag{6B.2}$$

where T_0, T_1 and C_0, C_1 are the prescribed temperature and concentration for the lower and upper surfaces, respectively, and $z = x_3/l$. Thus,

$$\mathbf{H}_1 = \xi_1 e, \quad \mathbf{H}_2 = \xi_2 e \quad \text{and} \quad \mathbf{H} = -e,$$

where ξ_1, ξ_2 are the normalized gradients of temperature and concentration and e is a unit vector in the direction of z increasing. One sets $\xi_1 = -1$ for heated from below and $\xi_2 = -1$ for $C_0 > C_1$, $\xi_2 = +1$ for $C_0 < C_1$, and $\mathbf{E} = 0$.

(a) *Euler Equations.* The appropriate Euler equations are (5A.14, 15, 16) with $s = 1$ (density increases with concentration), $R_\lambda = 0$, $\mu_1 \tilde{R}_\lambda = \tilde{\mathscr{R}}_\lambda$ (eigenvalue) and

$\mu_2 R_\lambda = \mathscr{C}$ (preassigned). Then with

$$g_1 e = \tilde{R}_\lambda \hat{G}_1 = -\tilde{\mathscr{R}}_\lambda \frac{(1+\lambda_1)}{2\sqrt{\lambda_1}} e,$$

$$g_2 e = \tilde{R}_\lambda \hat{G}_2 = \mathscr{C} \frac{(1+\xi_2 \lambda_2)}{2\sqrt{\lambda_2}} e,$$

we have

$$(g_1 \phi + g_2 \psi) e = -\nabla p + \Delta v, \tag{6B.3}$$

$$g_1 e \cdot v = \Delta \phi, \tag{6B.4}$$

$$g_2 e \cdot v = \Delta \psi \tag{6B.5}$$

for functions v, ϕ, ψ and p, which take on boundary values compatible with (3.11, 12, 13, 14 and 15).

(b) *Linear Theory and Exchange of Stability.* We wish to specify possible ranges for subcritical instabilities. For this purpose, we need the results of the linear theory as these define the critical limit. The linear problem may be obtained from (2.12–20) by setting the quadratic terms in the difference variables in equations (2.12–14) to zero. Then, under the change of variables of Chapter 3, we have

$$\sigma v = -\nabla p + e(\mathscr{R}_L \theta - \mathscr{C} c) + \Delta v, \tag{6B.6}$$

$$Pr\, \sigma \theta = \mathscr{R}_L e \cdot v + \Delta \theta, \tag{6B.7}$$

$$Sc\, \sigma c = -\mathscr{C} \xi_2 e \cdot v + \Delta c, \tag{6B.8}$$

for solenoidal v and boundary conditions (3.11–15). Here $\mathscr{R}_L$ is a principal eigenvalue, $\mathscr{C}$ is a (sufficiently small) preassigned value, $\sigma = \sigma_r + i\sigma_i$ where σ_r is a growth rate and σ_i a frequency. Equations (6B.6–8) arise from substitution of the time-separable modes $u(x, y, z, \tau) \to e^{\sigma\tau} u(x, y, z)$ (complex $u(x, y, z)$). In stability considerations, the locus of characteristic parameters determining the marginal stability limit, *i.e.*, $\sigma_r = 0$, is important. If $\sigma_r > 0$, the physical system is said to be unstable, and if $\sigma_r < 0$, the system is said to be stable with respect to infinitesimal disturbances. $\sigma_r = 0$ is the marginal state. More structure is given to these stability classifications by considering σ_i. Oscillatory or overstable disturbances are associated with states of the system for which $\sigma_i \neq 0$, and stationary disturbances are associated with states for which $\sigma_i = 0$. The critical value for the marginal state may be defined through the numbers

$$\hat{\mathscr{R}}_L(\mathscr{C}) = \mathscr{R}_L(\mathscr{C}, \sigma_i^*) = \min_{\sigma_i \in (-\infty, \infty)} \mathscr{R}_L(\mathscr{C}, \sigma_i). \tag{6B.9}$$

