

Parameter and Domain Dependence of Eigenvalues of Elliptic Partial Differential Equations

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It is our purpose in this paper to exploit the technique of parameter differentiation for studying the calculus of eigenvalues. The study of the domain dependence of eigenvalues is emphasized, but not exclusively, and dependence of eigenvalues on other parameters of the generating boundary-value problem is also considered.

The problem of the domain dependence of eigenvalues as a special case in the calculus of variations of domain functionals has been treated by RAYLEIGH [1], HADAMARD [2], GARABEDIAN & SCHIFFER [3,4], PÓLYA & SCHIFFER [5], and SCHIFFER [6], among others. The earliest general technique seems to be due to HADAMARD. This technique, in the spirit of classical analysis, is used to study how the domain functional changes under an infinitesimal change of domain. HADAMARD assumed that the domain D is bounded by a closed, smooth surface ∂D . He deforms ∂D into a surface ∂D^* by pushing each point ζ on ∂D by an amount δN in the direction of the exterior normal. The method is purely formal, though for sufficiently regular surfaces, it can be justified (*cf.* GARABEDIAN & SCHIFFER). The essential defect in the HADAMARD kinematics is clearly the restriction of the domains D to those associated uniquely at every boundary point ζ with an exterior normal. This difficulty, however, was overcome by GARABEDIAN & SCHIFFER [4], who developed an interior kinematics which involves straightforward mapping of neighboring domains and avoids the normal shift. These authors applied the interior kinematics to obtain a number of important results including the second variation with domain of the GREEN'S function for the Laplace equation. They were also able to demonstrate that for boundary-value problems which can be characterized as extremum problems, the local variations of the extremum (say the eigenvalue) with domain is equivalent to differentiation of the extremal functional with respect to a parameter characterizing the interior kinematics.

It is essentially this idea of parameter differentiation with which we are concerned in this paper. The idea itself is not totally new and restricted versions of it have been applied successfully to a number of problems (*cf.* LIN [7], JOSEPH [8], among others). In this paper we develop and apply an extended version of the theory of parameter differentiation and use it to obtain a number of new results as well as to re-derive some important known results. An extended treatment of the dependence of the eigenvalues of the HELMHOLTZ equation on domain and other parameters is given. We regard the eigenvalues as generated by a one-parameter family of boundary-value problems and make no essential use of the variational characterization of

eigenvalues. Formulas giving the first and second derivatives of the eigenvalues for an arbitrary deformation of the domain are obtained. Derivatives of the generating eigenfunctions appear only for the second derivative and then only in the special combination*

$$\lambda \int_D [\varphi^{(1)}]^2 - \int_D [\nabla \varphi^{(1)}]^2 - \beta \oint_{\partial D} [\varphi^{(1)}]^2, \quad \varphi^{(1)} = \frac{d\varphi}{dt},$$

which under widely applicable conditions is demonstrably nonpositive.

Various convexity results of PÓLYA & SCHIFFER may be obtained from the formula for the second derivative without the use of either transplantation techniques or the POINCARÉ minimax principle. From these equations one may deduce monotonicity and convexity theorems and otherwise bound eigenvalues [cf. PÓLYA & SCHIFFER].

First and second derivatives of eigenvalues are not all that one may obtain by parameter differentiation. It is, in fact, possible to generate an infinity of boundary-value problems governing the successively higher derivatives of the one-parameter family of eigenfunctions, and these may be manipulated to produce corresponding formulas for higher derivatives of the eigenvalues. Several important results follow from this procedure. A direct consequence of the hierarchy of equations is that derivatives of eigenvalues of order greater than two may be expressed in terms of derivatives of eigenfunctions two orders lower. This, coupled with the fact that the boundary-value problems for the derivatives of the eigenfunctions may be solved successively for particular parameter values, allows the calculation of the value of the ν^{th} derivative of the eigenvalue $\lambda(t)$ from the values of the r^{th} derivatives ($r \leq \nu - 2$) of the eigenfunction $\varphi(t)$. This makes it possible, in principle, to obtain the global behavior of both $\varphi(t)$ and $\lambda(t)$ as a Taylor series developed around some particular value $t = t_0$ for which the hierarchy of equations may explicitly be resolved. As a practical matter, it is frequently possible to generate the first three or more coefficients for the Taylor series from extremely simple calculations**. The higher-order coefficients, of course, require more extensive calculations, but the number of such coefficients which can be obtained is limited only by computational difficulties.

Before undertaking the detailed mathematical work, we should like to establish a notation and to state a few general assumptions which are necessary for the results which follow.

We assume that the boundary is smooth enough so that the divergence theorem holds. We also assume that the eigenfunctions are analytic in the parameter under consideration. The eigenvalue equations themselves are elliptic and nonsingular

* In writing integrals we shall ordinarily omit infinitesimal volume and surface elements. Surface integrals are indicated by a circle drawn through the integral sign.

** An example of the power of these methods is the elegant representation (derived in Section 4) of the principal eigenvalue for the elliptical membrane of eccentricity e

$$\lambda(e) = \lambda_1 - \frac{\lambda_1}{2} e^2 - \frac{\lambda_1}{16} \left(3 - \frac{\lambda_1}{2}\right) e^4 - \frac{\lambda_1}{16} \left(3 - \frac{\lambda_1}{2}\right) e^6 + 0 (e^8)$$

where λ_1 is the lowest eigenvalue for the circular membrane. The first four terms of the Taylor series give a remarkably accurate representation of $\lambda(e)$ for all values of e (cf. Table I).

with coefficients which are analytic in the domains of the independent variables and parameters considered. Only bounded regions are considered, and the boundary conditions treated guarantee an infinitely denumerable set of eigenvalues. Under these strong conditions, $\lambda(t)$ is continuously differentiable (see [4]) and even analytic. This is indeed the case, if by $\lambda(t)$ one means the eigenvalue associated with some particular eigenfunction $\varphi(t)$, which we identify in the conventional way as $\lambda_{v_i}(t_0)$ and $\varphi_{v_i}(t_0)$, where at $t=t_0$ the λ_v are arranged in order of increasing magnitude, and $i=1, 2, 3, \dots$. M represents the M eigenfunctions with one eigenvalue λ_v . The difficulty here is that the v^{th} eigenvalue at $t=t_0$ is not necessarily v^{th} at t (see PÓLYA & SCHIFFER (p. 292) for a full discussion). There is not this difficulty for those values of t for which the eigenvalues do not change place in the ordering. This is generally true of all simple eigenvalues and in particular of the lowest eigenvalue, but it does restrict somewhat the usefulness of our results for higher eigenvalues. Our understanding then of the results which follow is that they apply to higher eigenvalues $\lambda_v(t)$ where the ordering of the eigenvalues is given at some $t=t_0$.

In sum, we propose now to establish a formalism which leads to Taylor series expansions of a one parameter family of eigenfunctions $\varphi(t)$ and its eigenvalues $\lambda(t)$ from the values of the derivatives $\varphi^{(v)}(t_0)$ generated as solutions to a hierarchy of boundary-value problems valid when $t=t_0$.

1. Variational Kinematics for the Helmholtz Equation

We consider the dependence of the eigenvalues λ of the three-dimensional operator in D

$$\Delta \varphi = \lambda \varphi = 0 \quad (1.1)$$

subject to a normalizing condition

$$\int_D \varphi^2 = 1 \quad (1.2)$$

and to

$$(N \cdot \nabla) \varphi + \beta \varphi = 0 \quad (1.3)$$

with piecewise continuous $\beta \geq 0$ on the boundary ∂D of D , on the domain D and on the values of the function β .

We imagine a one (t) parameter family of domains $D(t)$ and functions $\beta(\mathbf{r}, t)$. Reference coordinates $x_i(t_0) = X_i$ are defined in $D(t_0)$, and current coordinates $x_i(t)$ are defined in $D(t)$. It is convenient to regard this parameter as the time and imagine an evolution for (1.1), (1.2) and (1.3). We then assert the following

Theorem 1. *Let $\varphi(t)$ be a one-parameter family of solutions of (1.1) and (1.3) which is analytic in t . Then the v^{th} derivative of $\varphi(t)$ satisfies the differential equation*

$$\Delta \varphi^{(v)} + \lambda \varphi^{(v)} + \sum_{r=1}^v \binom{v}{r} [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(v-r)} = 0 \quad (1.4)$$

in $D(t)$ and the boundary condition

$$\frac{\partial \varphi^{(v)}}{\partial N} + \beta \varphi^{(v)} + \sum_{r=1}^v \binom{v}{r} [(N \cdot \nabla)^{(r)} + \beta^{(r)}] \varphi^{(v-r)} = 0 \quad (1.5)$$

on $\partial D(t)$. Here $\Delta^{(r)}$ and $(N \cdot \nabla)^{(r)}$ are defined in $D(t)$ and on $\partial D(t)$ by (summation convention holds for scripts i, j, l)

$$\Delta^{(r)} = \frac{d^r \left(\frac{\partial^2}{\partial x_i^2} \right)}{dt^r} = \left(\frac{d^r g^{ij}}{dt^r} \right) \frac{\partial^2}{\partial X_i \partial X_j} + \left(\frac{d^r \left[\frac{1}{J} \frac{\partial}{\partial X_i} (J g^{ij}) \right]}{dt^r} \right) \frac{\partial}{\partial X_j},$$

and

$$(N \cdot \nabla)^{(r)} = \frac{d^r \left(N_i \frac{\partial}{\partial x_i} \right)}{dt^r} = \left[\frac{d^r \left(N_i \frac{\partial X_j}{\partial x_i} \right)}{dt^r} \right] \frac{\partial}{\partial X_j},$$

where

$$g^{ij} = \frac{\partial X_i}{\partial x_i} \frac{\partial X_j}{\partial x_i}, \quad J = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}$$

are the metric tensor and Jacobian of the deformation, respectively.

