

Evaluation of Tietjens Function in Stability Calculations

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Tietjens function is re-expressed as a ratio of rapidly converging power series of its (complex) argument which may be utilized to replace tables or graphs in the calculation of critical Reynolds numbers from asymptotic theory.

THE Tietjens function is an auxiliary function in the asymptotic theory of the Orr-Sommerfeld equation. It arises naturally in connection with the viscous solutions of that equation (for instance, Lin,¹ Stuart,² Shen,³ Reid,⁴ and others).

In the formulation of the eigenvalue problem for the Orr-Sommerfeld system, the derivatives of the aforementioned viscous solutions at the wall are encountered. These quantities are conveniently expressed in terms of the Tietjens function F , which is defined as

$$F(z) = \frac{\int_{-\infty}^{-z} d\gamma \int_{\infty}^{\gamma} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)} \left[\frac{2}{3}(i v)^{\frac{3}{2}} \right] dv}{-z \int_{\infty}^{-z} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)} \left[\frac{2}{3}(i v)^{\frac{3}{2}} \right] dv}, \quad (1)$$

where

$$z = (\alpha R W')^{\frac{1}{2}} (y_e - y_1) \quad (2)$$

$$D(z) = -\sqrt{2} (1 - i) \left[\left(1 + \frac{i}{\sqrt{3}} \right) \sum_{k=1}^{\infty} \frac{z^{3k-1} e^{-\frac{1}{2}k\pi i}}{(3)^{2k-2/3} (2k - \frac{2}{3}) \Gamma(k) \Gamma(k + \frac{1}{3})} + \frac{2}{\sqrt{3}} \sum_{k=1}^{\infty} \frac{z^{3k-2} e^{-\frac{1}{2}k\pi i}}{(3)^{2k-4/3} (2k - \frac{4}{3}) \Gamma(k) \Gamma(k - \frac{1}{3})} - \frac{1}{2} \left(i + \frac{1}{\sqrt{3}} \right) \right]. \quad (5)$$

The series (4) and (5) are valid everywhere in the finite z plane, in particular on the positive real axis, and converge rapidly even for relatively large real values (< 10) of z .

To facilitate the proof of Eq. (3), we first rewrite $F(z)$ by making use of the partial integration

$$\begin{aligned} \int_{\infty}^{-z} d\gamma \int_{\infty}^{\gamma} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)} \left[\frac{2}{3}(i v)^{\frac{3}{2}} \right] dv \\ = -z \int_{\infty}^{-z} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)} \left[\frac{2}{3}(i v)^{\frac{3}{2}} \right] dv \\ - \int_{\infty}^{-z} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)} \left[\frac{2}{3}(i v)^{\frac{3}{2}} \right] dv. \end{aligned} \quad (6)$$

and the symbols are those conventionally used in stability theory (cf. Lin¹ and Stuart²). The function $F(z)$ has a simple pole at $z = 0$. It is usual that for neutral stability the value of $y = y_1$ at the wall is less than $y = y_e$ at the point at which $W = c$. In this case z is a real, positive quantity. In the sequel we consider complex z .

In light of the foregoing, it is evident that the Tietjens function plays an important role in asymptotic hydrodynamic stability theory. Tabulations of the Tietjens function $F(z)$ and of the modified Tietjens function $\mathcal{F} = (1 - F)^{-1}$ have been given by Tietjens,⁵ Lin,⁶ Holstein,⁷ Lock,⁸ and Miles.⁹ Among these, the tables of Miles are the most extensive and also the most accurate.

With the background of the Tietjens function thus established, consideration may now be given to its reduction to series form.

The result to be derived here is

$$F(z) = 1 + [N(z)/z D(z)], \quad (3)$$

where

$$\begin{aligned} N(z) = \frac{1}{\sqrt{2}} (1 + i) \\ \cdot \left[\left(1 + \frac{i}{\sqrt{3}} \right) \sum_{k=1}^{\infty} \frac{(3k - 2) z^{3k-3} e^{-\frac{1}{2}k\pi i}}{(3)^{2k-5/3} \Gamma(k) \Gamma(k + \frac{1}{3})} + \frac{2}{\sqrt{3}} \sum_{k=1}^{\infty} \frac{(3k - 3) z^{3k-4} e^{-\frac{1}{2}k\pi i}}{(3)^{2k-7/3} \Gamma(k) \Gamma(k - \frac{1}{3})} \right], \end{aligned} \quad (4)$$

After substitution into Eq. (1), there emerges

$$F(z) = 1 + \frac{N(z)}{z D(z)},$$

$$N(z) = \int_{\infty}^{-z} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)} \left[\frac{2}{3}(i v)^{\frac{3}{2}} \right] dv, \quad (7)$$

$$D(z) = \int_{\infty}^{-z} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)} \left[\frac{2}{3}(i v)^{\frac{3}{2}} \right] dv.$$