If the only solutions of the linear problem when $\sigma_r = 0$ are those for which $\sigma_i = 0$, the principle of exchange of stability is said to hold. Conditions under which exchange of stability holds may sometimes be inferred from the relation

$$\sigma_i = -\mathscr{C} \frac{\mathscr{I}m\{\langle c\bar{w}\rangle + \xi_2 \langle \bar{c} w\rangle\}}{\langle |v|^2\rangle + Pr\langle |\theta|^2\rangle + Sc\langle |c|^2\rangle}, \tag{6B.10}$$

which has been given by SANI [9]. Equation (6B.10) may be obtained as the imaginary part of the sum of the three relations

$$\begin{aligned} \sigma\langle|\boldsymbol{v}|^2\rangle &= \mathscr{R}_L\langle\theta\,\bar{w}\rangle - \mathscr{C}\langle c\,\bar{w}\rangle + \langle\bar{\boldsymbol{v}}\cdot\Delta\boldsymbol{v}\rangle\,,\\ Pr\,\sigma\langle|\theta|^2\rangle &= \mathscr{R}_L\langle\bar{\theta}\,w\rangle + \langle\bar{\theta}\,\Delta\,\theta\rangle\,,\\ Sc\,\sigma\langle|c|^2\rangle &= -\mathscr{C}\,\xi_2\langle\bar{c}\,w\rangle + \langle\bar{c}\,\Delta\,c\rangle\,, \end{aligned} \tag{6B.11}$$

where $\boldsymbol{v}=(u,v,w)$, $\langle\cdot\;\cdot\rangle$ is a bilinear integral over $\mathscr{V}$ and the overbar means conjugate. Equations (6B.11) are obtained from (6B.6, 7, 8) by obvious multiplications and integration. The last term of each of the equations (6B.11) is real, because Δ is self-adjoint with respect to $\langle\;\rangle$ and any of the boundary conditions (3.11, 12, 13 and 14).

(c) *Heated Below and Salty Above.* In this situation, $\xi_2=1$, so that by (6B.10), $\sigma_i=0$ and exchange of stability holds. The best values of λ_1 and λ_2 are obtained from (5C.1, 2) as $\lambda_1=1$ and $\lambda_2=1$. With $\xi_2=1$, $\sigma_i=0$, $\lambda_1=\lambda_2=1$, the Euler equations (6B.3, 4, 5) are the linear equations (6B.7, 8, 9) with $\mathscr{R}_L(\mathscr{C},0)=\hat{\mathscr{R}}_L(\mathscr{C})$. Moreover, the boundary conditions are identical. Hence,

Theorem 8. *The motionless state of a Boussinesq fluid concentrated above and heated below is (subcritically) stable and unique provided only that*

$$\mathscr{R} < \tilde{\mathscr{R}}(\mathscr{C}) = \mathscr{R}_L(\mathscr{C},0) \tag{6B.12}$$

for any preassigned $\mathscr{C}\geqq 0$.

It is easily shown (in (d) below) that when the temperature and concentration differences vanish at the boundaries of a fluid layer, the values $\mathscr{R}_L(\mathscr{C},0)$ are given by

$$\mathscr{R}_L^2+\mathscr{C}^2 = Ra^* \equiv \begin{cases} 1708 & \text{(rigid-rigid surface)}\\ \;657 & \text{(free-free surface)}\\ 1100 & \text{(free-rigid surface)}\,. \end{cases} \tag{6B.13}$$

(d) *Heated and Salty Below.* The energy problem here, as in 6A, is singular, and the blind consequence of (5C.2) is the unacceptable value $\lambda_2=-1$. As in 6A, it is, nonetheless easy to find the best values of λ_1 and λ_2. Consider the energy problem (6B.3–5) in the layer geometry with the boundary conditions

$$u=v=w=\phi=\psi=0|_{z=0,1}\,, \tag{6B.14}$$

and, as an implication of div $\boldsymbol{v}=0$,

$$\frac{dw}{dz}\equiv D\,w=0|_{z=0 \text{ or } z=1} \qquad \text{(rigid-surface)} \tag{6B.15a}$$

or

$$D^2\,w=0|_{z=0 \text{ or } z=1} \qquad \text{(free-surface)}\,. \tag{6B.15b}$$

Form the curl of the curl of (6B.6) to find that

$$\Delta_{xy}(\varphi_1\,\phi+\varphi_2\,\psi)=\Delta^2\,w\,, \tag{6B.16}$$

where

$$\Delta_{xy}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

is tangential at the boundaries. The 6th order problem,

$$(\mathcal{G}_1^2+\mathcal{G}_2^2)\Delta_{xy}w=\Delta^3 w, \tag{6B.17}$$

where

$$w=\Delta^2 w=0 \tag{6B.15}$$

holds at $z=0, 1$, is readily derivable from (6B.16), (6B.4), (6B.5) and the boundary condition (6B.14).