Let $\varphi(t)$ satisfy (1.4) and (1.5). Then $\lambda^{(v)} = d^v \lambda / dt^v$ is given by

a) ($v=0$)

$$\begin{aligned} \lambda^{(1)} \int_{D(t)} \varphi^2 = & \oint_{\partial D(t)} \left[\beta \varphi^2 (\delta_{ij} - N_i N_j) \frac{\partial V_i}{\partial x_j} + \beta^{(1)} \varphi^2 \right] + \\ & + \int_{D(t)} [(\nabla \varphi)^2 - \lambda \varphi^2] (\nabla \cdot \mathbf{V}) - \int_{D(t)} D_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j}, \end{aligned} \quad (1.7)$$

b) ($v=1$)

$$\begin{aligned} \frac{1}{2} \int_{D(t)} \varphi [\Delta^{(2)} + \lambda^{(2)}] \varphi - \frac{1}{2} \oint_{\partial D(t)} \varphi [(N \cdot \nabla)^{(2)} + \beta^{(2)}] \varphi + \\ + \int_{D(t)} \{ [\nabla \varphi^{(1)}]^2 - \lambda [\varphi^{(1)}]^2 \} + \oint_{\partial D(t)} \beta [\varphi^{(1)}]^2 + \\ + \int_{D(t)} (\nabla \cdot \mathbf{V}) \varphi [\Delta^{(1)} + \lambda^{(1)}] \varphi + \oint_{\partial D(t)} (N_i N_j - \delta_{ij}) \frac{\partial V_i}{\partial x_j} \varphi [(N \cdot \nabla)^{(1)} + \beta^{(1)}] \varphi = 0, \end{aligned} \quad (1.8)$$

c) ($v \geq 1$)

$$\begin{aligned} \sum_{r=2}^{v+1} \frac{v!}{(v+1-r)! r!} \left\{ \int_{D(t)} \varphi [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(v+1-r)} - \oint_{\partial D(t)} \varphi [(N \cdot \nabla)^{(r)} + \beta^{(r)}] \varphi^{(v+1-r)} \right\} + \\ + \sum_{r=1}^v \binom{v}{r} \left\{ \int_{D(t)} [\varphi (\nabla \cdot \mathbf{V}) + \varphi^{(1)}] [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(v-r)} + \right. \\ \left. + \oint_{\partial D(t)} [(N_i N_j - \delta_{ij}) \frac{\partial V_i}{\partial x_j} \varphi - \varphi^{(1)}] [(N \cdot \nabla)^{(r)} + \beta^{(r)}] \varphi^{(v-r)} \right\} = 0 \end{aligned} \quad (1.9)$$

where $\mathbf{r}(t)$ is the position vector of the deformation of D ,

$$D_{ij} = \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i},$$

and $\mathbf{V} = d\mathbf{r}/dt$.

For $v \geq 1$, $\lambda^{(v+1)}(t_0)$ may be calculated from (1.9) if the $\varphi^{(r)}(t_0)$ ($r \leq v-1$) are known.

Proof. Equations (1.1) and (1.3) are referred to the reference coordinates, and (1.4) and (1.5) follow by differentiation using

$$\frac{d^v(AB)}{dt^v} = \sum_{r=0}^v \binom{v}{r} A^{(r)} B^{(v-r)}.$$

The derivation of (1.7), (1.8) and (1.9) are carried out as an appendix to this paper.

Theorem 1 is important because it opens the possibility of calculating eigenvalues and eigenfunctions for t by solving (1.4) and (1.5) when $t=t_0$. The guarantee that this procedure will give a unique result is the subject of

Theorem 2. Define a boundary-value problem for $\varphi^{(v)}(t)$ when $t=t_0$ by (1.4), (1.5) and

$$\frac{d^v}{dt^v} \int_{D(t)} \varphi^2 = \sum_{r=0}^v \sum_{l=0}^{v-r} \binom{v}{r} \binom{v-r}{l} \int_{D(t_0)} J^{(r)} \varphi^{(l)} \varphi^{(v-r-l)} = 0. \tag{1.6}$$

Then, if the solution of (1.1), (1.2) and (1.3) is unique when $t=t_0$, the solution of (1.4), (1.5) and (1.6) is also unique when $t=t_0$.

Proof. The result follows by induction, for we suppose the result to be true for $\tau \leq v-1$ and consider the difference between two solutions satisfying (1.4) and (1.5). This satisfies

$$A \delta \varphi^{(v)} + \lambda \delta \varphi^{(v)} = 0, \tag{1.10a}$$

$$\frac{\partial \delta \varphi^{(v)}}{\partial N} + \beta \delta \varphi^{(v)} = 0, \tag{1.10b}$$

in a neighborhood of t_0 . By comparison with (1.1) and (1.3), we have

$$\delta \varphi^{(v)}(t_0) = A \varphi(t_0). \tag{1.11}$$

On the other hand, equation (1.6), which may be written as

$$\sum_{r=1}^v \sum_{l=0}^{v-r} \binom{v}{r} \binom{v-r}{l} \int_{D(t_0)} J^{(r)} \varphi^{(l)} \varphi^{(v-r-l)} + \sum_{l=1}^{v-1} \binom{v}{l} \int_{D(t)} \varphi^{(l)} \varphi^{(v-l)} + 2 \int_{D(t)} \varphi \varphi^{(v)} = 0,$$

is equivalent, by subtraction, to the requirement that

$$\int_{D(t)} \varphi \delta \varphi^{(v)} = 0, \tag{1.12}$$

implying, through (1.2) and (1.9), that $A=0$. This proves the Theorem.

Various consequences of Theorems 1 and 2 will be explored in the applications which follow in subsequent sections. We should, however, like to remark on several general features of equations (1.7) and (1.8). Equation (1.8) may be made a basis for the various convexity results which have been otherwise obtained by PÓLYA & SCHIFFER. The technique here is to note that the quantity

$$\lambda \int_{D(t)} [\varphi^{(1)}]^2 - \int_{D(t)} [\nabla \varphi^{(1)}]^2 - \oint_{\partial D(t)} \beta [\varphi^{(1)}]^2$$

is essentially non-positive. This requires an extremum characterization of the eigenvalues and that one observe that $\varphi^{(1)}$ is an admissible trial function under the unstable boundary condition (1.3). The resulting differential inequality, under a suitable change of variables, can often be made to yield a convexity result (*cf.* Sections 2 and 4 of this paper). We shall not develop this feature of the theory in any great detail, as the convexity results can, in any event, be elegantly obtained by the PÓLYA-SCHIFFER method.

The kinematics of the deformation is made transparent in (1.7). The kinematic measures which influence the value of $\lambda^{(1)}$ are the dilatation $(V \cdot V)$, the strain rate

$$D_{ij} = \left(\frac{\partial V_i}{\partial X_j} + \frac{\partial V_j}{\partial X_i} \right)$$

and the area dilatation

$$(\delta_{ij} - N_i N_j) \frac{\partial V_i}{\partial X_j}.$$

It is apparent that rigid motions of the domain cannot influence the value of λ . Further, we know, *a priori*, that the variation of λ must depend only on the shift of the boundary ∂D of D and not on the interior values of the field; for the deformation of ∂D alone determines the varied domain D^* with eigenvalue λ^* . This fact is concretely a consequence of the relation

$$\begin{aligned} \int_D (V \cdot V) [(V\varphi)^2 - \lambda \varphi^2] - \int_D \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} D_{ij} \\ = \oint_{\partial D} (N \cdot V) [(V\varphi)^2 - \lambda \varphi^2] - 2 \oint_{\partial D} \frac{\partial \varphi}{\partial N} (V \cdot V) \varphi, \end{aligned} \quad (1.13)$$

which holds for all φ satisfying (1.1). With $\varphi = 0$ on ∂D ,

$$V\varphi = \frac{\partial \varphi}{\partial N} N$$

and (1.13) and (1.7) combine to give

$$\lambda^{(1)} \int_D \varphi^2 = - \oint_{\partial D} (V\varphi)^2 (V \cdot N),$$

a relation which was obtained with HADAMARD'S method by GARABEDIAN & SCHIFFER [4] (Equation (3.4) with $V \cdot N = -\delta v$).

As we have already remarked, Theorems 1 and 2 open the possibility of calculating the eigenvalues for a parameter range as a Taylor series around a reference-parameter value. The coefficients for this series are determined by recursive solution of the hierarchy of boundary-value problems governing the eigenfunction derivatives for that parameter value. The formalism has other secondary potentialities, principally for bounding eigenvalues. All these possibilities we explore in the context of the applications which follow.

2. Variation of the Boundary Constant

As a first application, we examine the influence of a variation in the values of $\beta(r, t)$ for a fixed domain. The relevant equations are (1.1), (1.2), and (1.3) for φ . For the eigenfunction derivatives, we have from (1.4) and (1.5) of Theorem 1 that

$$\Delta \varphi^{(v)} + \lambda \varphi^{(v)} + \sum_{r=1}^v \binom{v}{r} \lambda^{(r)} \varphi^{(v-r)} = 0, \tag{2.1}$$

$$\frac{\partial \varphi^{(v)}}{\partial N} + \beta \varphi^{(v)} + \sum_{r=1}^v \binom{v}{r} \beta^{(r)} \varphi^{(v-r)} = 0. \tag{2.2}$$

For the eigenvalue derivatives, we have from (1.7), (1.8) and (1.9) of Theorem 2 that

$$\lambda^{(1)} = \oint \beta^{(1)} \varphi^2, \tag{2.3}$$

$$\frac{\lambda^{(2)}}{2} = \frac{1}{2} \oint_{\partial D} \beta^{(2)} \varphi^2 + \lambda \int_D [\varphi^{(1)}]^2 - \int_D [\nabla \varphi^{(1)}]^2 - \oint_{\partial D} \beta [\varphi^{(1)}]^2, \tag{2.4}$$

$$\sum_{r=2}^{v+1} \frac{v!}{(v+1-r)! r!} \left\{ \lambda^{(r)} \int_D \varphi \varphi^{(v+1-r)} - \oint_{\partial D} \beta^{(r)} \varphi \varphi^{(v+1-r)} \right\} + \sum_{r=1}^v \binom{v}{r} \left\{ \lambda^{(r)} \int_D \varphi^{(1)} \varphi^{(v-r)} - \oint_{\partial D} \beta^{(r)} \varphi^{(1)} \varphi^{(v-r)} \right\} = 0. \tag{2.5}$$

It is clear from (2.3) that if β is monotone, so is λ . Similarly, (2.4) shows that if β is convex ($\beta^{(2)} \leq 0$), then λ is convex ($\lambda^{(2)} \leq 0$), reproducing the result obtained by PÓLYA & SCHIFFER when β is a specified constant 0 or ∞ on sub-elements of ∂D . If this last condition holds, β itself, rather than t , may be used as a parameter and

$$\beta^{(1)} = 1, \quad \beta^{(r)} = 0 \quad (r > 1), \quad \lambda^{(r)} = \frac{d^r \lambda}{d\beta^r}.$$

The convexity result has been used to obtain lower bounds for eigenvalues (cf. WEINBERGER [9]). It is perhaps of some interest that lower bounds which follow from convexity may often be obtained by integration of the differential inequality following from manipulation of the formula for the first derivative of the eigenvalue. To illustrate,

$$\frac{d\lambda}{d\beta} = \oint \varphi^2 = \frac{1}{\beta} \left[\lambda - \int_D (\nabla \varphi)^2 \right], \tag{2.7}$$

$$\frac{d\lambda}{d\beta} - \frac{\lambda}{\beta} = - \int_D (\nabla \varphi)^2 \leq - \text{Min}_{\beta} \int_D (\nabla \varphi)^2 = -A$$

where

$$A = \left(\int_D (\nabla \varphi)^2 \right)_{\beta=0}$$

is a well known result of the calculus of variations. For the lowest eigenvalue, $A=0$.

