Reprinted from
THE PHYSICS OF FLUIDS
Volume 11, Number 4, April 1968

Subcritical Instability and Exchange of Stability in a Horizontal Fluid Layer

D. D. JOSEPH, R. J. GOLDSTEIN, AND D. J. GRAHAM
Institute of Technology
University of Minnesota, Minneapolis, Minnesota
(Received 5 October 1967; final manuscript received 15 January 1968)

Rayleigh numbers calculated from linear and energy theories do not coincide when internal heat sources are present. For free boundaries exchange of stability applies, but energy theory nonetheless deems possible the existence of subcritical instabilities.

The stability of an initially quiescent fluid layer which is heated below and by uniform heat sources is considered. The problem, in one of it's many forms, has been treated by Sparrow, Goldstein, and Jonsson,[1] Joseph and Shir[2] and Krishnamurti[3]. This note adds a calculation of the linear and energy stability limits for free-surface boundary conditions. Special interest accrues to the free-surface case for to it a principle of exchange of stability applies. For the Bénard problem (no heat sources) two special circumstances coincide: (1) The principle of exchange of stability holds. (2) Energy theory excludes subcritical instabilities. The central point of this note is that this coincidence is fortuitous for there exists a situation (heat sources) for which exchange of stability holds but for which the stability and instability limits do not coincide. That subcritical instabilities do actually occur in this range deemed open by energy theory is suggested by the calculations and experiments of Krishnamurti.[3]

Consider the stability of an initially quiescent horizontal fluid layer which supports a temperature gradient in the vertical. The gradient is induced by a prescribed temperature difference ΔT (heated below) across the layer plus a uniform distribution of heat sources of intensity S in the fluid. The steady temperature of the quiescent state is given by

$$T - T_m = -\frac{S}{2k}\left(z^2 - \frac{L^2}{4}\right) - \frac{\Delta T}{L} z, \qquad (1)$$

where T_m is the arithmetic mean of the boundary temperatures, k is the thermal conductivity and L is the distance between plates. It is also possible to regard (1) as the solution of the conduction equation for a fixed value ΔT and a mean temperature $T_m = S\kappa t/k$ which increases linearly in time t(κ is the thermal diffusivity).[3] For this situation a solution to the conduction equation, $\partial T/\partial t = \kappa\partial^2 T/\partial z^2$, can be separated into $T = T_m + T_s(z)$. Integration of the conduction equation for $T_s(z)$ then gives (1) written for $T \to T_s$, $T_m \to T_{ms} = 0$ and $\Delta T_s = \Delta T$. It then follows that $T(t, z)$ is also represented by (1).

The stability of the conduction solution can be treated within the framework of the Boussinesq equations

$$P\Delta \mathbf{u} + P\mathbf{i}\theta - \nabla p = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}, \qquad (2)$$

$$\Delta\theta + R(\mathbf{i}\cdot\mathbf{u})(1 - N_s + 2N_s z) = \frac{\partial\theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta, \qquad (3)$$

$$\nabla\cdot\mathbf{u} = 0. \qquad (4)$$

Here $\mathbf{u}$ is the velocity, θ is the difference between the temperature of the altered state and the conduction state, $\mathbf{i}$ is a unit vector antiparallel to gravity, P is the Prandtl number, $N_s = sL^2/[2\kappa(T_1 - T_2)]$ is a heat source parameter and $R = \alpha\beta d^4 g/\nu\kappa$ is a Rayleigh number as conventionally designated.[2] Boundary conditions are set as

$$\theta = \mathbf{u}\cdot\mathbf{i} = \frac{\partial^2(\mathbf{i}\cdot\mathbf{u})}{\partial z^2} = 0 \quad \text{at} \quad z = 0, 1. \qquad (5)$$

The boundary-value problem (2), (3), (4) and (5) governs the difference between an altered motion and the conduction solution. There are several ways to make deductions about this postulated motion:

(a) Linearize the equations. The linear equations give sufficient conditions (a critical Rayleigh number, $R_c[N_s]$) for instability, but are silent on questions of stability. For free surfaces the linearized equations allow only stationary neutral solutions[3] (i.e., exchange of stability holds).

