

## Eigenvalue bounds for the Orr–Sommerfeld equation

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Estimates of the eigenvalues  $C$  belonging to the manifold of solutions of the Orr–Sommerfeld equation are constructed by application of elementary isoperimetric inequalities. The inequalities also lead to a considerable improvement on the estimate of  $(\alpha R)$  regions of linear stability given by Synge.

As is well known (e.g. see Synge (1938) or Lin (1955, pp. 31–2)) the real and imaginary parts of the eigenvalue

$$C = c_r + ic_i,$$

of the Orr–Sommerfeld problem

$$(U - C)(\phi'' - \alpha^2\phi) - U''\phi = -\frac{i}{\alpha R}(\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi), \quad (1a)$$

$$\phi(0) = \phi(1) = \phi'(0) = \phi'(1) = 0, \quad (1b)$$

necessarily satisfy

$$c_i = \{Q - \bar{Q} - (\alpha R)^{-1}(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)\} / (I_1^2 + \alpha^2 I_0^2), \quad (2)$$

and

$$c_r = \left\{ \int_0^1 [U|\phi'|^2 + (\alpha^2 U + \frac{1}{2}U'')|\phi|^2] dy \right\} / (I_1^2 + \alpha^2 I_0^2), \quad (3)$$

where

$$I_2^2 = \int_0^1 |\phi''|^2 dy, \quad I_1^2 = \int_0^1 |\phi'|^2 dy,$$

$$I_0^2 = \int_0^1 |\phi|^2 dy, \quad Q = \frac{i}{2} \int_0^1 U' \phi \bar{\phi}' dy,$$

and  $\alpha$ ,  $R$  are real non-negative parameters. Consider functions  $U(y)$  with two continuous derivatives and deduce results from (2) and (3) for elements  $\phi$  of a complex-valued Hilbert space,  $\bar{H}$ , completed under the norm  $I_2^2$  by the addition of limit points of sequences of four times continuously differentiable functions satisfying (1b).

The first result is obtained from the estimate

$$\begin{aligned} c_i &\leq \frac{q I_0 I_1 - (\alpha R)^{-1}(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)}{I_2^2 + \alpha^2 I_0^2}, \quad q = U'_{\max} \\ &= \max_{y \in [0, 1]} |U'(y)|, \end{aligned} \quad (4a, b)$$

which has been given by Synge (1938) and follows from Schwarz's inequality in an obvious way.

**THEOREM 1.** *Let  $C(\alpha, R)$  be any eigenvalue of (1a, b). Then*

$$c_i \leq \frac{q}{2\alpha} - \frac{\pi^2 + \alpha^2}{\alpha R}. \quad (5)$$

Moreover, no amplified disturbances ( $c_i > 0$ ) of (1a, b) exist if

$$\alpha R q < f(\alpha) \equiv \max [M_1, M_2], \quad (6)$$

where

$$M_1 = (4.73)^2 \pi + 2^{\frac{3}{2}} \alpha^3, \quad (6a)$$

and

$$M_2 = (4.73)^2 \pi + 2\alpha^2 \pi. \quad (6b)$$

In the limit  $\alpha R \rightarrow \infty$  the estimate (5) reduces to a known inviscid result of Høiland (1953). It is clear that the inviscid estimate always dominates the amplification rate in the real fluid. Of course, for finite  $\alpha R$ , (5) is a real improvement over the inviscid estimate.

The estimate (6) greatly improves the result

$$\alpha R q < g(\alpha) \equiv \max [A_1, A_2, A_3], \quad (7)$$

$$A_1 = 2^{\frac{3}{2}} \alpha^2 (\alpha^2 + 1)^{\frac{1}{2}}, \quad (7a)$$

$$A_2 = (2\alpha^2 + 1)^{\frac{1}{2}} (4\alpha^4 + 1)^{\frac{1}{2}}, \quad (7b)$$

$$A_3 = 2^{\frac{3}{2}} \alpha (1 - \alpha^2 + \alpha^3 + \alpha^4)^{\frac{1}{2}}, \quad (7c)$$

which has been given by Synge (1938). For example, from (6) we calculate

$$45.6 \cong \min_{\alpha} \frac{f(\alpha)}{\alpha}, \quad \text{whereas (7) gives } 2.74 \cong \min_{\alpha} \frac{g(\alpha)}{\alpha}$$

(see figure 1). The estimate (6) (and (7)) gives sufficient conditions for stability to small-amplitude disturbances governed by (1a, b). The application of Squires' theorem, which leads to (1a, b), restricts deductions from (2) to two-dimensional disturbances periodic in the stream direction. In the equivalent non-linear problem it is the cross-stream disturbances which give the energy functional its minimum value (Joseph 1966).