It is known (CHANDRASEKHAR [1]) that the smallest eigenvalue of the problem (6B.17) is given by

$$\mathcal{G}_1^2+\mathcal{G}_2^2=Ra^*; \tag{6B.18}$$

that is

$$\tilde{\mathcal{R}}_\lambda^2\,\frac{(1+\lambda_1)^2}{4\lambda_1}+\mathscr{C}^2\,\frac{(1-\lambda_2)^2}{4\lambda_2}=Ra^*.$$

The largest of the values $\mathcal{R}_\lambda^2(\lambda_1,\lambda_2,\mathscr{C})=\mathcal{R}^2(\mathscr{C})$ is clearly that for which $\lambda_1=\lambda_2=1$ and $\tilde{\mathcal{R}}^2(\mathscr{C})=Ra^*$. We have just proved

Theorem 9. *The motionless state of a Boussinesq fluid layer (satisfying* (6B.15)) *heated and concentrated below is stable and unique provided only that*

$$\mathcal{R}<\tilde{\mathcal{R}}(\mathscr{C})=\sqrt{Ra^*} \tag{6B.19}$$

independent of $\mathscr{C}\geqq 0$.

To obtain the $(\mathcal{R},\mathscr{C})$ regions open to subcritical instabilities, one needs the critical values $\tilde{\mathcal{R}}_L(\mathscr{C})$ of linear theory (6B.9). These values are not, with $\xi_2=-1$, necessarily associated with $\sigma_i=0$, because (6B.10) does not imply exchange of stability. Marginal steady solutions ($\sigma=0$) are possible when exchange of stability does not hold, but the eigenvalues $\mathcal{R}_L(\mathscr{C},0)$ need not satisfy (6B.9). The values $\mathcal{R}_L(\mathscr{C},0)$ are, in fact, given by

$$\mathcal{R}_L^2-\mathscr{C}^2=Ra^*, \tag{6B.20}$$

as may be readily verified by constructing the argument leading from (6B.3–5) to (6B.18) to the linear equations (6B.6–8) when $\sigma=0$. For a given value $\mathscr{C}$, equations (6B.19) and (6B.20) give the interval

$$Ra^*\leqq\mathcal{R}^2<Ra^*+\mathscr{C}^2 \tag{6B.21}$$

as a candidate for subcritical instabilities. The best of such intervals,

$$Ra^*\leqq\mathcal{R}^2<\hat{\mathcal{R}}_L^2(\mathscr{C}), \tag{6B.22}$$

follows from the solution of (6B.9).

For free-free surfaces, it has been shown by SANI [9] and by VERONIS [16] that

$$\lim_{Sc\to\infty}\hat{\mathcal{R}}_L^2(\mathscr{C})\to Ra^* \tag{6B.23}$$

independent of $\mathscr{C}$. In (e) below, we shall examine some of the free-surface results, for with these boundary conditions, the linear problem is easily resolved, and one can obtain the explicit values for the numbers $\mathcal{R}_L(\mathscr{C})$.

That the result (6B.23) holds generally is very strongly suggested by the following argument: One can easily prove that *for any solution of* (6B.6–8) *and* (3.11–15), *we have*

$$|\sigma_i| \leqq \frac{\mathscr{C}}{\sqrt{Sc}}. \tag{6B.24}$$

To prove (6B.24), we note that when $\xi_2 = -1$, equation (6B.10) implies that

$$|\sigma_i| \leqq \frac{2\mathscr{C}\langle|c|\,|w|\rangle}{\langle|w|^2\rangle + Sc\langle|c|^2\rangle}. \tag{6B.25}$$

The inequality $2\sqrt{Sc}\langle|c|\,|w|\rangle \leqq \langle|w|^2\rangle + Sc\langle|c|^2\rangle$ together with (6B.25) implies (6B.24).

