Subcritical convective instability

Part 1. Fluid layers

By DANIEL D. JOSEPH AND C. C. SHIR

J. Fluid Mech. (1966), *vol.* 26, *part* 4, *pp.* 753–768
Printed in Great Britain

Subcritical convective instability

Part 1. Fluid layers

By DANIEL D. JOSEPH AND C. C. SHIR

Department of Aeronautics and Engineering Mechanics, University of Minnesota

(Received 17 February 1966)

This paper elaborates on the assertion that energy methods provide an always mathematically rigorous and a sometimes physically precise theory of subcritical convective instability. The general theory, without explicit solutions, is used to deduce that the critical Rayleigh number is a monotonically decreasing function of the Nusselt number, that this decrease is very slow if the Nusselt number is large, and that a fluid layer heated from below and internally is definitely stable when $RA < \widetilde{RA}(N_s) > 1708/(N_s+1)$ where N_s is a heat source parameter and $\widetilde{RA}$ is a critical Rayleigh number. This last problem is also solved numerically and the result compared with linear theory. The critical Rayleigh numbers given by energy theory are slightly less than those given by linear theory, this difference increasing from zero with the magnitude of the heat-source intensity. To previous results proving the non-existence of subcritical instabilities in the absence of heat sources is appended this result giving a narrow band of Rayleigh numbers as possibilities for subcritical instabilities.

1. Introduction

The energy method judges stability or instability of a given fluid motion by whether the energy of a disturbance of the given motion grows or decays. If the values of certain stability parameters are below critical values, the energy decreases and the hydrodynamic system is called stable. Reynolds (1895) used an equation for the global kinetic energy of simple perturbation flows to estimate values of the critical Reynolds number. Orr (1907) used the same equation to formulate a variational problem for finding the critical Reynolds number. The calculation procedures and the results associated with this older use of the energy method are summarized by Bateman, Dryden & Murnaghan (1932). These early results of the energy method gave such conservative estimates for the critical Reynolds number, that the method was neglected for many years.

The modern theory of energy dates from the work of Thomas (1942) and Serrin (1959). In the modern theory one considers the global energy of a difference motion. The global energy, kinematic conditions and boundary constraints are used in two lines of deduction. The first of these leads to a universal stability criterion, universal in the sense that specific details of the basic motion and details of the flow geometry need not be completely specified. A second line of deduction leads to the formulation of a maximum problem and achieves a sharper result by making more efficient use of known details of the basic flow. The procedure is elegantly developed in Serrin's (1959) paper. The results are valuable because they apply to a difference motion and, therefore, guarantee stability to finite disturbances. It is of importance that the method can give results which are not too conservative, as is obvious from Serrin's (1959) calculation of the stability limits for Couette flow between rotating cylinders. That even stronger results are possible was demonstrated in Joseph's (1965, 1966; hereafter referred to as I and II) extension of the method to accommodate convective motions governed by the non-linear equations of Boussinesq. It is with these convective motions, particularly with those which start from rest, that the most powerful and physically meaningful results of the energy method can be associated. These results are briefly listed below:

(i) There exists a neighbourhood of the origin of the Rayleigh–Reynolds number plane in which all Boussinesq flows which satisfy certain natural boundary conditions and which can be contained in a sphere of a diameter d are universally stable. No matter how large the disturbance, it will eventually die away (see I).

(ii) A variational technique which uses the details of the basic motion to be studied can be defined and used to extend the region of certain stability. The object of these calculations is the specification of the largest region in the Rayleigh–Reynolds number plane in which the basic motion is certainly stable. The boundary of this largest region is called the optimum stability boundary. If (Re, Ra) lie within the optimum boundary, then the energy of the difference motion decays to zero as time goes to infinity, and stability (in the mean) is rigorously guaranteed (see II).

(iii) A general description of the optimum stability boundary in terms of maximizing functions of the Euler–Lagrange equations can be constructed. This description leads to the result that the Reynolds number is a decreasing function of the Rayleigh number, for small Rayleigh numbers, on the optimum stability boundary. It also leads to a general *a priori* criterion for non-existence of subcritical instabilities (see II).

(iv) Rigid rotation cannot destabilize the class of flows which satisfy the criterion for non-existence of subcritical instabilities (see II).

(v) Plane Couette flow heated from below is stable to arbitrary disturbances when $Re^2 + Ra < 1708$. For $Ra = 0$ and $Re < 41{\cdot}3$, the flow is stable, replacing the celebrated value $Re = 88{\cdot}6$ given by Orr (1907) (see II).

It is our view that energy theory complements linear theory. Linear theory, roughly speaking, gives conditions under which hydrodynamic systems are definitely unstable. It cannot with certainty conclude stability. Energy theory gives conditions under which hydrodynamic systems are definitely stable. It cannot with certainty conclude instability. Comparison of the stability limits as given by energy and linear theory yields the range of values of relevant stability parameters in which subcritical instabilities of the hydrodynamic system are possible.