A lower bound for λ is produced by integration of (2.7). Thus, the lower bound ($\beta_1 \geq \beta$)

$$[\lambda(\beta) - A] \geq \frac{\beta}{\beta_1} [\lambda(\beta_1) - A] \quad (2.8)$$

follows by convexity or by direct integration of (2.7). A serious flaw in this procedure is that it requires a not easily obtained bit of missing information, namely, a value of $\lambda(\beta_1)$ for finite β_1 .

Bounding eigenvalues is, however, only of secondary interest in this study. A main value of the formalism, which we shall now examine in the context of a simple example, is the possibility of finding the exact values of $\lambda(\beta)$ as a Taylor series

$$\lambda(\beta) = \lambda(0) + \lambda^{(1)}\beta + \frac{\lambda^{(2)}}{2!}\beta^2 + \dots \quad (2.9)$$

developed around a particular value of β , say zero.

The coefficients for (2.9) are to be supplied by (2.3), (2.4), and (2.5). The eigenfunction derivatives which appear in these equations are to be obtained successively as solutions of the boundary-value problems (2.1) and (2.2) subject to the constraint that the v^{th} derivative of the normalizing condition (1.2) is zero. To illustrate, let us obtain the first four terms of (2.9) for the lowest eigenvalue of a circular membrane of unit radius constrained on the boundary by springs with constant β . To obtain the first four terms, we need the lowest eigenfunction and its first two derivatives with respect to β . These are to be determined as solutions to the boundary value problems

$$\Delta \varphi + \lambda \varphi = 0, \quad (2.10a)$$

$$\frac{\partial \varphi}{\partial r} + \beta \varphi = 0, \quad (2.10b)$$

$$\int_0^1 r dr \int_0^{2\pi} \varphi^2 d\vartheta = 1; \quad (2.10c)$$

$$\Delta \varphi^{(1)} + \lambda \varphi^{(1)} + \lambda^{(1)} \varphi = 0, \quad (2.11a)$$

$$\frac{\partial \varphi^{(1)}}{\partial r} + \beta \varphi^{(1)} + \varphi = 0, \quad (2.11b)$$

$$\int_0^1 r dr \int_0^{2\pi} \varphi \varphi^{(1)} = 0; \quad (2.11c)$$

$$\Delta \varphi^{(2)} + \lambda \varphi^{(2)} + 2\lambda^{(1)} \varphi^{(1)} + \lambda^{(2)} \varphi = 0, \quad (2.12a)$$

$$\frac{\partial \varphi^{(2)}}{\partial r} + \beta \varphi^{(2)} + 2\varphi^{(1)} = 0, \quad (2.12b)$$

$$\int_0^1 dr r \int_0^{2\pi} \{[\varphi^{(1)}]^2 + [\varphi^{(2)} \varphi]\} d\vartheta = 0. \quad (2.12c)$$

The derivatives of the eigenvalues evaluated at $\beta=0$ are then given by

$$\lambda^{(1)} = \int_0^{2\pi} \varphi^2 d\vartheta, \quad (2.13)$$

$$\frac{\lambda^{(2)}}{2} = \int_0^1 r dr \int_0^{2\pi} d\vartheta \{ \lambda [\varphi^{(1)}]^2 - [\nabla \varphi^{(1)}]^2 \} - \beta \int_0^{2\pi} [\varphi^{(1)}]^2 = \int_0^{2\pi} \varphi \varphi^{(1)} d\vartheta, \quad (2.14)$$

$$\frac{\lambda^{(3)}}{6} = \int_0^{2\pi} [\varphi^{(1)}]^2 d\vartheta - \lambda^{(1)} \int_0^1 r dr \int_0^{2\pi} [\varphi^{(1)}]^2, \quad (2.15)$$

$$\frac{\lambda^{(4)}}{6} = -\beta \int_0^{2\pi} [\varphi^{(2)}]^2 d\vartheta - \int_0^1 r dr \int_0^{2\pi} d\vartheta \{ [\nabla \varphi^{(2)}]^2 - \lambda [\varphi^{(2)}]^2 + 2\lambda^{(2)} [\varphi^{(1)}]^2 \}. \quad (2.16)$$

Equations (2.10), (2.11), and (2.12) are to be solved for $\beta=0$. From (2.10) and (2.13), we readily determine

$$\varphi = \frac{1}{\sqrt{\pi}}, \quad \lambda = 0, \quad \lambda^{(1)} = 2. \quad (2.17)$$

From (2.11), (2.14), and (2.15)

$$\varphi^{(1)} = \frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} - r^2 \right), \quad \lambda^{(2)} = -1, \quad \lambda^{(3)} = \frac{1}{2}, \quad (2.18)$$

and from (2.12) and (2.16)

$$\varphi^{(2)} = \frac{1}{8\sqrt{\pi}} \left(r^4 - \frac{1}{2} \right), \quad \lambda^{(4)} = -\frac{1}{4}. \quad (2.19)$$

Inserting these values into (2.9), we obtain

$$\lambda(\beta) = 2\beta - \frac{1}{2}\beta^2 + \frac{1}{12}\beta^3 - \frac{1}{96}\beta^4 + \dots \quad (2.20)$$

for the eigenvalue and

$$\varphi(\beta) = \frac{1}{\sqrt{\pi}} \left\{ 1 + \frac{1}{2} \left(\frac{1}{2} - r^2 \right) \beta + \frac{1}{16} \left(r^4 - \frac{1}{2} \right) \beta^2 + \dots \right\} \quad (2.21)$$

for the eigenfunction.

It would not be difficult to generate more coefficients for (2.20) and (2.21) by solving the boundary-value problems for the higher order derivatives of φ . Of course, for these illustrative purposes, we have chosen a particularly easy example. The locus of $\lambda(\beta)$ values is given exactly by the transcendental equation

$$\sqrt{\lambda} J_1(\sqrt{\lambda}) = \beta J_0(\sqrt{\lambda}), \quad (2.22)$$

where J_0 and J_1 are Bessel functions. We remark that (2.20) can be obtained, with difficulty, from (2.22). In the sequel, we shall consider more difficult problems of a similar nature.

3. Dilatation Without Extension or Shear

We consider here a purely dilatational deformation defined by

$$\begin{aligned} x_i &= X_i t, & V_i &= \frac{x_i}{t}, \\ (N \cdot \nabla)^{(1)} &= -\frac{1}{t} (N \cdot \nabla), & (N \cdot \nabla)^{(2)} &= \frac{2}{t^2} (N \cdot \nabla), \\ \Delta^{(1)} &= -\frac{2}{t} \Delta, & \Delta^{(2)} &= \frac{6}{t^2} \Delta. \end{aligned} \quad (3.1)$$

We wish only to examine the first and second derivative of the eigenvalue $\lambda(\mu)$ where $\mu = t^{-2}$. From (1.7)

$$\frac{d\lambda}{d\mu} = \frac{\lambda}{\mu} - \oint \left(\frac{\beta}{2\mu} - \frac{d\beta}{d\mu} \right) \varphi^2, \quad (3.2)$$

and from (1.11)

$$\frac{d^2\lambda}{d\mu^2} = -2 \left\{ \oint \beta \varphi_\mu^2 + \int (\nabla \varphi_\mu)^2 - \lambda \int \varphi_\mu^2 \right\} - \oint \left(\frac{\beta}{4\mu^2} - \frac{1}{\mu} \frac{d\beta}{d\mu} + \frac{1}{2} \frac{d^2\beta}{d\mu^2} \right) \varphi^2, \quad (3.3)$$

where $\varphi_\mu = d\varphi/d\mu$.

It is clear that for zero-boundary values, we have the exact relation

$$\frac{d\lambda}{d\mu} = \frac{\lambda}{\mu},$$

and nothing is to be gained from considering higher derivatives. From (3.3) we deduce hypotheses which insure the convexity of $\lambda(\mu)$; *i.e.*,

$$\frac{d\beta}{d\mu} \leq 0, \quad \frac{d^2\beta}{d\mu^2} \geq 0.$$

We also remark that in applications with a distribution $\beta(\mathbf{r}, t)$ on ∂D , it would be usual for

$$\frac{\partial \beta}{\partial t} = 0, \quad \frac{d\beta}{dt} = \mathbf{V} \cdot \nabla \beta.$$

4. Pure Extension

As our next application, we shall consider the pure stretch deformation

$$x_1 = X_1 t, \quad x_2 = X_2, \quad x_3 = X_3. \quad (4.1)$$

The convexity implications of this deformation with zero boundary-values have been explored by PÓLYA & SCHIFFER. We also restrict our considerations to the zero boundary-value case. To obtain the relevant formulas in convenient form, we observe that

$$\Delta^{(r)} = \frac{d^r}{dt^r} \left\{ \frac{1}{t^2} \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \frac{\partial^2}{\partial X_3^2} \right\} = t^{2r} \frac{d^r}{dt^r} \left(\frac{1}{t^2} \right) \frac{\partial^2}{\partial x^2}.$$

It is clear that (equation A.13)

$$0 = \int [\varphi \Delta^{(1)} \varphi^{(v)} - \varphi^{(v)} \Delta^{(1)} \varphi] = \sum_{r=1}^v \binom{v}{r} \int (\nabla \cdot \nabla) \varphi [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(v-r)},$$

so that this term may be set to zero in equation (1.9). We next observe that (1.4) is also valid if the indicated derivatives are with respect to $\mu = t^{-2}$. Then

$$\Delta^{(r)} = \frac{1}{\mu} \frac{d^r \mu}{d\mu^r} \frac{\partial^2}{\partial x_1^2} = \delta_{r1} \frac{1}{\mu} \frac{\partial}{\partial x_1^2}, \quad \delta_{r1} = \begin{cases} 0 & (r \neq 1) \\ 1 & (r = 1) \end{cases},$$

and the relevant equations are

$$\Delta \varphi^{(v)} + \lambda \varphi^{(v)} + \frac{v}{\mu} \frac{\partial^2 \varphi^{(v-1)}}{\partial x_1^2} + \sum_{r=1}^v \binom{v}{r} \lambda^{(r)} \varphi^{(v-r)} = 0, \quad (4.2)$$

$$\varphi^{(v)} = 0 \quad \text{on } \partial D, \quad (4.3)$$

$$\frac{d^v}{d\mu^v} \int \varphi^2 = 0 \quad (4.4)$$

for the boundary-value problems and

$$\lambda^{(1)} = \frac{1}{\mu} \int_{D(t)} \left(\frac{\partial \varphi}{\partial x_1} \right)^2, \quad (4.5)$$

$$\begin{aligned} \frac{\lambda^{(2)}}{2} &= - \int_{D(t)} \{ [\nabla \varphi^{(1)}]^2 - \lambda [\varphi^{(1)}]^2 \} \\ &= \frac{1}{\mu} \int_{D(t)} \frac{\partial \varphi}{\partial x_1} \frac{\partial \varphi^{(1)}}{\partial x_1} - \lambda^{(1)} \int_{D(t)} \varphi \varphi^{(1)}, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \frac{\lambda^{(v+1)}}{v(v+1)} &= \frac{1}{\mu} \int_{D(t)} \frac{\partial \varphi^{(1)}}{\partial x_1} \frac{\partial \varphi^{(v-1)}}{\partial x_1} - \lambda^{(1)} \int_{D(t)} \varphi^{(v-1)} \varphi^{(1)} \\ &\quad - \sum_{r=2}^v \frac{(v-1)!}{(v-r)! r!} \lambda^{(r)} \left\{ \int_{D(t)} \varphi^{(1)} \varphi^{(v-r)} + \int_{D(t)} \frac{\varphi \varphi^{(v+1-r)}}{(v+1-r)} \right\}, \end{aligned} \quad (4.7)$$

for the eigenvalue derivatives.