(b) Form the energy functional and associated Euler equations for the system. The Euler equations are

$$\tfrac{1}{2}R_\lambda \left\{\lambda \frac{[N_s(1 - 2z) - 1]}{(N_s + 1)} - 1\right\}(\mathbf{i}\cdot\mathbf{u}) = \lambda\nabla^2\theta \qquad (6)$$

and

$$\tfrac{1}{2}R_\lambda \left\{\lambda \frac{[N_s(1 - 2z) - 1]}{(N_s + 1)} - 1\right\}\mathbf{i}\theta = -\nabla p + \nabla^2\mathbf{u} \qquad (7)$$

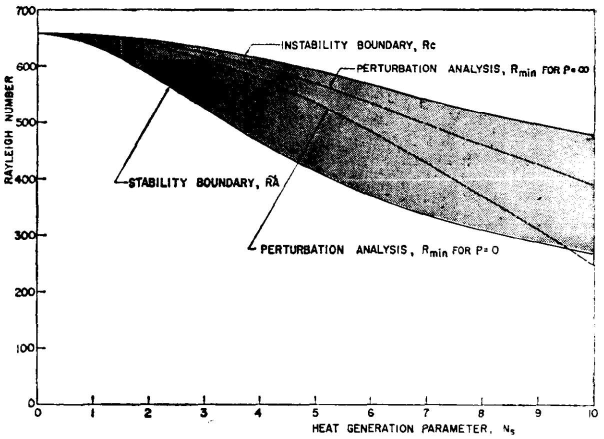

FIG. 1. Regions of stability and instability for a fluid layer with free surfaces heated from below and internally. The linear analysis gives values of the Rayleigh number $[R_c(N_s)]$ above which infinitesimally small disturbances will be amplified. The stability boundary, from the energy theory, gives Rayleigh numbers $[\widetilde{RA}(N_s)]$ below which even large disturbances will not be amplified. In the shaded region subcritical solutions of the Boussinesq equations cannot be excluded. The perturbation analysis[3] seems to indicate the existence of at least one family of such subcritical solutions.

subject to (4) and (5).[2,5] Stability is guaranteed when the square root of the Rayleigh number is less than the smallest positive eigenvalue of (6) and (7) for fixed N_s and any $\lambda > 0$. The largest of these smallest values $R_\lambda(\lambda, N_s)$ is the energy limit

$$\tilde{R}(N_s) = \max_{\lambda > 0} R_\lambda .$$

The quiescent state is stable to disturbances of any magnitude provided that

$$R < [\tilde{R}(N_s)]^2 \equiv \tilde{R}A .$$

The value

$$\lambda = -\int_0^1 dz\,(\mathbf{i}\cdot\mathbf{u})\theta \cdot \left[\int_0^1 dz \left(\frac{2N_s(1-z)}{N_s+1} - 1\right)(\mathbf{i}\cdot\mathbf{u})\theta\right]^{-1}$$

which makes R_λ maximum evidently reduces to unity for $N_s = 0$. For this case energy and linear theory coincide, exchange of stability also applies, and the conduction solution is subcritically stable. It is clear that the simultaneous applicability of the nonexistence of subcritical instabilities and the principle of exchange of stability when $N_s = 0$ is a coincidence peculiar to that limit. The energy equations give limits sufficient for stability but are silent on questions of instability.

(c) Krishnamurti[3] has given a perturbation analysis for this problem. Using a double perturbation series in the amplitude of the motion and N_s, detailed finite amplitude results are obtained. Subcritical flow is present only for hexagonal plan forms and this flow is stable. Subcritical motion can persist for values of the Rayleigh number $R_{\min} \leq R < R_c$ where

$$R_{\min} - R_c = -\frac{4N_s^2}{9\pi^2} \cdot \frac{\{5.92^2 + 2(5.92)(1.1838)/P + (1.1838)^2/P^2\}}{[6.89 + (0.0457)/P + (0.0709)/P^2]}$$

The nature of the perturbation series as well as the possible existence of other subcritical solutions have yet to be established. All subcritical solutions, of whatever form, must be within the shaded region of Fig. 1.

The integration of the linear and energy equations was carried out using a standard Runge-Kutta procedure.[6,7]

The work of D. D. Joseph was partially supported under the National Science Foundation Grant GK 1838, while the activities of R. J. Goldstein and D. J. Graham were partially supported by National Aeronautics and Space Administration Grant NGR 24-005-091 and NSF Grant GK 1737.

---

[1] E. Sparrow, R. J. Goldstein, and V. Jonsson, J. Fluid Mech. 18, 513 (1964).
[2] D. Joseph and C.C. Shir, J. Fluid Mech. 26, 753 (1966).
[3] R. Krishnamurti, Ph.D. thesis, University of California at Los Angeles (1967).
[4] G. Veronis, Astrophys. J. 137, 641 (1963).
[5] D. Joseph, Arch. Ratl. Mech. Anal. 22, 163 (1966).
[6] D. Harris and W. Reid, J. Fluid Mech. 20, 95 (1964).
[7] E. Sparrow, J. Appl. Math. and Phys. 15, 638 (1964).