In the two parts of this paper we propose to elaborate on the assertion that energy methods provide an always mathematically rigorous and a sometimes physically precise theory of subcritical convective instability. We shall restrict our attention to those initially quiet motions for which the theory seems physically precise. The general formulation of the theory is given in I and II, but since the method and associated criteria are neither conventional nor widely known, we have included in part 1 a somewhat longer than usual review of previously published work.

The principal result of part 1 is that a horizontal layer of fluid heated from below and *internally* will not be unstable to arbitrary disturbances for Rayleigh numbers which are only slightly less than those given by small perturbation theory and which increase from zero with the heat-source intensity. That is, there is only a narrow band of Rayleigh numbers for which subcritical instabilities are possible. Similar results for convection in spherical shells are obtained in part 2. It is shown that no subcritical instabilities are possible in spherical shells when the gravity and temperature-gradient variations are identical. Even when subcritical instabilities are possible, they may, as in the cases treated by Chandrasekhar (1961), be confined to a narrow band of Rayleigh numbers. The important implications of these facts and their relation to the often invoked principle of exchange of stability are explored in the conclusion of part 2.

2. Energy identities for the difference motion

The essential elements of the energy method as this is applied to Boussinesq fluids evolve from deductions made from the energy identities†

$$\frac{dK}{d\tau} = \frac{d}{d\tau}\int \tfrac{1}{2}v^2 = -\int (Re\,\mathbf{v}\,.\,\boldsymbol{\epsilon}\,.\,\mathbf{v} + \sqrt{Ra}\,\mathbf{f}\,.\,\mathbf{v}\theta + 2\mathbf{e}:\mathbf{e}), \tag{1}$$

and

$$Pr\frac{d\Theta}{d\tau} = Pr\frac{d}{d\tau}\int \tfrac{1}{2}\theta^2 = -\int (\sqrt{Ra}\,\nabla\psi\,.\,\mathbf{v}\theta + \nabla\theta:\nabla\theta) - \oint h\theta^2. \tag{2}$$

Here $\mathscr{V} = \mathscr{V}(\tau)$ is a region of space (which may change with time $t = d^2\tau/\nu$) occupied by the basic fluid motion; $\mathbf{u} = \mathbf{V}^* - \mathbf{V} = \mathbf{v}\sqrt{(\alpha g\kappa/\nu\beta)}$ and $\theta = T^* - T$ are, respectively, the differences of velocity and temperature between the disturbed (starred) and undisturbed (unstarred) motion;

$$(\mathbf{D})_{ij} = \tfrac{1}{2}(V_{i,j} + V_{j,i}) = m(\boldsymbol{\epsilon})_{ij},$$
$$d = \mathbf{D}^* - \mathbf{D} = \mathbf{e}\sqrt{(\alpha g\kappa/d^2\nu\beta)}, \quad \nabla T = \beta\nabla\psi \quad \text{and} \quad \mathbf{g}(\mathbf{r},\tau) = g\mathbf{f}$$

are, respectively, the strain-rate tensor of the basic and difference motions, the gradient of the temperature of the basic fluid motion and the prescribed field force (typically, gravity) vector. The constants $-m$, β and g are maximum values of the characteristic values of $\mathbf{D}$, ∇T and $\mathbf{g}$ in the time interval $[0, \tau]$ respectively. The constants κ, α and ν are, respectively, the thermometric coeffi-

† In writing integrals we shall omit infinitesimal volume elements; moreover, all integrals are understood to be extended over the entire (dimensionless) region, except for integrals over $\mathscr{S}$, the boundary of $\mathscr{V}$, which are indicated by a circle drawn through the integral sign.

cient, the coefficient of thermal expansion and the kinematic viscosity. All lengths are measured in units of a fixed reference length d and

$$Ra = \alpha\beta g d^4/\nu\kappa, \quad Re = d^2 m/\nu \quad \text{and} \quad Pr = \nu/\kappa.$$

Consistent with the requirement that the two flows satisfy the same conditions at the boundary $\mathscr{S}$ of $\mathscr{V}$ are

$$\mathbf{v} = 0 \quad \text{(rigid surface, velocity } \mathbf{V} \text{ prescribed)}, \tag{3}$$

or

$$(\boldsymbol{\epsilon}.\mathbf{N})\times\mathbf{N} = 0, \quad \mathbf{v}.\mathbf{N} = 0 \quad \text{(free surface, normal velocity } \mathbf{V}.\mathbf{N} \text{ prescribed)}, \tag{4}$$

and a Robin condition

$$\frac{\partial\theta}{\partial N} + h\theta = 0 \tag{5}$$

for the temperature. Here $\mathbf{N}$ is the outward normal to $\mathscr{S}$, $h(\mathbf{r},\tau) \geqslant 0$ is piecewise continuous function of position (Nusselt number), $\boldsymbol{\epsilon}.\mathbf{N}$ is proportional to the viscous part of the surface tractions which are assumed entirely normal. A mixture of these conditions may prevail on subelements of $\mathscr{S}$.