The monotonicity and convexity of λ are inferred by inspection of (4.5) and (4.6).

The convexity result can be used in a number of ways. For example, if one stretches a circle of unit radius into an infinite strip of height 2, one may assert with confidence that the lowest eigenvalue of all of the ellipses generated in the deformation is greater than the straight line between the eigenvalue for the strip ($\lambda(0) = \pi^2/4$) and the eigenvalue for the circle ($\lambda(1) = \lambda_1 = 5.783186$) (cf. WEINBERGER). Similarly, one may obtain lower bounds for ellipsoids generated in the one-sided stretching of a sphere into an infinite cylinder.

We wish also to consider this transformation from a circle to a strip but from a different point of view. We wish to develop the eigenvalues for the ellipses generated by stretching ($\mu < 1$) from data given in the circle ($\mu = 1$). The result is a potentially exact Taylor-series development of $\lambda(\mu)$ around the circle. To

illustrate, we shall show that

$$\frac{d\lambda}{d\mu} = \frac{\lambda}{2}, \quad (4.8)$$

$$\frac{d^2\lambda}{d\mu^2} = -\frac{\lambda}{8} \left(3 - \frac{\lambda}{2}\right), \quad (4.9)$$

$$\frac{d^3\lambda}{d\mu^3} = -3 \frac{d^2\lambda}{d\mu^2}. \quad (4.10)$$

The exact development of the eigenvalues $\lambda(\mu)$ for the one-parameter family of ellipses is then

$$\begin{aligned} \lambda(\mu) = & \lambda_1 + \frac{\lambda_1}{2} (\mu - 1) - \frac{\lambda_1}{16} \left(3 - \frac{\lambda_1}{2}\right) (\mu - 1)^2 + \\ & + \frac{\lambda_1}{16} \left(3 - \frac{\lambda_1}{2}\right) (\mu - 1)^3 + R_4(\mu), \end{aligned} \quad (4.11)$$

where R_4 is the remainder after four terms, and $\lambda_1 = 5.183186$ is the first root of $J_0(\sqrt{\lambda})$. The agreement between (4.11) without remainder and the true values of $\lambda(\mu)$ over the entire μ range is remarkable (see the Table and the discussion at the end of this section). We remark that it is, in principle, possible to obtain all terms in (4.11) and, in fact, to obtain more terms. Also, the mathematics required for one-sided deformation of spheres is perhaps slightly less involved than the simple calculation we carry out below.

To obtain (4.8), (4.9) and (4.10), we need only solve (4.2), (4.3) and (4.4) when $\nu = 1$ and then carry out the integrations indicated in (4.5), (4.6), and (4.7). For the lowest eigenvalue in a circle satisfying (1.1), (1.2) and $\varphi = 0$ at $r = \sqrt{x^2 + y^2} = 1$, we have

$$\varphi = \frac{J_0(\sqrt{\lambda} r)}{\sqrt{\pi} J_1(\sqrt{\lambda})}, \quad (4.12)$$

where $\sqrt{\lambda} \cong 2.405$ is the first root of J_0 . The first derivative of $\lambda(\mu)$ at $\mu = 1$ is then obtained from (4.5) as

$$\frac{d\lambda}{d\mu} = \iint_{D(1)} \left(\frac{\partial\varphi}{\partial x}\right)^2 dx dy = \pi \int_0^1 \left(\frac{d\varphi}{dr}\right)^2 r dr = \frac{\lambda}{2}.$$

To obtain $\varphi^{(1)} = \psi$ when $\mu = 1$, we must satisfy ($x_1 = x$, $x_2 = y$)

$$\Delta\psi + \lambda\psi + \frac{\partial^2\varphi}{\partial x^2} + \frac{\lambda}{2}\varphi = 0 \quad (r < 1), \quad (4.13a)$$

$$\psi = 0 \quad (r = 1), \quad (4.13b)$$

$$\frac{d}{d\mu} \iint_{D(1)} \varphi^2 dx dy = 0 = -\frac{1}{2\mu} + 2 \iint_{D(1)} \psi\varphi dx dy. \quad (4.13c)$$

The solution of the boundary-value problem (4.13) is

$$\psi = \frac{\varphi}{4} + \frac{\sqrt{\lambda} \cos 2\vartheta}{4\sqrt{\pi} J_1(\sqrt{\lambda})} \left\{ r J_1(r\sqrt{\lambda}) - \frac{J_1(\sqrt{\lambda})}{J_2(\sqrt{\lambda})} J_2(r\sqrt{\lambda}) \right\}, \quad (4.14)$$

where $\vartheta = \tan^{-1} y/x$. Given the solution (4.14), we are in a position to calculate the second and third derivatives of $\lambda(\mu)$ for $\mu=1$. From (4.6) and (4.14), we find

$$\frac{1}{2} \frac{d^2 \lambda}{d\mu^2} = - \iint_{D(1)} (\nabla \psi)^2 dx dy + \lambda \iint_{D(1)} \psi^2 dx dy = -\frac{\lambda}{16} \left(3 - \frac{\lambda}{2} \right), \quad (4.15)$$

and from (4.7) (with $\nu=2$), (4.14) and (4.13c), we find

$$\begin{aligned} \frac{1}{6} \frac{d^3 \lambda}{d\mu^3} &= \iint_{D(1)} \left(\frac{\partial \psi}{\partial x} \right)^2 dx dy - \frac{d\lambda}{d\mu} \iint_{D(1)} \psi^2 dx dy - \\ &\quad - \frac{d^2 \lambda}{d\mu^2} \iint_{D(1)} \varphi \psi dx dy = -\frac{1}{2} \frac{d^2 \lambda}{d\mu^2}. \end{aligned} \quad (4.16)$$

Yet higher derivatives of λ could be obtained from successive solutions of (4.2), (4.3), and (4.4) with $\nu > 1$. This procedure, in addition to providing more terms for the already accurate expression (4.11) for $\lambda(\mu)$, leads to an expansion of the lowest eigenfunction of the ellipse as a power series in $(\mu-1)$ with spatially variable coefficients represented by products of Bessel and trigonometric coefficients. The sum

$$\varphi(Xt, y, \lambda(t)) = \varphi(X, y, \lambda(1)) + (\mu-1) \psi(X, y, \lambda(1)) + \dots$$

gives the first two terms of this expansion.

This section is concluded with explanatory remarks for Table 1, in which we

Table. *Lowest Eigenvalue for the Elliptical Membrane*

e	$\lambda(\mu)$ after DAYMOND [9]	$\lambda - R_4(\mu)$ (Eq. 4.17)
0	5.783186	5.783186
0.1	5.754266	5.754266
0.2	5.667458	5.667457
0.3	5.522610	5.522597
0.4	5.319435	5.319368
0.5	5.057443	5.057227
0.6	4.735796	4.735306
0.7	4.352812	4.352288
0.8	3.904443	3.906245
0.9	3.37582	3.394464
1	2.467401	2.813226

compare the Taylor development (4.11) without remainder with the known variation of λ as given by DAYMOND [9]. DAYMOND's results are expressed in terms of the eccentricity (e) of the ellipse (the ratio of minor to major axes = $\sqrt{1-e^2}$) for

ellipses of area π . To put our results in eccentricity notation, we note that

$$\frac{\text{minor axis}}{\text{major axis}} = \frac{1}{t} = \sqrt{1-e^2} = \mu^{\frac{1}{2}},$$

$$\{\lambda - R_4(\mu)\} = \lambda_1 - \frac{\lambda_1}{2} e^2 - \frac{\lambda_1}{16} \left(3 - \frac{\lambda_1}{2}\right) e^4 - \frac{\lambda_1}{16} \left(3 - \frac{\lambda_1}{2}\right) e^6. \quad (4.17)$$

The values $(\lambda - R_4)$ are to be compared with the values $\lambda^*(e) \sqrt{1-e^2}$ where $\lambda^*(e)$ is the principal eigenvalue for the ellipse of eccentricity (e) and area π . The factor $\sqrt{1-e^2}$ follows from a conversion of $\lambda^*(e)$ to its value $\lambda(\mu) = \lambda^*(e) \sqrt{1-e^2}$ in an ellipse of area $\pi/t = \pi \sqrt{1-e^2}$.

5. Pure Shear

As a final application of our theory to the Helmholtz equation, we consider the variation of eigenvalues in a two-dimensional membrane fixed at the boundary parallelogram and sheared in accord with the deformation (see Fig. 1)

$$x = X, \quad y = Y + X \gamma, \quad \gamma = \tan \alpha$$

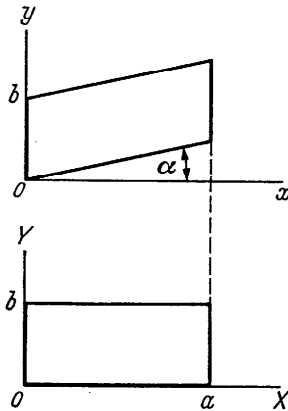


Fig. 1. Shear Deformation

where X, Y are space variables in a rectangle ($\gamma=0$) of sides a, b . This application is important for three reasons. First, neither monotonicity nor convexity follow by our methods or by transplantation. Second, the eigenfunctions for the varied domain are unknown. Third, the problem has a structure which leads to precise statements relative to variation of higher eigenvalues and eigenfunctions.