I should like to make one final remark relative to (6) (or (7)) before proving the theorem. The estimates (6) and (7) share with true neutral curves the feature that upper and lower branches of these curves bend to the right. For each fixed value of  $R$  there exists a value  $\alpha_{\min}$  below which there is linear stability and a value  $\alpha_{\max}$  above which there is linear stability. Above  $\alpha_{\max}$  the disturbance vorticity is so great that dissipation overcomes production of energy. On the other hand (6) and (7) also indicate that for every finite  $\alpha$  there exists an  $R(\alpha)$  above which linear stability cannot be deduced. This feature is not shared by true neutral curves. These exhibit a cut-off  $\alpha = \alpha_c$  above which there is always linear stability, independent of  $R$ . In asymptotic theory neutral curves are formed from a truncated version of (1a):

$$(U - c_r)(\phi'' - \alpha^2 \phi) - U'' \phi = -(i/\alpha R) \phi^{iv}$$

and (1b). The bending to the right of the upper branch of (6) (or (7)) is clearly a consequence of the dissipation integrals

$$2\alpha^2 I_1^2 + \alpha^4 I_0^2.$$

These are precisely the terms which are dropped in the asymptotics. It follows that dissipation of disturbance vorticity cannot explain the cut-off  $\alpha$  and that this qualitative behaviour of the neutral curve, on the one hand, and (6) or (7), on the other, have nothing in common.

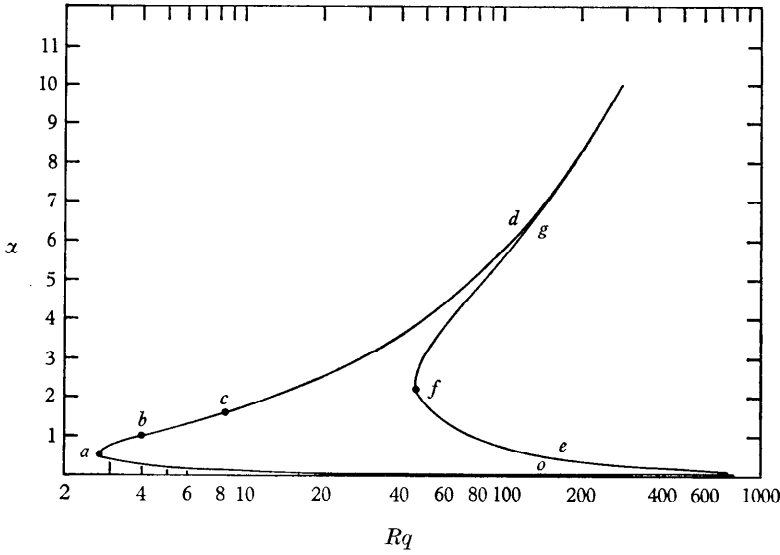


FIGURE 1. Linear stability bounds for the Orr–Sommerfeld equation. The region of certain linear stability lies to the left of the curve  $eg$  (equations (6)):  $ef$  is the graph of  $M_2(\alpha)/\alpha$ ,  $fg$  is the graph of  $M_1(\alpha)/\alpha$ . The curve  $od$  is the graph of Synge’s bounds (equations (7)):  $oa$  is the graph of  $A_2(\alpha)/\alpha$ ;  $ab$  is the graph of  $A_3(\alpha)/\alpha$ ;  $bc$  is the graph of  $A_1(\alpha)/\alpha$ ;  $cd$  is again the graph of  $A_3(\alpha)/\alpha$ .