Suppose exchange of stability does not hold. Then there exists

$$\sigma_i = \frac{\alpha^*\mathscr{C}}{\sqrt{Sc}}, \qquad -1 \leqq \alpha^* \leqq 1,\ \alpha^* \neq 0,$$

and with $\sigma_r = 0$, (6B.6–8) may be rewritten as

$$\Delta \boldsymbol{v} + \mathscr{R}_L \theta \boldsymbol{e} - \nabla p = \mathscr{C} c \boldsymbol{e} + \frac{\alpha^* \mathscr{C} \boldsymbol{v}}{\sqrt{Sc}},$$

$$\Delta\theta + \mathscr{R}_L \boldsymbol{e}\cdot\boldsymbol{v} = \frac{\alpha^*\mathscr{C}\,Pr}{\sqrt{Sc}}\,\theta,$$

and

$$\alpha^*\mathscr{C} c = \frac{1}{\sqrt{Sc}}\{\mathscr{C}\boldsymbol{e}\cdot\boldsymbol{v} + \Delta c\}.$$

It is apparent that if, in the limit $Sc \to \infty$, there exists a regular solution for which $\alpha^*\sqrt{Sc} \to \infty$, then

$$c \to 0,$$
$$\Delta\boldsymbol{v} + \mathscr{R}_L\theta\boldsymbol{e} - \nabla p \to 0,$$
$$\Delta\theta + \mathscr{R}_L\boldsymbol{e}\cdot\boldsymbol{v} \to 0,$$

and

$$\lim_{Sc\to\infty} \hat{\mathscr{R}}_L(\mathscr{C}) \to \mathscr{R}_L(0,0), \tag{6B.26}$$

for any domain and any of the boundary conditions (3.11–15). Of course, relative to conditions (6B.14, 15), $\mathscr{R}_L^2(0,0) = Ra^*$, the range (6B.22) open to subcritical instabilities collapses, and we once again have the strongest result.

For free surfaces on which temperature and concentration are prescribed, complete explicit results relative to the linear problem are known ([9] and [16]). Formal calculations for the nonlinear problem leading to subcritical solutions in the region deemed open by energy theory have also been given by Sani [10] and by Veronis [16]. The comparison of these results with those of energy theory complete this investigation.

(e) *Heated and Salty Below-Free Surfaces.* The linear problem (6B.6−8) (with $\sigma_r=0$) relative to the boundary conditions (6B.14) and (6B.15b) has a solution in elementary functions. The set (6B.6−8) can be reduced readily to a set of ordinary differential equations in the Fourier transforms of the functions. These equations involve only even derivatives and, by virtue of the boundary conditions, may be solved by the series

$$\sum A_N \sin N\pi z\,.$$

Substitution of this series into the transformed equations gives an easy set of algebraic equations from which the coefficient A_N may be eliminated. This leads to the dispersion relation $F(\sigma_i,\mathcal{R},\mathcal{C},k^2)=0$, which is solved for preassigned $\mathcal{C}$ to obtain the smallest $\mathcal{R}=\hat{\mathcal{R}}_L(\mathcal{C})$ over σ_i and the wave number k (see [9] or [16] for details).

The relevant result may be expressed as

$$\hat{\mathcal{R}}_L^2(\mathcal{C})=\begin{cases}\mathcal{R}_L^2(\mathcal{C},0)=Ra^*+\mathcal{C}^2 & (\gamma>1),\\ \mathcal{R}_L^2(\mathcal{C},0)=Ra^*+\mathcal{C}^2 & \left(\gamma<1,\ \dfrac{\mathcal{C}}{\mathcal{C}^*}<1\right)\\ Ra^*(1+Sc^{-1})(1+\gamma)+\dfrac{1+Sc}{1+\gamma Sc}\,\gamma^2\mathcal{C}^2 & \left(\gamma<1,\ \dfrac{\mathcal{C}}{\mathcal{C}^*}>1\right),\end{cases} \tag{6B.27}$$

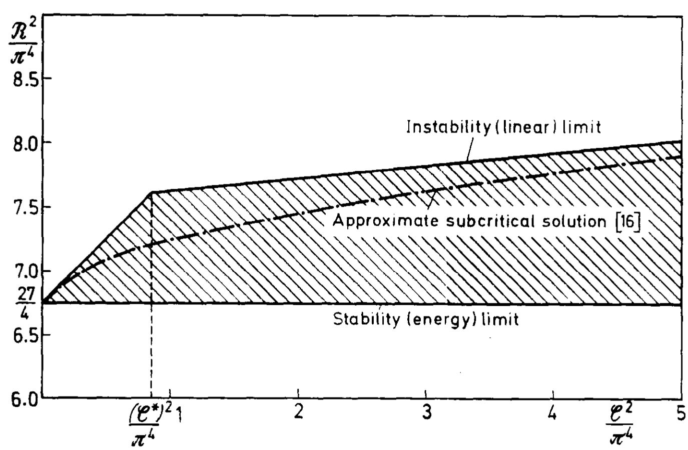

Fig. 2. Stability and instability limits for a free layer heated and concentrated (salty) below. The energy limit (Theorem 9) is given as