To reduce $N(z)$ and $D(z)$ to series form, we use the decomposition

$$\begin{aligned} H_{\nu}^{(1)}(t) &= J_{\nu}(t) + i Y_{\nu}(t) \\ &= [J_{-\nu}(t) - e^{-\nu\pi i} J_{\nu}(t)] / i \sin \nu\pi, \end{aligned} \quad (8)$$

(where t is complex) in conjunction with the series representation of the Bessel function J_ν ,

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}t)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}. \quad (9)$$

When combined with (8), this produces

$$H_{\frac{1}{2}}^{(1)}(t) = \left(1 + \frac{i}{\sqrt{3}}\right) \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}t)^{2k+\frac{1}{2}}}{\Gamma(k+1)\Gamma(k+\frac{4}{3})} - \frac{2i}{\sqrt{3}} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}t)^{2k-\frac{1}{2}}}{\Gamma(k+1)\Gamma(k+\frac{2}{3})}$$

or

$$H_{\frac{1}{2}}^{(1)}[\frac{2}{3}(iv)^{\frac{1}{2}}] = \left(1 + \frac{i}{\sqrt{3}}\right) \sum_{k=0}^{\infty} \frac{(-1)^k (i)^{3k+\frac{1}{2}} (v)^{3k+\frac{1}{2}}}{(3)^{2k+\frac{1}{2}} \Gamma(k+1)\Gamma(k+\frac{4}{3})} - \frac{2i}{\sqrt{3}} \sum_{k=0}^{\infty} \frac{(-1)^k (i)^{3k-\frac{1}{2}} (v)^{3k-\frac{1}{2}}}{(3)^{2k-\frac{1}{2}} \Gamma(k+1)\Gamma(k+\frac{2}{3})}, \quad (10)$$

and $v^{\frac{1}{2}}H_{\frac{1}{2}}^{(1)}[\frac{2}{3}(iv)^{\frac{1}{2}}]$ is an entire function of v .

Now, we are in a position to reduce

$$D(z) = \int_{\infty}^{-z} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)}[\frac{2}{3}(iv)^{\frac{1}{2}}] dv = \int_0^{-z} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)}[\frac{2}{3}(iv)^{\frac{1}{2}}] dv - \int_0^{\infty} v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)}[\frac{2}{3}(iv)^{\frac{1}{2}}] dv \quad (11)$$

to series form. The first integral on the right-hand side of (11) is obtained by direct integration of the power series representation (10). The second integral may be obtained by substituting $\frac{2}{3}v^{\frac{1}{2}} = z$ and using the relation [Bateman,¹⁰ Eq. (27), p. 51]

$$\int_0^{\infty} H_{\frac{1}{2}}^{(1)}[ze^{3\pi i/4}] dz = \left(i + \frac{1}{\sqrt{3}}\right) e^{-5\pi i/4}. \quad (12)$$

The result of the just described operations is the series representation, Eq. (5).

To reduce $N(z)$ to series form, we utilize Miles'⁹ observation that

$$f(v) = v^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)}[\frac{2}{3}(iv)^{\frac{1}{2}}] \quad (13)$$

satisfies the differential equation

$$f''(v) - v f(v) = 0 \quad (14)$$

with the boundary condition

$$f(\infty) = 0. \quad (15)$$

It follows that

$$N(z) = \int_{\infty}^{-z} v f(v) dv = -i f'(-z). \quad (16)$$

The series representation for $N(z)$, Eq. (4), is obtained by differentiating the power series representation of $f(-z)$ as embodied in Eqs. (10) and (13).

This completes the proof of the representation of the Tietjens function by Eq. (3). We close with a few remarks about the numerical evaluation of (3).

A sample calculation of a table of values of $F_r(z)$ and $F_i(z)$ for $z = 1.0$ (0.2) 10 was performed in less than 20 sec on a CDC 1604 digital computer.¹¹ The criterion for the convergence of each of the four series appearing in Eq. (3) was set so that the calculations were terminated when

$$|S_{N+1} - S_N|/S_{N+1} \leq 10^{-8},$$

where S_N is the partial sum of the absolute value of each of the terms of a given series. For $z = 1$, the four series had converged after 7 terms, whereas for $z = 10$, 28 terms were necessary. The calculation which was carried out with six-figure accuracy agrees with Miles'⁹ four-figure results within the significance of the latter. Convergence characteristics of (3) were not investigated for $z > 10$; for these values, the asymptotic representation of $F(z)$ given by Miles is adequate.

The representation (3) circumvents the need for tables of $F(z)$ when stability calculations are carried out with the aid of a digital computer. The convergence of the series is sufficiently rapid so that (3) may be included as a standard element in any computer program for calculating eigenvalues from the asymptotic solutions of the Orr-Sommerfeld equation.

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