(1) and (2) follow from the integration of suitably multiplied differential equations (made dimensionless) governing the difference motion over the volume $\mathscr{V}$. It is, of course, necessary that the boundary terms which arise from application of the divergence theorem vanish; a condition which is assured (see 1 and Serrin (1959)) when $\mathscr{S}$ is closed, when the geometry is such that disturbances are sufficiently spatially periodic or when the disturbances are sufficiently localized. The equations for the difference motion (in physical variables) are formed by subtracting the Boussinesq equations for the basic (unstarred) flow,

$$\frac{d\mathbf{V}}{dt} = -\frac{\nabla p}{\rho_0} + \{1-\alpha(T-T_0)\}\mathbf{g} + 2\nu\nabla.\mathbf{D}, \tag{6}$$

$$\frac{dT}{dt} = \kappa\nabla^2 T + Q(\boldsymbol{\chi}, t), \tag{7}$$

$$\nabla.\mathbf{V} = 0, \tag{8}$$

from the same equations for the disturbed (starred) flow. Here T_0 and $Q(\boldsymbol{\chi}, t)$ are, respectively, a prescribed reference temperature and a prescribed heat-source function.

Subsequent deductions about the stability of the difference motion are extracted from the energy identities (1) and (2), the boundary constraints (3), (4) and (5) and kinematic constraint

$$\operatorname{div}\mathbf{v} = 0. \tag{9}$$

The local non-linear conservation equations do not play a direct role in further construction of the theory.

Two lines of deduction which start from the energy identities are possible. The first of these leads to a criterion for universal stability. The universal criterion does not depend on details of the motion or geometry of the basic flow. When satisfied, the criterion guarantees asymptotic stability in the sense of an exponential decay of disturbances of any magnitude (see I for details). The region of certain stability can, however, be extended by a sharper criterion which makes more efficient use of details of the basic flow. This leads to a second line of deduction which we call 'the problem of the optimum stability boundary'.

The problem of the optimum stability boundary as this is formulated in II consists of finding the largest region in Rayleigh–Reynolds number plane in which a given fluid motion is surely stable. In § 3 below we shall briefly review the structure of this problem. It should be stressed that this particular parameter emphasis is representative. In § 4 below a different parameter emphasis is considered and the stability boundary is defined in a heat-source parameter N_s, Rayleigh number plane.

3. The problem of the optimum stability boundary

To begin we introduce a coupling parameter and define an 'energy'

$$E_\lambda = K + \lambda Pr\Theta.$$

The requirement that this energy be positive is equivalent to the restriction that $\lambda > 0$. We next simplify the problem by introducing another positive parameter $\mu(0 \leqslant \mu \leqslant \infty)$ by the relation $Re = \mu\sqrt{Ra}$. We regard μ as preassigned and use it to eliminate explicit dependence on the Reynolds number. Introduce the notation

$$\begin{aligned} I_1 &= \int(\mu\mathbf{v}.\boldsymbol{\epsilon}.\mathbf{v} + \mathbf{f}.\mathbf{v}\theta), & I_2 &= \int\nabla\psi.\mathbf{v}\theta, \\ D &= 2\int\mathbf{e}:\mathbf{e}, & \mathscr{D} &= \int\nabla\theta.\nabla\theta + \oint h\theta^2, \\ I_\lambda &= I_1 + \lambda I_2, & D_\lambda &= D + \lambda\mathscr{D}, \end{aligned}$$

and form the inequality

$$\begin{aligned} \frac{dE_\lambda/d\tau}{D_\lambda} &= -1 + \sqrt{Ra}(-I_\lambda/D_\lambda) \leqslant -1 + \sqrt{Ra}\max(-I_\lambda/D_\lambda) \\ &= -1 + \sqrt{Ra}/\rho, \end{aligned}$$

or

$$dE_\lambda/d\tau \leqslant -(1 - \sqrt{Ra}/\rho)\,D_\lambda, \tag{10}$$

where

$$\rho^{-1} = \rho^{-1}(\lambda, \mu) = \max(-I_\lambda/D_\lambda).$$

From the inequality (11) one obtains the following result:

Let the inequalities

$$\tfrac{1}{2}a^2\int v^2 \leqslant D(\mathbf{v}, \mathbf{v}),$$
$$\tfrac{1}{2}Prb^2\int\theta^2 \leqslant \mathscr{D}(\theta, \theta),$$

with $a^2 > 0$, and $b^2 > 0$, hold. Then, if for any fixed values $\lambda > 0$ and $\mu \geqslant 0$, $\sqrt{Ra} < \rho(\lambda, \mu)$ in the time interval $[0, \tau]$, we have

$$E_\lambda(\tau) \leqslant E_\lambda(0)\exp\{-(1 - \sqrt{Ra}/\rho)\,\xi^2\tau\}, \tag{11}$$

where $E_\lambda(0)$ is the initial energy of the difference motion and $\xi^2 = \min(a^2, b^2)$. If $\sqrt{Ra} < \rho$ for all τ then $E_\lambda \to 0$, and the flow is asymptotically stable in the mean.†

† This result, which was derived jointly by Joseph and Serrin (see II), constitutes a firm basis for the maximum problem defining the numbers $\rho^{-1}(\lambda, \mu)$. It reduces the problem of finding limits sufficient for stability to a standard problem in the calculus of variations.