It is convenient to regard γ rather than t as the parameter of this deformation. Then the v^{th} derivative ($v \geq 1$) of the eigenfunction satisfies

$$\Delta \varphi^{(v)} + \lambda \varphi^{(v)} - 2v \frac{\partial^2 \varphi^{(v-1)}}{\partial x \partial y} + v(v-1) \frac{\partial^2 \varphi^{(v-2)}}{\partial y^2} + \sum_{r=1}^v \binom{v}{r} \lambda^{(r)} \varphi^{(v-r)} = 0, \quad (5.1a)$$

$$\varphi^{(v)} = 0 \quad \text{on} \quad \partial D \quad (5.1b)$$

$$\int \frac{d^v}{d\gamma^v} \varphi^2 = \sum_{r=0}^v \binom{v}{r} \int \varphi^{(r)} \varphi^{(v-r)} = 0, \quad (5.1c)$$

and the eigenvalue derivatives are given by

$$\lambda^{(1)} = -2 \int \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y}, \quad (5.2)$$

$$\begin{aligned} \lambda^{(2)} &= 2 \int \left(\frac{\partial \varphi}{\partial y} \right)^2 - 4 \int \frac{\partial \varphi}{\partial y} \frac{\partial \psi}{\partial x} \\ &= 2 \int \left(\frac{\partial \varphi}{\partial y} \right)^2 + 2\lambda \int [\varphi^{(1)}]^2 - \int [\nabla \varphi^{(1)}]^2, \end{aligned} \quad (5.3)$$

$$\begin{aligned} \frac{\lambda^{(v+1)}}{v(v+1)} &= (v-1) \int \frac{\partial \varphi^{(1)}}{\partial y} \frac{\partial \varphi^{(v-2)}}{\partial y} + \int \frac{\partial \varphi}{\partial y} \frac{\partial \varphi^{(v-1)}}{\partial y} - \\ &- 2 \int \frac{\partial \varphi^{(1)}}{\partial y} \frac{\partial \varphi^{(v-1)}}{\partial x} - \lambda^{(1)} \int \varphi^{(1)} \varphi^{(v-1)} - \\ &- \sum_{r=2}^v \frac{(v-1)!}{(v-r)! r!} \lambda^{(r)} \left[\int \varphi^{(1)} \varphi^{(v-r)} + \int \varphi \varphi^{(v+1-r)} / (v+1-r) \right]. \end{aligned} \quad (5.4)$$

As our first deduction we obtain from (5.2) the rough inequality

$$e^{-\gamma} \leq \frac{\lambda(\gamma)}{\lambda(0)} \leq e^{\gamma}, \quad (5.5)$$

which follows easily from the integration of the pair of inequalities

$$\frac{d\lambda}{d\gamma} = -2 \int \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y} = \begin{cases} -\int \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \right)^2 + \lambda \leq \lambda, \\ \int \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} \right)^2 - \lambda \geq \lambda \end{cases} \quad (5.6)$$

between $\gamma=0$ and γ . The estimate is very crude, but it applies to every eigenvalue $\lambda_M(\gamma)$ which was M^{th} in an ordering at $\gamma=0$. If the ratio of the sides a/b is irrational, then every eigenvalue belongs to one eigenfunction which when $\gamma=0$ is given as

$$\varphi_M(\gamma=0) = \frac{2}{\sqrt{b a}} \sin \frac{k \pi x}{a} \sin \frac{l \pi y}{b} \quad (5.7)$$

with

$$\lambda_M(\gamma=0) = \pi^2 \left\{ \frac{k^2}{a^2} + \frac{l^2}{b^2} \right\}. \quad (5.8)$$

Equation (5.5) is deficient in two ways: First, we know that as $\tan \alpha = \gamma \rightarrow \infty$, $\lambda \rightarrow \infty$ like $(b^2 \cos^2 \alpha)^{-2}$ and not exponentially. Hence, the large γ behavior of the L.H.S. of (5.5) is not a good estimate. In addition, the small γ behavior of the L.H.S. of (5.5) is crude in that λ increases like γ^2 near $\gamma=0$. This last remark is a consequence (1) of the fact that $\lambda(\gamma)$ is an even, analytic function, so that odd derivatives of the Taylor-series representation of $\lambda(\gamma)$ about $\gamma=0$ vanish, and (2) of

the relation

$$\left(\frac{d^2 \lambda}{d\gamma^2}\right)_{\gamma=0} = \frac{128 l^2 k^2 a^2}{\pi b^3} \sum_{\substack{N=1 \\ N \neq k}}^{\infty} \frac{[1 - (-1)^{N+k}]^2}{[N^2 - k^2]^4} N^2 \times \begin{cases} (l \text{ even}) \begin{cases} -\gamma_N \tan \gamma_N \frac{b}{2} & \text{(i)} \\ \alpha_N \tanh \alpha_N \frac{b}{2} & \text{(ii)} \end{cases} \\ (l \text{ odd}) \begin{cases} \gamma_N \cot \gamma_N \frac{b}{2} & \text{(i)} \\ \alpha_N \coth \alpha_N \frac{b}{2} & \text{(ii)} \end{cases} \end{cases} \quad (5.9)$$

where (i) $\gamma_N > 0$, (ii) $\alpha_N > 0$

$$\gamma_N^2 = -\alpha_N^2 = \pi^2 \left(\frac{k^2 - N^2}{a^2} + \frac{l^2}{b^2} \right),$$

which is a result to be obtained.

To obtain (5.9), we must solve the boundary value problem for $\psi = \psi_M$ ($\varphi = \varphi_M$, $\lambda = \lambda_M$)

$$\Delta \psi + \lambda \psi - 2 \frac{\partial^2 \varphi}{\partial x \partial y} = 0, \quad (5.10a)$$

$$\psi(0, y) = \psi(a, y) = \psi(x, 0) = \psi(x, b) = 0, \quad (5.10b)$$

$$\int_0^a \int_0^b \varphi \psi \, dx \, dy = 0. \quad (5.10c)$$

The solution to (5.10, a, b, c) is easily obtained as

$$\psi = \frac{2l\pi a}{(ab)^{\frac{3}{2}}} x \sin \frac{k\pi x}{a} \cos \frac{l\pi y}{b} + \sum_{N=1}^{\infty} B_N(y) \sin \frac{N\pi x}{a} \quad (5.11a)$$

where

$$B_N(y) = B_N(0) \begin{cases} (l \text{ even}) \begin{cases} \frac{\cos \gamma_N(y-b/2)}{\cos \gamma_N b/2} & \left(N^2 < k^2 + \frac{a^2}{b^2} l^2\right) \\ \frac{\cosh \alpha_N(y-b/2)}{\cosh \alpha_N b/2} & \left(N^2 > k^2 + \frac{a^2}{b^2} l^2\right) \end{cases} \\ (l \text{ odd}) \begin{cases} -\frac{\sin \gamma_N(y-b/2)}{\sin \gamma_N b/2} & \left(N^2 < k^2 + \frac{a^2}{b^2} l^2\right) \\ -\frac{\sinh \alpha_N(y-b/2)}{\sinh \alpha_N b/2} & \left(N^2 > k^2 + \frac{a^2}{b^2} l^2\right) \end{cases} \end{cases} \quad (5.11b)$$

and

$$B_N(0) = \begin{cases} -\frac{8lk a^2 N}{\pi(ab)^{\frac{3}{2}}} \frac{[(-1)^{N+k} - 1]}{[N^2 - k^2]^2} & (N \neq k) \\ -\frac{l\pi a^2}{(ab)^{\frac{3}{2}}} & (N = k). \end{cases} \quad (5.11c)$$

Equation (5.9) follows from the evaluation of

$$\frac{1}{2} \left(\frac{d^2 \lambda}{d\gamma^2} \right)_{\gamma=0} = \iint \left(\frac{\partial \varphi}{\partial y} \right)^2 dx dy - 2 \iint \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} dx dy. \tag{5.12}$$

To carry out this evaluation, we note that (5.12) can be reduced as follows:

$$\begin{aligned} \frac{1}{2} \frac{d^2 \lambda}{d\gamma^2} &= \frac{l^2 \pi^2}{b^2} + \frac{8l^2 k \pi^2}{b(a b)^2} \int_0^b \sin^2 \frac{l \pi y}{b} dy \int_0^a x \sin \frac{k \pi x}{a} \cos \frac{k \pi x}{a} dx - \\ &\quad - \frac{4k \pi}{a \sqrt{b a}} \sum_{N=1}^{\infty} \int_0^b \frac{dB_N}{dy} \sin \frac{l \pi y}{b} dy \int_0^a \sin \frac{N \pi x}{b} \cos \frac{k \pi x}{b} dx \\ &= \frac{4k}{\sqrt{b a}} \sum_{\substack{N=1 \\ N \neq k}}^{\infty} \frac{N[1 - (-1)^{N+k}]}{N^2 - k^2} \int_0^b \left\{ \frac{d}{dy} \left(B_N \sin \frac{l \pi y}{a} \right) - \frac{l \pi}{b} B_N \cos \frac{l \pi y}{a} \right\} dy \\ &= \frac{4lk a^2}{\pi b \sqrt{b a}} \sum_{\substack{N=1 \\ N \neq k}}^{\infty} \frac{N[1 - (-1)^{N+k}]}{(N^2 - k^2)^2} \left\{ \cos l \pi \frac{dB_N(b)}{dy} - \frac{dB_N(0)}{dy} \right\} \end{aligned}$$

where use of the fact that B_N satisfies

$$\frac{d^2 B_N}{dy^2} + \pi^2 \left\{ \frac{(k^2 - N^2)}{a^2} + \frac{l^2}{b^2} \right\} B_N = 0 \tag{5.14}$$

has been made in carrying out the integration.

Equation (5.9) is combined with the derivative of (5.11 b) to produce (5.9).

For arbitrary values of $k, l, a/b$, the sign of (5.9) is not obvious. The expression can be shown to be positive for the lowest eigenvalue ($\kappa = l = 1$) when $(3 > a^2/b^2)$. For this case, (5.9, ii), with l odd, holds.

It should be noted that for this deformation, $\lambda(\gamma) = \lambda(-\gamma)$ and

$$\lambda(\gamma) = \lambda(0) + \frac{1}{2} \frac{d^2 \lambda}{d\gamma^2} \gamma^2 + \frac{1}{4!} \frac{d^4 \lambda}{d\gamma^4} \gamma^4 + \dots, \tag{5.15}$$

so that the first two terms of the Taylor series should give an increasingly good estimate of the values $\lambda(\gamma)$ as $\gamma < 1$ ($\alpha < \pi/4$). Yet more accurate values of $\lambda(\gamma)$ could, of course, be obtained through the solution of the relatively easy, though tedious, problem defined by (5.1 a, b, c) with $\nu = 2$. The value of $\lambda^{(4)} = d^4 \lambda / d\gamma^4$ could then be obtained from (5.4) with $\nu = 3$. So, too, may all higher-order terms of (5.15) be calculated from the recursive solution *ad infinitum* of boundary-value problems defined in the rectangle.