*Proof of Theorem 1*

In the real-valued Hilbert space  $H$  corresponding to  $\bar{H}$  the following inequalities hold:

$$I_1^2 \geq \pi^2 I_0^2, \tag{8a}$$

$$I_2^2 \geq \pi^2 I_1^2, \tag{8b}$$

$$I_2^2 \geq (4.73)^4 I_0^2. \tag{8c}$$

The value  $(4.73)^4$  is the smallest eigenvalue of a vibrating rod with displacement  $\phi$  satisfying (1b) (Rayleigh 1878). These inequalities also hold in  $\bar{H}$ . Consider

$$\phi = a + ib,$$

where  $a, b \in H$ . Then  $I_1^2(a) \geq \pi^2 I_0^2(a)$ ,  $I_1^2(b) \geq \pi^2 I_0^2(b)$  (9)

and by addition  $I_1^2(\phi) \geq \pi^2 I_0^2(\phi)$ .

Equations (8b) and (8c) follow in the same way.

I next use the estimates (8) to establish the necessary preliminary inequalities.

$$2\alpha I_0 I_1 \leq I_1^2 + \alpha^2 I_0^2, \tag{10a}$$

$$I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 = (I_2^2 + \alpha^2 I_1^2) + \alpha^2 (I_1^2 + \alpha^2 I_0^2) \geq (\pi^2 + \alpha^2) (I_1^2 + \alpha^2 I_0^2), \tag{10b}$$

$$\frac{I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2}{I_0 I_1} \geq \left\{ \begin{array}{l} (4.73)^2 \pi + \frac{2\alpha^2}{I_1 I_0} \left[ \left( I_1 - \frac{\alpha}{\sqrt{2}} I_0 \right)^2 + \frac{2\alpha}{\sqrt{2}} I_0 I_1 \right] \\ (4.73)^2 \pi + \frac{2\alpha^2 I_1^2}{I_1 I_0} \end{array} \right\} \geq \left\{ \begin{array}{l} M_1 \\ M_2 \end{array} \right\}. \quad (10c)$$

The estimate (5) follows easily from (4), (10a) and (10b). To obtain (6) consider the set of values  $\{\alpha R\}$  for which the left-hand side of (4) is negative. Then apply the estimate (10c).

I turn next to estimates of the wave speed  $c_r$ . The result here is as follows.

**THEOREM 2.** *Let  $C(\alpha, R)$  be any eigenvalue of (1a, b). Then the following inequalities hold:*

$$(a) \left. \begin{array}{l} U''_{\min} \geq 0, \\ U_{\min} < c_r < U_{\max} + \frac{U''_{\max}}{2(\pi^2 + \alpha^2)}. \end{array} \right\} \quad (11a)$$

$$(b) \left. \begin{array}{l} U''_{\min} \leq 0 \leq U''_{\max}, \\ U_{\min} + \frac{U''_{\min}}{2(\pi^2 + \alpha^2)} < c_r < U_{\max} + \frac{U''_{\max}}{2(\pi^2 + \alpha^2)}. \end{array} \right\} \quad (11b)$$

$$(c) \left. \begin{array}{l} U''_{\max} \leq 0, \\ U_{\min} + \frac{U''_{\min}}{2(\pi^2 + \alpha^2)} < c_r < U_{\max}. \end{array} \right\} \quad (11c)$$

Here  $U_{\max}$ ,  $U''_{\max}$ ,  $U_{\min}$  and  $U''_{\min}$  are maximum and minimum values on the range of  $U(y)$  and  $U''(y)$  for  $y \in [0, 1]$ .

The estimates (11) considerably improve Pai's (1954) extension,

$$\frac{U''_{\min}}{2\alpha^2} + U_{\min} < c_r < \frac{U''_{\max}}{2\alpha^2} + U_{\max},$$

of the wave speed bounds given by Synge (1938) for Couette and Poiseuille flow.

These estimates restrict the wave speed  $c_r$  for all parallel motions to an interval only slightly larger than the range of  $U$ . Unlike previous estimates (Synge 1938; Pai 1954) (11) shows that  $c_r$  is bounded above and below, uniformly in  $\alpha$ .