Proof. Let the assumed inequalities hold. Then

$$E_\lambda = \tfrac{1}{2}\int(v^2+\lambda Pr\theta^2) \leqslant a^{-2}D(\mathbf{v},\mathbf{v})+\lambda b^{-2}\mathscr{D}(\theta,\theta) \leqslant \xi^{-2}D_\lambda,$$

which may be combined with (10) to produce

$$dE_\lambda/d\tau \leqslant -(1-\surd Ra/\rho)\,D_\lambda \leqslant -\xi^2(1-\surd Ra/\rho)\,E_\lambda. \tag{12}$$

This last inequality is then integrated on $[0,\tau]$, proving (11) and the theorem.

The hypotheses of the theorem are not very restrictive. It is clear that $\frac{1}{2}a^2$ is the smallest of the eigenvalues associated with the vector Helmholtz equation for $\mathbf{v}$ and the conditions (3), (4) and (10). The quantity $\frac{1}{2}Pr\,b^2$ is similarly identified as the least eigenvalue of the scalar Helmholtz equation for θ and the condition (5). In nearly all situations encountered in applications, the existence of a positive, least-eigenvalue can be assumed and in many instances proved (see II for references).

Roughly speaking then, stability is guaranteed if $\surd Ra < \rho(\lambda,\mu)$. This leads naturally to the formulation of a maximum problem for the number $1/\rho$. This number is to be sought as the maximum value of the expression

$$-\frac{I_\lambda}{D_\lambda} = \frac{-I_1(\mathbf{v},\theta)-\lambda I_2(\mathbf{v},\theta)}{D(\mathbf{v},\mathbf{v})+\lambda\mathscr{D}(\theta,\theta)} \tag{13}$$

over a field of twice-continuously differentiable functions θ and $\mathbf{v}$, satisfying (3), (4), (5) and (9).

This maximum problem generates a field of values $\rho(\lambda,\mu)$ for each (λ,μ) parameter pair. Since for each fixed value of μ the flow is stable provided only that $\surd Ra < \rho(\lambda,\mu)$, we may select λ so as to give the best possible limit for stability. Since this best limit is clearly that for which Ra is largest, we seek the largest of the values of $\rho(\lambda,\mu)$ over λ for a fixed μ, and define

$$R(\mu) = \max_{\lambda>0}\rho(\lambda,\mu). \tag{14}$$

The locus of values $R(\mu)$ gives the optimum stability boundary, $F(\surd\widetilde{Ra},\widetilde{Re}) = 0$, parametrically through the equations $\surd\widetilde{Ra} = R(\mu)$ and $\widetilde{Re} = \mu R(\mu)$. The value of $\lambda = \lambda(\mu)$, which is associated with the maximum ρ, i.e.

$$R(\mu) = \rho\{\lambda(\mu),\mu\}$$

is called the best value for the coupling parameter λ. If this best value is assumed finite, then it may be found as a root of the equation

$$\left(\frac{\partial\rho(\lambda,\mu)}{\partial\lambda}\right)_\mu = 0. \tag{15}$$

It follows that

$$\frac{dR}{d\rho} = \left(\frac{\partial\rho}{\partial\mu}\right)_\lambda + \left(\frac{\partial\rho}{\partial\lambda}\right)_\mu\frac{d\lambda}{d\mu} = \left(\frac{\partial\rho(\lambda,\mu)}{\partial\mu}\right)_\lambda, \tag{16}$$

and $R(\mu)$ is an envelope of the curves $\rho\,(\text{const.},\mu)$ depending on the parameter $\lambda = \text{const.}$

A summary statement of the structure of the problem may be readily grasped from figure 1. In this figure, I is the region of universal stability. The problem of the optimum stability boundary is posed so as to delineate a larger region of certain stability (II) by using the details of the basic motion. The stability boundary $F(\sqrt{\widetilde{Ra}}, \widetilde{Re}) = 0$ is determined as follows: we first fix μ. This determines a ray from the origin. A set of maximizing eigenvalues $1/\rho$ are then found for different λ and the fixed μ. The λ which produces the maximum value of ρ on the given ray determines the critical value $R(\mu)$. The corresponding critical Reynolds number is given parametrically by $\mu R(\mu)$. The stability boundary $F(\sqrt{\widetilde{Ra}}, \widetilde{Re}) = 0$ is generated as μ takes on allowed values in the first quadrant.