6. Higher Order and Non-self-adjoint Systems

The formalism developed for the Helmholtz equation in preceding sections can easily be extended to higher order systems and even to non-self-adjoint problems. In this section we briefly sketch some of these extensions.

First we consider self-adjoint boundary value problems of arbitrary order characterized by

$$L\varphi=0 \quad \text{in } D, \quad (6.1)$$

$$A_i\varphi=0 \quad \text{on } \partial D \quad (6.2)$$

and some suitable normalizing condition. Here the differential operator L can depend on a parameter t and eigenvalue $\lambda(t)$ but, for simplicity, we let the boundary operators A_i be free of t . Differentiation of (6.1) and (6.2) with respect to t leads to a boundary-value problem

$$L\varphi^{(v)} + \sum_{r=1}^v \binom{v}{r} L^{(r)}\varphi^{(v-r)}=0 \quad \text{in } D, \quad (6.3)$$

$$A_i\varphi^{(v)}=0 \quad \text{on } \partial D, \quad (6.4)$$

for the v^{th} derivative of φ . To (6.3) and (6.4) it will ordinarily be necessary, to obtain a unique solution, to append the condition that the v^{th} derivative of the normalizing condition vanish.

To obtain formulas for derivatives of the eigenvalues, one uses the adjoint property

$$(\varphi^{(\mu)}, L\varphi^{(v)})=(\varphi^{(v)}, L\varphi^{(\mu)}) \quad (6.5)$$

and observes that

$$(\varphi^{(v)}, L\varphi)=0$$

to produce

$$\sum_1^v \binom{v}{r} (\varphi, L^{(r)}\varphi^{(v-r)})=0, \quad (6.6)$$

which gives the v^{th} derivative of the eigenvalue in terms of lower order derivatives of the eigenfunctions.

Similar statements hold also for non-self-adjoint problems. This can be readily grasped from the following remarks on the Orr-Sommerfeld equation and its adjoint. As is well known, the linear stability of parallel motions may be obtained from the smallest R value of the problem

$$(V-C)L\varphi - V''\varphi = -\frac{i}{\alpha R} L^2\varphi, \quad (6.7)$$

$$L^2 = \frac{d^2}{dy^2} - \alpha^2,$$

$$\varphi(\pm 1) = \frac{d\varphi(\pm 1)}{dy} = 0, \quad (6.8)$$

or its adjoint (see, for example, W. REID [10])

$$L\{(V-C)\chi\} - V''\chi = -\frac{i}{\alpha R} L^2\chi, \quad (6.9)$$

$$\chi(\pm 1) = \frac{d\chi(\pm 1)}{dy} = 0. \quad (6.10)$$

The constants α , R and the basic distribution of velocity V are real quantities. The constant C and functions φ and χ are complex.

Consider, for simplicity, the variation of R with C at constant α . Then with $(\alpha R)^{-1} = B$, and M and M^+ defined by (6.11) and (6.13), we have

$$\begin{aligned} M\varphi^{(v)} &= (V-C)L\varphi^{(v)} - V''\varphi^{(v)} + iBL^2\varphi^{(v)} \\ &= -\sum_{r=1}^v \binom{v}{r} \{C^{(r)}L + iB^{(r)}L^2\} \varphi^{(v-r)}, \end{aligned} \quad (6.11)$$

$$\varphi^{(v)}(\pm 1) = \frac{d\varphi^{(v)}(\pm 1)}{dy} = 0, \quad (6.12)$$

$$\begin{aligned} M^+\chi^{(v)} &= L\{(V-C)\chi^{(v)}\} - V''\chi^{(v)} + iBL^2\chi^{(v)} \\ &= -\sum_{r=1}^v \binom{v}{r} \{C^{(r)}L + iB^{(r)}L^2\} \chi^{(v-r)}, \end{aligned} \quad (6.13)$$

$$\chi^{(v)}(\pm 1) = \frac{d\chi^{(v)}(\pm 1)}{dy} = 0. \quad (6.14)$$

It is clear that in general

$$\int_{-1}^1 \chi^{(\mu)} M\varphi^{(v)} dy = \int_{-1}^1 \varphi^{(v)} M^+\chi^{(\mu)} dy,$$

and, in particular,

$$0 = \int_{-1}^1 \varphi^{(v)} M^+\chi dy = \int_{-1}^1 \chi M\varphi^{(v)} dy = -\sum_{r=1}^v \binom{v}{r} \int_{-1}^1 \chi \{C^{(r)}L + iB^{(r)}L^2\} \varphi^{(v-r)}. \quad (6.15)$$

If (complex) C be regarded as the parameter, following partial integration and utilization of the relation

$$C^{(r)} = \delta_{r,1} \quad (r \geq 1),$$

we have

$$0 = -v \int_{-1}^1 (D\chi D\varphi^{(v-1)} + \alpha^2 \chi \varphi^{(v-1)}) dy + i \sum_{r=1}^v \binom{v}{r} \frac{d^r B}{dC^r} \int_{-1}^1 L\chi L\varphi^{(v-r)} dy. \quad (6.16)$$

Equation (6.16) gives the v^{th} derivative of B in terms of lower-order quantities.

7. Applications to Stability Theory

In this final section, we use the technique of parameter differentiation to obtain several results relevant to the theory of stability of the Boussinesq equations. The relevant boundary-value problem is fully discussed in [7]. A brief recapitulation of this problem follows:

$$\nabla \cdot \mathbf{v} = 0, \quad (7.1)$$

$$\text{Re}_\lambda \mathbf{v} \cdot \boldsymbol{\varepsilon} + \frac{R}{2\sqrt{\lambda}} (\lambda \nabla \psi + \mathbf{f}) \varphi = -\nabla P + \Delta \mathbf{v}, \quad (7.2)$$

$$\frac{R}{2\sqrt{\lambda}} (\lambda \nabla \psi + \mathbf{f}) \cdot \mathbf{v} = \Delta \varphi, \quad (7.3)$$

in D , and

$$\mathbf{v} = 0 \quad (\text{rigid surface}), \tag{7.4}$$

$$\mathbf{e} \cdot \mathbf{N} \times \mathbf{N}, \quad \mathbf{v} \cdot \mathbf{N} = 0 \quad (\text{free surface}), \tag{7.5}$$

$$\frac{\partial \varphi}{\partial N} + h \varphi, \tag{7.6}$$

on ∂D . Here we consider $\text{Re}_\lambda(R, \lambda)$ as the eigenvalue and R and λ as preassigned parameters*. The dimensionless field quantities are: \mathbf{v} , a difference velocity; \mathbf{e} , the symmetric part of $\nabla \mathbf{v}$; φ , a difference temperature; P , a difference pressure; $\nabla \psi$, gradient of the temperature field of the basic state; \mathbf{f} , the normalized gravity vector; $\boldsymbol{\varepsilon}$, the symmetric part of the dyadic gradient of the basic state-velocity field; h , a piecewise continuous function on ∂D ; \mathbf{N} , normal to ∂D .

The parameter R corresponds to $\sqrt{\text{Ra}}$ where Ra is a Rayleigh number. Our problem then is to find the minimum positive value of the Reynolds number $\widetilde{\text{Re}}_\lambda = \text{Min Re}_\lambda$ over the discrete positive set Re_λ . For each fixed λ , $\widetilde{\text{Re}}_\lambda(R)$ is a stability boundary. This leaves λ as a free parameter to be chosen so as to obtain the optimum stability boundary, *i.e.*,

$$\widetilde{\text{Re}}(R) = \text{Max}_{\lambda > 0} \widetilde{\text{Re}}_\lambda(R). \tag{7.7}$$

We note that the differential equations (7.1), (7.2), (7.3), are the Euler-Lagrange equations for a standard variational problem for which (7.18), (7.19), and (7.20) are the functionals (*cf.* [7]).

It has been shown ([7], Theorem 3) that

$$\frac{d\widetilde{\text{Re}}}{dR} = \frac{\partial \widetilde{\text{Re}}_\lambda}{\partial R} < 0, \tag{7.8}$$

for sufficiently small $R > 0$. This result can be extended as a

Theorem. *Let \mathbf{v} and φ solve the boundary value problem (7.1), (7.2), (7.3), (7.4) or (7.5) and (7.6). Further, let \mathbf{v} and φ give the least positive eigenvalue $\widetilde{\text{Re}}_\lambda$ for fixed values of $R > 0$ and $\lambda > 0$. Then $\widetilde{\text{Re}}_\lambda(R)$ is a monotone-decreasing function of R , convex from above. If $\nabla \psi = \mathbf{f}$, then $\widetilde{\text{Re}}(R)$ is also monotone-decreasing and convex from above.*

To prove this theorem, we note that $\mathbf{v}^{(1)}$ and $\varphi^{(1)}$ satisfy a boundary-value problem

$$\nabla \cdot \mathbf{v}^{(1)} = 0, \tag{7.9}$$

$$\text{Re}_\lambda^{(1)} \mathbf{v} \cdot \boldsymbol{\varepsilon} + A^{(1)} \varphi + \widetilde{\text{Re}}_\lambda \mathbf{v}^{(1)} \cdot \boldsymbol{\varepsilon} + A \varphi^{(1)} = -\nabla P^{(1)} + \Delta \mathbf{v}^{(1)}, \tag{7.10}$$

$$A^{(1)} \cdot \mathbf{v} + A \cdot \mathbf{v}^{(1)} = \Delta \varphi^{(1)}, \tag{7.11}$$

* It is more convenient for our purposes here to parametrize with R and λ than with the parameter choice (μ, λ) in [7]. In that reference $R_\lambda \sim R$, $\mu R_\lambda \sim \text{Re}_\lambda$ with $R_\lambda(\mu, \lambda)$ regarded as the eigenvalue. The results enunciated in that reference and this are, of course, independent of the parametrization used.

where

$$A = R \frac{(\lambda \nabla \psi + f)}{2\sqrt{\lambda}},$$

along with homogeneous boundary conditions

$$v^{(1)} = 0 \quad (\text{rigid surface}), \tag{7.12}$$

$$(e^{(1)} \cdot N) \times N, \quad v^{(1)} \cdot N = 0 \quad (\text{free surface}), \tag{7.13}$$

$$\frac{\partial \varphi^{(1)}}{\partial N} + h \varphi^{(1)} = 0. \tag{7.14}$$

For the boundary conditions on the second derivatives, we differentiate (7.12), (7.13), and (7.14), replacing superscript (1) with (2). The differential equations for $v^{(2)}$ and $\varphi^{(2)}$ are obtained by differentiation of (7.9), (7.10), and (7.11) as

$$\nabla \cdot v^{(2)} = 0, \tag{7.15}$$

$$\begin{aligned} \widetilde{R}e_{\lambda}^{(2)} v \cdot \varepsilon + A^{(2)} \varphi + 2 \widetilde{R}e_{\lambda}^{(1)} v^{(1)} \cdot \varepsilon + 2 A^{(1)} \varphi^{(1)} + \\ + \widetilde{R}e_{\lambda} v^{(2)} \cdot \varepsilon + A \varphi^{(2)} = -\nabla P^{(2)} + \Delta v^{(2)}, \end{aligned} \tag{7.16}$$

$$A^{(2)} \cdot v + 2 A^{(1)} \cdot v^{(1)} + A \cdot v^{(2)} = \Delta \varphi^{(2)}. \tag{7.17}$$