$$\tilde{\mathcal{R}}^2(\mathcal{C})=\frac{27}{4}\,\pi^4$$

and is independent of $\mathcal{C}$. The linear limit defines the upper boundary of the subcritical shaded region and is the graph of (6B.28) for $\gamma=0.1$ and $(\mathcal{C}^*)^2=.86\,\pi^4$. When $\mathcal{C}<\mathcal{C}^*$, the linear limit occurs as a stationary solution with

$$\hat{\mathcal{R}}_L^2(\mathcal{C})=\frac{27}{4}\,\pi^4+\mathcal{C}^2$$

independent of Pr and Sc. The value $\gamma=0.1$ suffices to calculate the tentative finite amplitude subcritical solution (6B.32) given by VERONIS [16].

where

$$Ra^* = \frac{27}{4}\pi^4 = 657, \quad \gamma = \frac{Pr}{Sc} \quad \text{and} \quad \mathscr{C}^* = \sqrt{\frac{Ra^*}{Sc}\frac{(1+\gamma Sc)}{(1-\gamma)}}.$$

Possible subcritical solutions then must necessarily lie on the interval (6B.22) with $Ra^* = 657$ and $\hat{\mathscr{R}}_L^2(\mathscr{C})$ given by (6B.27). These subcritical regions are shaded in Figures 2, 3 and 4. In Figure 2, the slope of the linear limit $\hat{\mathscr{R}}_L^2(\mathscr{C})$ is discontinuous

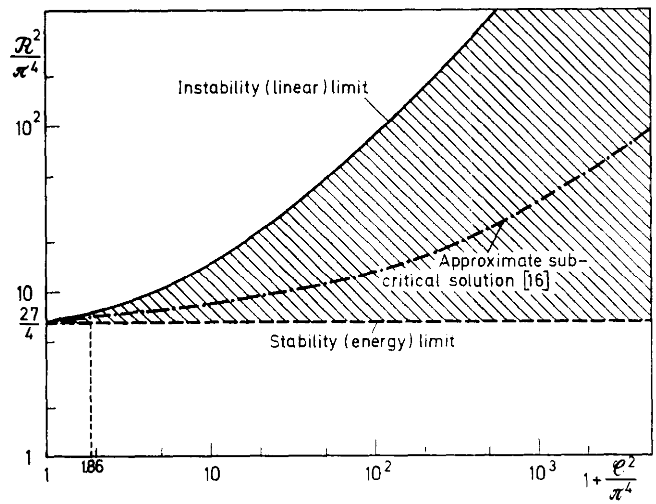

Fig. 3. Stability and instability limits. This graph is the extension of Figure 2 for larger values of $\mathscr{C}$.

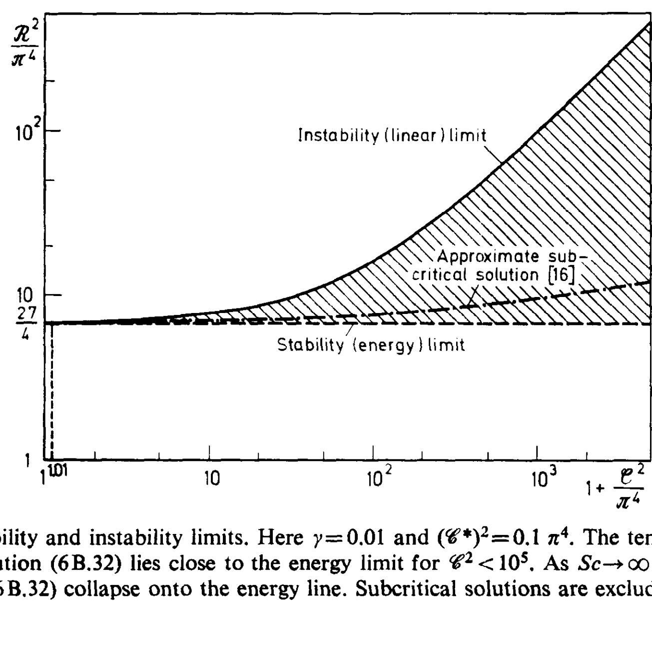

Fig. 4. Stability and instability limits. Here $\gamma = 0.01$ and $(\mathscr{C}^*)^2 = 0.1\,\pi^4$. The tentative finite amplitude solution (6B.32) lies close to the energy limit for $\mathscr{C}^2 < 10^5$. As $Sc \to \infty$, both the linear limit and (6B.32) collapse onto the energy line. Subcritical solutions are excluded in this limit.