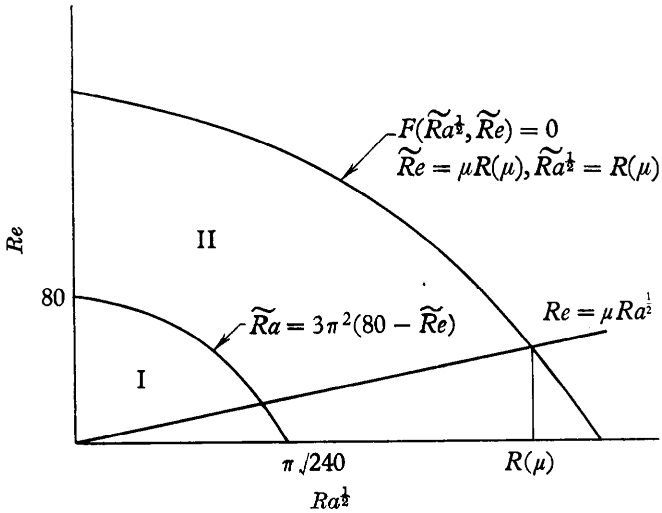

FIGURE 1. Stability regions and stability boundary. All flows for which boundary temperatures and velocities are prescribed and which can be contained in a sphere of diameter d are stable in I. By using known details of the basic motion this region is extended to the boundary of the largest region, that is, the optimum boundary $F(\sqrt{\widetilde{Ra}}, \widetilde{Re}) = 0$. Eigenvalues $\rho(\lambda, \mu)$ lie in the stable regions I and II; $R(\mu) = \max_{\lambda>0} \rho(\lambda, \mu)$.

It should be noted that in the preceding development we have chosen to suppress the possible dependence of the system on parameters other than μ. This is not an essential feature of theory, and these and subsequent remarks apply when there are other parameters which characterize the basic state. It is precisely these other parameters with which we are concerned in the applications which follow.

4. Generalization and solution of the problem of the optimum stability boundary

The maximum problem (13) is easily formulated in the framework of variational calculus. We require that

$$-\{I_1(\mathbf{v}, \theta) + \lambda I_2(\mathbf{v}, \theta)\} = \max = 1/\rho(\lambda, \mu) \tag{17}$$

hold for a class of twice-continuously-differentiable functions $\mathbf{v}$ and θ satisfying (3), (4), (5) and (9) and the normalizing condition

$$D(\mathbf{v}, \mathbf{v}) + \lambda \mathscr{D}(\theta, \theta) = 1. \tag{18}$$

Lagrange multipliers R_λ and $P(x,y,z,t)$ are then introduced, and (17), (3), (4), (5), (9) and (18) are reformulated in a system of partial differential equations by requiring

$$\delta\left\{I_1(\mathbf{v},\theta)+\lambda I_2(\mathbf{v},\theta)-\frac{2p}{\int R_\lambda}\nabla\cdot\mathbf{v}+\frac{1}{R_\lambda}(D(\mathbf{v},\mathbf{v})+\lambda\mathscr{D}(\theta,\theta))\right\}=0. \qquad (19)$$

The Euler–Lagrange equations corresponding to (19) are

$$\tfrac{1}{2}(R_\lambda/4\lambda)(\lambda\nabla\psi+\mathbf{f})\cdot\mathbf{v}=\nabla^2\theta, \qquad (20)$$

$$\mu R_\lambda\mathbf{v}\cdot\boldsymbol{\epsilon}+\tfrac{1}{2}R_\lambda(\lambda\nabla\psi+\mathbf{f})\theta=-\nabla p+\nabla^2\mathbf{v}, \qquad (21)$$

which are to be solved subject to (3), (4), (5) and (9). It is easy to establish that for any normalized solution of the Euler–Lagrange equations and side conditions

$$-I_1(\mathbf{v},\theta)-\lambda I_2(\mathbf{v},\theta)=1/R_\lambda. \qquad (22)$$

Hence it follows that the values

$$\rho(\lambda,\mu)=\min R_\lambda(\mu) \qquad (23)$$

for any of the positive set of eigenvalues R_λ. Also

$$R(\mu)=\max_{\lambda>0}\rho(\lambda,\mu)=\max_{\lambda>0}(\min R_\lambda(\mu)). \qquad (24)$$

Given the solutions to the maximum problem, i.e. the numbers $\rho^{-1}(\lambda,\mu)$ and the corresponding eigenfunctions $\tilde{\mathbf{v}}$, $\tilde{\theta}$, the problem of finding the best value for the coupling parameter λ and the associated stability boundary may be easily resolved (see II). The technique used to resolve this problem yields the result that

$$\lambda=\frac{\int\mathbf{f}\cdot\tilde{\mathbf{v}}\tilde{\theta}}{\int\nabla\psi\cdot\tilde{\mathbf{v}}\tilde{\theta}}. \qquad (25)$$

This result is independent of the nature of the basic state, and applies not only when the system dependence on parameters other than the Rayleigh and Reynolds number is suppressed, that is, when $\lambda=\lambda(\mu)$, but also generally.