Multiply (7.2) by v , (7.3) by φ , integrate and add to obtain

$$\widetilde{R}e_{\lambda} \int v \cdot \varepsilon \cdot v + 2 \int A \cdot v \varphi = -[D(v, v) + \mathcal{D}(\varphi, \varphi)], \tag{7.18}$$

where

$$D(v, v) = 2 \int e : e \tag{7.19}$$

and

$$\mathcal{D}(\varphi, \varphi) = \int_D \nabla \varphi \cdot \nabla \varphi + \oint_{\partial D} h \varphi^2. \tag{7.20}$$

Multiply (7.10) by v and (7.2) by $v^{(1)}$, integrate and subtract to produce

$$\int A^{(1)} \cdot v \varphi - \widetilde{R}e_{\lambda}^{(1)} \int v \cdot \varepsilon \cdot v + \int A \cdot (\varphi v^{(1)} - \varphi^{(1)} v) = 0. \tag{7.21}$$

Multiply (7.11) by φ and (7.3) by $\varphi^{(1)}$, integrate and subtract to produce

$$\int A \cdot (v \varphi^{(1)} - v^{(1)} \varphi) - \int A^{(1)} \cdot v \varphi = 0. \tag{7.22}$$

Combine (7.21) and (7.22) to produce

$$\widetilde{R}e_{\lambda}^{(1)} \int v \cdot \varepsilon \cdot v + 2 \int A^{(1)} \cdot v \varphi = 0, \tag{7.23}$$

which is the formula governing the first variation of parameters.

We now obtain an inequality governing the second variation of parameters:

$$\widetilde{R}e_{\lambda}^{(2)} \int v \cdot \varepsilon \cdot v + 2 \int A^{(2)} \cdot v \varphi \geq 0. \tag{7.24}$$

This is constructed as follows: Multiply (7.15) by v , (7.2) by $v^{(2)}$, integrate and subtract to find

$$\begin{aligned} \widetilde{R}e_{\lambda}^{(2)} \int v \cdot \varepsilon \cdot v + \int A^{(2)} \cdot v \varphi = -2 \int A^{(1)} \cdot v \varphi - \\ - 2 \widetilde{R}e_{\lambda}^{(1)} \int v \cdot \varepsilon \cdot v^{(1)} + \int A \cdot (\varphi v^{(2)} - v \varphi^{(2)}). \end{aligned} \tag{7.25}$$

Multiply (7.3) by $\varphi^{(2)}$, (7.17) by φ , integrate and subtract to find

$$\int A \cdot (v \varphi^{(2)} - v^{(2)} \varphi) - 2 \int A^{(1)} \cdot v \varphi - \int A^{(2)} \cdot v \varphi = 0. \tag{7.26}$$

Combining (7.25) and (7.26), we have

$$\operatorname{Re}_\lambda^{(2)} \int v \cdot \varepsilon \cdot v + 2 \int A^{(2)} \cdot v \varphi + 2 \left[\int A^{(1)} \cdot (v \varphi^{(1)} + \varphi v^{(1)}) + \widetilde{\operatorname{Re}}_\lambda^{(1)} \int v \cdot \varepsilon \cdot v \right] = 0. \tag{7.27}$$

The equation

$$\begin{aligned} & \widetilde{\operatorname{Re}}_\lambda^{(1)} \int v \cdot \varepsilon \cdot v^{(1)} + \int A^{(1)} (v^{(1)} \varphi + \varphi^{(1)} v) \\ & = - \left[\mathcal{D}(\varphi^{(1)}, \varphi^{(1)}) + D(v^{(1)}, v^{(1)}) + \widetilde{\operatorname{Re}}_\lambda \int v^{(1)} \cdot \varepsilon \cdot v^{(1)} + 2 \int A \cdot v^{(1)} \varphi^{(1)} \right], \end{aligned} \tag{7.28}$$

which is obtained by addition of the integral of $v^{(1)}$ into (7.10) and $\varphi^{(1)}$ into (7.11), is combined with (7.28) to produce

$$\begin{aligned} & 2 \left[\mathcal{D}(\varphi^{(1)}, \varphi^{(1)}) + D(v^{(1)}, v^{(1)}) + \widetilde{\operatorname{Re}}_\lambda \int v^{(1)} \cdot \varepsilon \cdot v^{(1)} + \right. \\ & \left. + 2 \int A \cdot v^{(1)} \varphi^{(1)} \right] = \widetilde{\operatorname{Re}}_\lambda^{(2)} \int v \cdot \varepsilon \cdot v + 2 \int A^{(2)} \cdot v \varphi. \end{aligned} \tag{7.29}$$

We next note that $v^{(1)}$ and $\varphi^{(1)}$ are admissible functions for the minimum problem discussed in [8], so that

$$- \frac{\left[\widetilde{\operatorname{Re}}_\lambda \int v^{(1)} \cdot \varepsilon \cdot v^{(1)} + 2 \int A \cdot v^{(1)} \varphi^{(1)} \right]}{\mathcal{D}(\varphi^{(1)}, \varphi^{(1)}) + D(v^{(1)}, v^{(1)})} \leq 1. \tag{7.30}$$

Equation (7.30) and (7.29) imply (7.24).

Now consider (7.23) and (7.24) for variable R and constant λ . From (7.23)

$$\frac{\partial \widetilde{\operatorname{Re}}_\lambda}{\partial R} \int v \cdot \varepsilon \cdot v + \frac{1}{\sqrt{\lambda}} \int (\lambda \nabla \psi + f) \cdot v \varphi = 0. \tag{7.31}$$

It is easy to show (cf. [8]) that

$$\frac{\partial \widetilde{\operatorname{Re}}_\lambda}{\partial R} = \frac{2 \mathcal{D}(\varphi, \varphi)}{R \int v \cdot \varepsilon \cdot v}. \tag{7.32}$$

From (7.32) one may deduce the inequality (7.8). From (7.24)

$$\frac{\partial^2 \widetilde{\operatorname{Re}}_\lambda}{\partial R^2} \int v \cdot \varepsilon \cdot v \geq 0, \tag{7.33}$$

which when combined with (7.32) gives

$$\frac{\partial}{\partial R} \left[\log \left(\frac{\partial \widetilde{\operatorname{Re}}_\lambda}{\partial R} \right) \right] \geq 0. \tag{7.34}$$

Equation (7.34) is then integrated over R ($R_1 \leq R$) to produce

$$\log \frac{(\partial \widetilde{\operatorname{Re}}_\lambda / \partial R)}{(\partial \widetilde{\operatorname{Re}}_\lambda / \partial R)_1} \geq 0 \tag{7.35}$$

and

$$\frac{(\partial \tilde{\text{Re}}_\lambda / \partial R)}{(\partial \tilde{\text{Re}}_\lambda / \partial R)_1} \geq 1. \quad (7.36)$$

For R_1 sufficiently small we have through (7.8) that

$$\frac{\partial \tilde{\text{Re}}_\lambda}{\partial R} \leq \left(\frac{\partial \tilde{\text{Re}}_\lambda}{\partial R} \right)_1 \leq 0, \quad (7.37)$$

so that $\tilde{\text{Re}}_\lambda(R)$ is a monotonic decreasing, function of R . Also, by (7.32), $\int \mathbf{v} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}$ is negative implying through (7.33) that

$$\frac{\partial^2 \tilde{\text{Re}}_\lambda}{\partial R^2} \leq 0, \quad (7.38)$$

which proves the convexity $\tilde{\text{Re}}_\lambda(R)$.

It is shown in [8] that when $\nabla \psi = \mathbf{f}$, then $\lambda=1$ solves (7.7). This completes the proof of the theorem. If the hypotheses of the theorem are satisfied, stability is guaranteed against inertially-nonlinear disturbances for all pairs (Re, R) lying under a straight line drawn between any two points of the optimum stability boundary.

As our last application, we should like to obtain a very simple monotonicity result which is of interest in the theory of convection. It is known that the onset of convection in an initially quiescent fluid layer heated from below is given correctly by the linear theory under the assumption that the fluid properties are constant. On the other hand, several important features of the resulting convection, e.g., the convection cell structure and the direction of flow within the cell, seem to depend in a fundamental way on the temperature dependence of the viscosity (see SEGEL, [11]). For this reason, we should like to know how the stability limit is influenced by the variation of viscosity with temperature. To this end we consider the linear stability of an initially quiescent fluid layer heated from below. This system is governed by

$$-R \mathbf{i} \varphi = -\nabla P + \nabla \cdot (\mathbf{v} \mathbf{e}), \quad (7.39)$$

$$-R(\mathbf{i} \cdot \mathbf{v}) = \nabla^2 \varphi, \quad (7.40)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (7.41)$$

and the boundary conditions (7.4) or (7.5) and (7.6). The eigenfunction derivatives satisfy the boundary conditions, (7.41) and

$$-R^{(1)} \varphi \mathbf{i} - \nabla \cdot (\mathbf{v}^{(1)} \mathbf{e}) = -\nabla P^{(1)} + \mathbf{i} R \varphi^{(1)} + \nabla \cdot (\mathbf{v} \mathbf{e}^{(1)}), \quad (7.42)$$

$$-R^{(1)} \mathbf{i} \cdot \mathbf{v} = \nabla^2 \varphi^{(1)} + R(\mathbf{i} \cdot \mathbf{v}^{(1)}). \quad (7.43)$$

In the linear theory, the dependent variables are perturbations rather than differences.

The unit vector \mathbf{i} is antiparallel to gravity. The kinematic viscosity ν is spatially variable, its variation depending on the temperature distribution of the rest state.