at the point $\mathscr{C}^*$. In the limit $Sc \to \infty$, $\gamma \to 0$, $\mathscr{C}^* \to 0$ and $\hat{\mathscr{R}}_L^2(\mathscr{C}) \to Ra^*$. If subcritical instabilities do exist, they are to be found in the range specified. The existence problem itself cannot be treated by deductions from energy identities. Such questions must necessarily start from local conservation equations, and, unfortunately, no rigorous results are presently available for this very difficult problem. There is available, however, an approximate solution (VERONIS [16]) of the nonlinear Boussinesq equations for the problem subject of this subsection.

VERONIS treated two-dimensional finite amplitude disturbances approximated by a so called "minimal representation". The governing equations are

$$\begin{gathered}\left(\frac{1}{Pr}\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2\psi=-\mathscr{R}^2\frac{\partial\theta}{\partial x}+\frac{1}{\gamma}\mathscr{C}^2\frac{\partial c}{\partial x}+\frac{1}{Pr}\mathscr{J}(\psi,\nabla^2\psi),\\ \left(\frac{\partial}{\partial t}-\nabla^2\right)\theta+\frac{\partial\psi}{\partial x}=\mathscr{J}(\psi,\theta),\\ \left(\frac{\partial}{\partial t}-\gamma\nabla^2\right)c+\frac{\partial\psi}{\partial x}=\mathscr{J}(\psi,c),\end{gathered}\tag{6B.28}$$

with boundary conditions

$$\psi=\frac{d^2\psi}{dz^2}=\theta=c=0, \quad \text{at} \quad z=0,1, \tag{6B.29}$$

where ψ is a stream function and

$$\mathscr{J}(f,g)=\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}.$$

From the physics of the problem, once the convection has set in, the mean temperature and concentration field must be distorted by the convective motions. Hence, a simple representation, which takes account of the finite amplitude motion plus the distortion of the temperature and concentration field, is the following:

$$\begin{aligned}\psi&=a_1(t)\sin k_x\pi x\sin\pi z,\\ \theta&=a_3(t)\cos k_x\pi x\sin\pi z+a_2(t)\sin 2\pi z,\\ c&=a_5(t)\cos k_x\pi x\sin\pi z+a_4(t)\sin 2\pi z,\end{aligned}\tag{6B.30}$$

where $\sin 2\pi z$ represents the mean field distortions.

A system of nonlinear first order differential equations for the $a_i(t)$ are obtained by substitution of (6B.30) into (6B.28). The equations for $a_i(t)$ follow from identifying coefficients of independent trigonometric functions. Terms proportional to $\sin 3\pi z$ in the last two of equations (6B.28) are merely ignored, so that one has no criterion by which inherent errors may be assessed.

A closed system of five ordinary differential equations results from this procedure. It is then easily verified that the system has a stationary equilibrium solution *

* SANI [10] solved the finite amplitude problem for roll cell disturbances by the approximate method of STUART & WATSON. In addition to the steady subcritical instability, he found oscillatory subcritical instabilities.

given by

$$\mathscr{R}^2 = \left[\mathscr{C}\gamma + \sqrt{(1-\gamma^2)\frac{27}{4}\pi^4}\right]^2. \tag{6B.31}$$

This solution clearly neighbors the energy limit

$$\frac{27}{4}\pi^4$$

for small values of γ and moderate values for $\mathscr{C}$. In Figures 2, 3, 4 we have plotted stability limits for several values of γ. For real $\mathscr{R}$, this stationary solution (6B.31) covers, as γ ranges from 0 to 1, the entire region bounded by the energy limit, on the one hand, and the linear limit, on the other. This feature of (6B.31) does suggest that the energy bound which does not depend on γ, is attained, and indicates the sense in which this bound is optimal. The comparison, unfortunately, is tentative because (6B.31) is an approximate result, and no proof of existence of subcritical solutions is presently available.

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