(25) is a direct consequence of the following relation:

$$\int\tilde{\mathbf{v}}\cdot\delta(\mu\sqrt{\lambda}\boldsymbol{\epsilon})\cdot\tilde{\mathbf{v}}+\sqrt{\lambda}\int\delta\mathbf{G}_\lambda\cdot\tilde{\mathbf{v}}\tilde{\theta}=-\delta(\sqrt{\lambda}/\rho)-(\sqrt{\lambda^3}/\rho)\oint\delta h\tilde{\theta}^2, \qquad (26)$$

where $\mathbf{G}_\lambda=\lambda\nabla\psi+\mathbf{f}$. For suppose that the λ which gives ρ its maximum value is finite, and the system depends on parameters α_i. Then the best values of λ will be found as a root of the equation

$$\frac{\partial\rho(\lambda,\mu,\alpha_i)}{\partial\lambda}=0, \qquad (27)$$

which implies, through (26), that

$$\mu\int\tilde{\mathbf{v}}\cdot\boldsymbol{\epsilon}\cdot\tilde{\mathbf{v}}+2\lambda\int\nabla\psi\cdot\tilde{\mathbf{v}}\tilde{\theta}=-1/R(\mu,\alpha_i).$$

One compares this with the maximum problem (13) to produce (25).

(25) is particularly valuable for estimating the best value of λ, when it is not possible to do this from *a priori* considerations. We shall demonstrate this

repeatedly in the applications which follow. When $\mathbf{f} = \nabla\psi$ then $\lambda = 1$, a fact which makes it possible to exclude the possibility of subcritical instabilities for a wide class of basic motions starting from rest (see II).

The solution of the problem of the best coupling parameter is but one application of (26). A number of interesting deductions, particularly how these bear upon the problem of finding the optimum stability boundary, may be made from (26). (See II and § 5.)

It remains then to establish (26). First change variables so that $\tilde{\theta} = \tilde{\phi}/\sqrt{\lambda}$, then consider two different solutions of the Euler–Lagrange equations and identify them with subscripts. Thus $\mathbf{v}_1 = \tilde{\mathbf{v}}_1$ and $\theta_1 = \tilde{\theta}_1$ satisfy the Euler–Lagrange equations for R_1, λ_1, ϵ_1, $\nabla\psi_1$, $\mathbf{f}_1$ and h_1. Equations (20) and (21) written for subscript one are multiplied by $\tilde{\phi}_2$ and $\tilde{\mathbf{v}}_2$, respectively, and integrated over $\mathscr{V}$. This procedure is repeated with the subscripts exchanged. In this way we are led to the four equations ($j = 1$, $i = 2$, and $j = 2$, $i = 1$)

$$\tfrac{1}{2}\int(\lambda_j\nabla\psi+\mathbf{f})\,.\,\mathbf{v}_j\phi_i = -(\lambda_j/R_j)\,\mathscr{D}(\phi_i,\phi_j), \tag{28}$$

$$\mu_j\sqrt{\lambda_j}\int\mathbf{v}_j\,.\,\boldsymbol{\epsilon}_j\,.\,\mathbf{v}_i+\tfrac{1}{2}\int(\lambda_j\nabla\psi+\mathbf{f})\,.\,\mathbf{v}_i\phi_j = -(\sqrt{\lambda_j}/R_j)D(\mathbf{v}_j,\mathbf{v}_i), \tag{29}$$

where $R_i = \rho(\lambda_i,\mu_i,h_i,\ldots)$. A linear combination of the equations can then be made to produce

$$\begin{aligned}\int\mathbf{v}_1\cdot(\mu_2\sqrt{\lambda_2}\,\boldsymbol{\epsilon}_2-\mu_1\sqrt{\lambda_1}\,\boldsymbol{\epsilon}_1)\,.\,\mathbf{v}_2+\tfrac{1}{2}\int(\mathbf{G}_{\lambda_2}-\mathbf{G}_{\lambda_1})\,(\mathbf{v}_2\phi_1+\mathbf{v}\,\phi_2)\\ = -(\sqrt{\lambda_2}/R_2-\sqrt{\lambda_1}/R_1)\,(D(\mathbf{v}_1,\mathbf{v}_2)+D(\phi_1,\phi_2))\\ -\oint(h_2\sqrt{\lambda_2}/R_2-h_1\sqrt{\lambda_1}/R_1)\,\phi_1\phi_2.\end{aligned} \tag{30}$$

Now let the solutions coalesce and use $\phi = \theta\sqrt{\lambda}$ (18) and (27) to produce (26).