Multiply (7.39) by $v^{(1)}$ and (7.42) by v , integrate and compare to obtain

$$-R \int \mathbf{i} \cdot (v^{(1)} \boldsymbol{\varphi} - v \boldsymbol{\varphi}^{(1)}) + \int v^{(1)} \mathbf{e} : \mathbf{e} + R^{(1)} \int (\mathbf{i} \cdot v) \boldsymbol{\varphi} = 0. \tag{7.44}$$

Multiply (7.40) by $\boldsymbol{\varphi}^{(1)}$ and (7.43) by $\boldsymbol{\varphi}$, integrate and compare to obtain

$$R^{(1)} \int \mathbf{i} \cdot v \boldsymbol{\varphi} = R \int \mathbf{i} \cdot (v \boldsymbol{\varphi}^{(1)} - v^{(1)} \boldsymbol{\varphi}). \tag{7.45}$$

Compare (7.44) and (7.45) to obtain

$$\int v^{(1)} \mathbf{e} : \mathbf{e} + 2 R^{(1)} \int (\mathbf{i} \cdot v) \boldsymbol{\varphi} = 0, \tag{7.46}$$

which may, using (7.40), be written as

$$\int v^{(1)} \mathbf{e} : \mathbf{e} = -2 R^{(1)} \mathcal{D}(\boldsymbol{\varphi}, \boldsymbol{\varphi})/R. \tag{7.47}$$

Now we shall use (7.47) to examine the consequences of a variable viscosity. Note first that the variation of the viscosity of Newtonian fluids with temperature may often be reasonably represented by a γ parameter family of functions of the form $v(\gamma \vartheta)$, where ϑ is a positive temperature difference and γ , a positive number. It is convenient to take as a reference the temperature at the coldest point of the system. At this point $\vartheta=0$ and $v(0)=1$. As examples of such representations for the viscosity, there are the relations $\exp(-\gamma \vartheta)$, which is widely used for oils, and $(1 \pm \gamma \vartheta)^{-1}$ which is a commonly adopted linear form for liquids and gases. It is usual that $v'(0) < 0$ for liquids, and $v'(0) > 0$ for gases.

Let γ be the parameter in (7.47). Then we have

$$\int v'(\gamma \vartheta) \vartheta \mathbf{e} : \mathbf{e} = -\frac{2}{R} \frac{dR}{d\gamma} \mathcal{D}(\boldsymbol{\varphi}, \boldsymbol{\varphi}), \tag{7.48}$$

and, for $\gamma=0$,

$$\frac{dR}{d\gamma} = -R v'(0) \int \vartheta \mathbf{e} : \mathbf{e} / 2 \mathcal{D}(\boldsymbol{\varphi}, \boldsymbol{\varphi}). \tag{7.50}$$

Equation (7.50) is a measure of the error in the Rayleigh number following from the assumption of constant viscosity ($\gamma=0$). This error, for small γ , is clearly proportional to the derivative of v with respect to its argument. It is clear that the critical Rayleigh numbers for small γ are higher for liquids and smaller for gases than those for which $\gamma=0$. The result however does depend on the location of the reference position for the unit viscosity and if this position is fixed in the channel interior the monotonicity result does not necessarily follow.

Appendix: Proof of Theorem 1.

Multiply (1.4) by $\boldsymbol{\varphi}^{(\mu)}$ and integrate over D using the divergence theorem and boundary conditions (1.5) to obtain

$$\begin{aligned} E_{v,\mu} &= \sum_{r=1}^v \binom{v}{r} \left\{ \int \boldsymbol{\varphi}^{(\mu)} [\Delta^{(r)} + \lambda^{(r)}] \boldsymbol{\varphi}^{(v-r)} - \oint \boldsymbol{\varphi}^{(\mu)} [(N \cdot \nabla)^{(r)} + B^{(r)}] \boldsymbol{\varphi}^{(v-r)} \right\} \\ &= \oint \beta \boldsymbol{\varphi}^{(v)} \boldsymbol{\varphi}^{(\mu)} + \int \nabla \boldsymbol{\varphi}^{(v)} \cdot \nabla \boldsymbol{\varphi}^{(\mu)} - \lambda \int \boldsymbol{\varphi}^{(v)} \boldsymbol{\varphi}^{(\mu)}. \end{aligned} \tag{A.1}$$

To obtain (1.7), we first note that

$$0 = E_{10} = \int \varphi [\Delta^{(1)} + \lambda^{(1)}] \varphi - \oint \varphi [(N \cdot \nabla)^{(1)} - \beta^{(1)}] \varphi. \quad (\text{A.2})$$

We make use of the following relations:

$$\frac{dJ}{dt} = J^{(1)} = \nabla \cdot \mathbf{V}, \quad (\text{A.3})$$

$$N_a^{(1)} = N_a \left(N_i N_j \frac{\partial V_i}{\partial x_j} \right) - \frac{\partial V_j}{\partial x_a} N_j, \quad (\text{A.4})$$

$$\left(\frac{\partial X_i}{\partial x_i} \right)^{(1)} = - \frac{\partial X_i}{\partial x_j} \frac{\partial V_j}{\partial x_i}, \quad (\text{A.5})$$

$$(g^{ij})^{(1)} = - \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) = -D_{ij}, \quad (\text{A.6})$$

$$(N \cdot \nabla)^{(1)} A = N_i N_j \frac{\partial V_i}{\partial x_j} \frac{\partial A}{\partial N} - N_j D_{ij} \frac{\partial A}{\partial x_i}, \quad (\text{A.7})$$

$$\Delta^{(1)} A = \frac{\partial(\nabla \cdot \mathbf{V})}{\partial x_i} \frac{\partial A}{\partial x_i} - \frac{\partial}{\partial x_i} \left[D_{ij} \frac{\partial A}{\partial x_j} \right], \quad (\text{A.8})$$

to find that (for any A and φ)

$$\begin{aligned} & \int \varphi \Delta^{(1)} A - \oint \varphi (N \cdot \nabla)^{(1)} A \\ &= \oint (\nabla \cdot \mathbf{V}) \varphi \frac{\partial A}{\partial N} - \int (\nabla \cdot \mathbf{V}) \left(\frac{\partial \varphi}{\partial x_i} \frac{\partial A}{\partial x_i} + \varphi \nabla^2 A \right) + \\ &+ \int D_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial A}{\partial x_j} - \oint N_i N_j \frac{\partial V_i}{\partial x_j} \varphi \frac{\partial A}{\partial N}. \end{aligned} \quad (\text{A.9})$$

With $A = \varphi$, we find, using (1.1) and (1.2), that

$$\begin{aligned} & \int \varphi \Delta^{(1)} \varphi - \oint \varphi (N \cdot \nabla)^{(1)} \varphi \\ &= \int D_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \int (\nabla \cdot \mathbf{V}) [\lambda \varphi^2 - (\nabla \varphi)^2] + \\ &+ \oint \beta \varphi^2 (N_i N_j - \delta_{ij}) \frac{\partial V_i}{\partial x_j}, \end{aligned}$$

which when combined with (A.2) proves equation (1.7):

We next show that with $\nu \geq 1$

$$\begin{aligned} & \sum_{r=2}^{\nu+1} \frac{\nu!}{(\nu+1-r)! r!} \left\{ \int \varphi [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(\nu+1-r)} - \oint \varphi [(N \cdot \nabla)^{(r)} + \beta^{(r)}] \varphi^{(\nu+1-r)} \right\} + \\ &+ \sum_{r=1}^{\nu} \binom{\nu}{r} \left\{ \int \varphi^{(1)} [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(\nu-r)} - \oint \varphi^{(1)} [(N \cdot \nabla)^{(r)} + \beta^{(r)}] \varphi^{(\nu-r)} \right\} + \\ &+ \int [\varphi \Delta^{(1)} \varphi^{(\nu)} - \varphi^{(\nu)} \Delta^{(1)} \varphi] - \oint [\varphi (N \cdot \nabla) \varphi^{(\nu)} - \varphi^{(\nu)} (N \cdot \nabla)^{(1)} \varphi] = 0. \end{aligned} \quad (\text{A.10})$$

To establish (A.10), we note that $E_{v+1,0} = E_{0,v+1}$ may be written as

$$\begin{aligned} & \lambda^{(1)} \int \varphi \varphi^{(v)} - \oint \beta^{(1)} \varphi \varphi^{(v)} + \int \varphi \Delta^{(1)} \varphi^{(v)} - \oint \varphi (N \cdot \mathcal{V})^{(1)} \varphi^{(v)} + \\ & + \sum_{r=2}^{v+1} \frac{v!}{(v+1-r)! r!} \{ \int \varphi [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(v+1-r)} - \\ & - \oint \varphi [(N \cdot \mathcal{V})^{(r)} + \beta^{(r)}] \varphi^{(v+1-r)} \} = 0. \end{aligned} \quad (\text{A.11})$$

Then, since $E_{1v} = E_{v1}$, we have

$$\begin{aligned} & \lambda^{(1)} \int \varphi \varphi^{(v)} - \oint \beta^{(1)} \varphi \varphi^{(v)} = - \int \varphi^{(v)} \Delta^{(1)} \varphi + \\ & + \oint \varphi^{(v)} (N \cdot \mathcal{V})^{(1)} \varphi + \sum_{r=1}^v \binom{v}{r} \{ \int \varphi^{(1)} [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(v-r)} - \\ & - \oint \varphi^{(1)} [(N \cdot \mathcal{V})^{(r)} + \beta^{(r)}] \varphi^{(v-r)} \}. \end{aligned} \quad (\text{A.12})$$

The first two terms of (A.11) are eliminated with (A.12) producing (A.10).

We eliminate the last two terms of (A.10) with the relation

$$\begin{aligned} & \int [\varphi \Delta^{(1)} \varphi^{(v)} - \varphi^{(v)} \Delta^{(1)} \varphi] - \oint [\varphi (N \cdot \mathcal{V})^{(1)} \varphi^{(v)} - \varphi^{(v)} (N \cdot \mathcal{V})^{(1)} \varphi] \\ & = \sum_{r=1}^v \binom{v}{r} \{ \int (\mathcal{V} \cdot \mathbf{v}) \varphi [\Delta^{(r)} + \lambda^{(r)}] \varphi^{(v-r)} + \\ & + \oint (N_i N_j - \delta_{ij}) \frac{\partial V_i}{\partial x_j} [(N \cdot \mathcal{V})^{(r)} + \beta^{(r)}] \varphi^{(v-r)} \}. \end{aligned} \quad (\text{A.13})$$

This relation may be verified by using (A.9) to form the L. H. S. of (A.9) followed by elimination $\mathcal{V}^2 \varphi^{(v)}$ and $\partial \varphi^{(v)} / \partial N$ with (1.4) and (1.5). This produces (1.9). To obtain (1.8) set $v=1$ in (1.9) and utilize the relation

$$E_{11} = \int \varphi^{(1)} [\Delta^{(1)} + \lambda^{(1)}] \varphi - \oint \varphi^{(1)} [(N \cdot \mathcal{V})^{(1)} + \beta^{(1)}] \varphi.$$

This gives (1.8) and completes the proof of Theorem 1.

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