I am unaware of calculations leading to negative wave speeds in situations covered by (11c). However, the negative wave speeds outside the range of  $U$  which are consistent with (11b) evidently do occur in Jeffery–Hamel flow in diverging channels with back flow (Eagles 1966).

*Proof of Theorem 2*

From (3) we find that

$$c_r = U(y_1) + \frac{\frac{1}{2}U''(y_2)}{(I_1^2/I_0^2) + \alpha^2}, \quad \infty \geq \frac{I_1^2}{I_0^2} \geq \pi^2, \quad (12)$$

where  $y_1, y_2 \in (0, 1)$  are mean values. It is clear that in the situations  $(a, b, c)$  specified in the theorem we have

$$(a) \quad 0 < \frac{U''(y_2)}{(I_1^2/I_0^2) + \alpha^2} < \frac{U''_{\max}}{\pi^2 + \alpha^2}, \quad (13a)$$

$$(b) \quad \frac{U''_{\min}}{\pi^2 + \alpha^2} < \frac{U''(y_2)}{(I_1^2/I_0^2) + \alpha^2} < \frac{U''_{\max}}{\pi^2 + \alpha^2}, \quad (13b)$$

$$(c) \quad \frac{U''_{\min}}{\pi^2 + \alpha^2} < \frac{U''(y_2)}{(I_1^2/I_0^2) + \alpha^2} < 0. \quad (13c)$$

The estimates (11) now follow easily from (12) and (13).

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## REVIEWS

**Physics of shock waves and high-temperature hydrodynamic phenomena, volume II.** By YA. B. ZEL'DOVICH and YU. P. RAIZER. (Translation edited by W. D. Hayes and R. F. Probstein.) Academic Press, 1967. 451 pp. £8 8s.

This volume translates the second half of Zel'dovich and Raizer's second Russian edition. (For a review of volume I, see *J. Fluid Mech.* **31**, 1968, p. 826.) Numbering of chapters and pages is consecutive with volume I, with the subject and author indexes at the end of volume II covering both volumes. The same high standards of technical accuracy and clarity of composition observed earlier characterize the six chapters included here. While the structure of subject-matter is such that volume II is directly usable by those interested in one of the particular applications covered, most active workers in high-speed or high-temperature fluid mechanics will doubtless wish to acquire both volumes of the translation.

Chapter VII, on shock-wave structure in gases, treats in succession the nature of the thin 'mechanical' shock front where viscosity dominates the scattering of directed momentum of the incident gas with conversion of kinetic energy into thermal motion, and the nature of the more complex processes of binary diffusion, relaxation, dissociation, ionization, and radiant heat exchange which give shock waves their full spectrum of intriguing profiles. In the case of flows with appreciable radiation-energy content, a useful distinction is made between sub- and super-critical waves, the demarcation arising at that condition when gas upstream of the mechanical front is heated by radiation to just the final equilibrium temperature. After treating the further phenomenon of radiation-dispersed shocks (i.e. no distinct mechanical front) the text wisely stops short of an exposé of relativistic effects and shocks in the presence of magnetic fields.

Physical and chemical kinetics in hydrodynamic processes are treated straightforwardly in chapter VIII. The material is largely an extension toward selected applications of the underlying fundamentals covered in volume I. While many of the sections deal with fields that have been thoroughly ploughed before, others concern problems of significant current interest. Two of these are the sudden expansion into vacuum of gas clouds as encountered in supernovae or meteor impact on the moon, and condensation of cosmic dust. Chapter IX treats radiative phenomena associated with air shocks generated by nuclear explosions. Particularly valuable to workers in fluid mechanics is the lucid treatment of air brightness and limiting luminosity. Here, as in many places throughout the book, simple sketches and tables giving physical property data aid the understanding considerably. Chapter X, entitled 'Thermal Waves', is actually a brief treatment of the theory of heat conduction. English language readers will probably prefer to refer to Carslaw and Jaeger's book for definitive material on this special topic.

Chapter XI, on shock waves in solids, is of major significance in the total work. Occupying nearly 100 pages, this chapter gives a comprehensive treatment to the basic differences between shocks in solid media and those in gases as these