5. On the destabilizing effect of the Robin condition

As a first application of the general theory, we investigate the effect of the Nusselt number (h) on the stability limits. For simplicity we fix the distribution $f(\mathbf{r})$ of $h = Nuf(\mathbf{r})$ but allow the magnitude Nu to vary. It will be recalled that h enters the problem through the Robin condition

$$\partial\theta/\partial N+Nuf(\mathbf{r})\theta = 0 \quad (0 \not\equiv f(\mathbf{r}) \geqslant 0),$$

on the temperature of the difference motion. The limits $Nu\to 0$, $Nu\to\infty$ imply, respectively, a prescribed heat flux or a prescribed temperature on the boundary $\mathscr{S}$ of $\mathscr{V}$. The prescribed temperature condition, like the prescribed displacement condition for vibrating systems, is most restrictive or, in other words, most stable. This fact has been abundantly verified by exact calculation from the linear equations (Sparrow, Goldstein & Jonsson 1964, Sani 1963) and is here recovered, for the non-linear case, as an easy application of (26).

Consider that all basic-state parameters except Nu are fixed and determine the effect of Nu on the stability limit $R(Nu)$. The Nusselt number now plays a role analogous to the Reynolds number in § 2. In particular (25) for the best λ is valid as is found from the requirement that

$$\left(\frac{\partial\rho(\lambda,Nu)}{\partial\lambda}\right)_{Nu} = 0 \tag{31}$$

for the best $\lambda = \lambda(Nu)$. Using (31) we find that

$$\frac{dR(Nu)}{dNu} = \frac{\partial\rho(\lambda, Nu)}{\partial\lambda}\frac{\partial\lambda}{\partial Nu} + \frac{\partial\rho(\lambda, Nu)}{\partial Nu} = \left(\frac{\partial\rho(\lambda, Nu)}{\partial Nu}\right)_\lambda. \tag{32}$$

This last partial derivative is easily formed from (26) as

$$\frac{dR(Nu)}{dNu} = \lambda R \oint f(\mathbf{r})\tilde{\theta}^2 > 0. \tag{33}$$

It follows that *the stability limit $R(Nu)$ increases monotonically with the Nusselt number.*

We observe that when Nu is large, the Robin condition will ordinarily force a large normal derivative and a relatively small value of θ on $\mathscr{S}$. Then the equation

$$\frac{dR(Nu)}{dNu} = -\frac{\lambda R}{2Nu}\oint\frac{\partial\tilde{\theta}^2}{\partial N}$$

will imply very slow changes of stability limit as the Nusselt number is decreased through very large values.

We also note that in many cases this last conclusion is valid for local perturbations of steady solutions of the Boussinesq equations. When $\mu = 0$ and $\mathbf{f} = \nabla\psi$, the Euler–Lagrange equations coincide with the linear perturbation equations with partial time derivatives set to zero (see II). This implies that no subcritical instabilities exist, and conclusions drawn from the energy method are sufficient for instability as well as stability. It follows, that for these cases less than perfect control of the thermal boundary condition will not introduce great error into experimental results which purport to verify stability limits for the prescribed temperature case (cf. Sani 1963).

Our next application of energy theory bears directly on the question of subcritical instabilities. It is to this question that we now turn.

6. Subcritical convective instability in fluid layers

In this section we apply the theory to obtain stability limits for transversally infinite fluid layers heated from below and internally. We shall assume that the initially quiet fluid layer is bounded above and below by rigid plates, and we locate the co-ordinate origin at the lower plate. The distance between plates is unity (measured in units of d). The unit vector $\mathbf{i}$ points in the direction of z increasing, and $\nabla\psi = \mathbf{i}\,d\psi/dz$ and $\mathbf{f} = -\mathbf{i}$. Under stated conditions (20) and (21) may be written as

$$\frac{\frac{1}{2}R_\lambda}{\lambda}\left(\lambda\frac{d\psi}{dz} - 1\right)\omega = \nabla^2\theta, \tag{34}$$

$$\frac{\frac{1}{2}R_\lambda}{\lambda}\left(\lambda\frac{d\psi}{dz} - 1\right)\mathbf{i}\theta = -\nabla p + \nabla^2\mathbf{v}. \tag{35}$$

When $d\psi/dz = -1$, $\lambda = 1$ (equation (25)) then (34) and (35) coincide with the classical Rayleigh–Jeffrey's problem. No subcritical instabilities exist even when the problem is generalized to include the Robin condition on the temperature (see II).

In the present application the distribution of temperature in the quiet state differs from linearity by virtue of a distribution of internal heat sources, which, for ease of comparison with known results of linear theory (Sparrow *et al.* 1964), is taken as uniform across the channel.

The temperature distribution corresponding to circumstances specified above is

$$T = -\tfrac{1}{2}(s/\kappa)x_3^2 + Ax_3 + B,$$

where s is the internal heat-source intensity, and κ is the thermal conductivity. This may be written in non-dimensional variables as

$$\frac{T-T_2}{T_1-T_2} = N_s(z-z^2)+(1-z), \tag{36}$$

where T_1 and T_2 are the temperatures of the bottom and top plate respectively, $z = x_3/d$ and $N_s = \frac{1}{2}sd^2/\kappa(T_1-T_2)$ is a heat-source parameter considered positive, i.e. restricted to cases for which $T_1 > T_2$. From (36)

$$\frac{dT}{dx_3} = \frac{T_1-T_2}{d}\{N_s(1-2z)-1\},$$

and with

$$\beta = \max.\left|\frac{dT}{dx_3}\right| = \frac{T_1-T_2}{d}(N_s+1),$$

we obtain

$$\frac{d\psi}{dz} = \frac{1}{\beta}\frac{dT}{dx_3} = \frac{2N_s(1-z)}{N_s+1} - 1. \tag{37}$$

For easy comparison with known results of linear theory, we have stated our results in terms of a Rayleigh number

$$RA = \frac{\alpha g d^3|T_1-T_2|}{\kappa\gamma} = Ra\frac{|T_1-T_2|}{\beta d} = \frac{Ra}{N_s+1}. \tag{38}$$

Our critical value is designated as $\widetilde{RA}$. The critical value of linear theory is called RA_c. We next use (37), (38) and the variable $\phi = \sqrt{\lambda}\theta$ to rewrite (34) and (35) as

$$\frac{\frac{1}{2}R_\lambda}{\sqrt{\lambda}}\left(\lambda\frac{\{N_s(1-2z)-1\}}{N_s+1} - 1\right)\omega = \nabla^2\phi, \tag{39}$$

$$\frac{\frac{1}{2}R_\lambda}{\sqrt{\lambda}}\left(\lambda\frac{\{N_s(1-2z)-1\}}{N_s+1} - 1\right)\mathbf{i}\phi = -\nabla p + \nabla^2\mathbf{v}. \tag{40}$$

These are to be solved subject to a prescribed temperature condition

$$\phi(0) = \phi(1) = 0$$

and a solenoidal velocity vanishing at $z = 0, 1$.

We next assert that on the optimum stability boundary

$$\widetilde{RA} \geqslant \frac{1708}{N_s+1}. \tag{41}$$

Stated in another way stability is guaranteed when

$$RA \leqslant \frac{1708}{N_s+1}. \tag{42}$$

The estimates (41) and (42) follow as a simple consequence of (26), which in the present context has the form

$$\int\left(\delta\lambda\nabla\psi+\lambda\nabla\frac{d\psi}{dN_s}\delta N_s\right).\tilde{\mathbf{v}}\tilde{\phi} = -\delta\left(\frac{\sqrt{\lambda}}{\rho}\right). \tag{43}$$

Also, when $\rho = R = \sqrt{\widetilde{Ra}}$ and $\lambda = \lambda(N_s)$ (see (25)),

$$\int(\lambda\nabla\psi+\mathbf{f}).\tilde{\mathbf{v}}\tilde{\phi} = -\frac{\sqrt{\lambda}}{R(N_s)} = 2\int\mathbf{f}.\tilde{\mathbf{v}}\tilde{\phi}. \tag{44}$$

The condition for the best $\lambda = \lambda(N_s)$ implies that

$$\frac{dR(Ns)}{dN_s} = \left(\frac{\partial\rho(\lambda(N_s),N_s)}{\partial N_s}\right)_\lambda,$$

which is easily evaluated from (43) as

$$\frac{\sqrt{\lambda}}{R^2}\frac{dR}{dN_s} = \lambda\int\nabla\left(\frac{d\psi}{dN_s}\right).\tilde{\mathbf{v}}\tilde{\phi}. \tag{45}$$

We next divide (45) by (44) and use $\mathbf{f} = -\mathbf{i}$ and

$$\nabla\frac{d\psi}{dN_s} = 2\mathbf{i}\frac{(1-z)}{(N_s+1)^2}$$

to obtain

$$\frac{1}{R}\frac{dR}{dN_s} = \frac{\lambda}{(N_s+1)^2}\frac{\int(1-z)\tilde{\omega}\tilde{\phi}}{\int\tilde{\omega}\tilde{\phi}}. \tag{46}$$

The formula (25) for the best λ has the form

$$\lambda\int\left(\frac{2N_s(1-z)}{N_s+1}-1\right)\tilde{\omega}\tilde{\phi} = -\int\tilde{\omega}\tilde{\phi},$$

implying that

$$\frac{\int(1-z)\tilde{\omega}\tilde{\phi}}{\int\tilde{\omega}\tilde{\phi}} = \frac{(\lambda-1)(N_s+1)}{2\lambda N_s}. \tag{47}$$

Use (47) in (46) and multiply the resulting equation by $R^2 = \widetilde{Ra}$ to produce

$$\frac{d\widetilde{Ra}}{dN_s} = \frac{\widetilde{Ra}(\lambda-1)}{(N_s+1)N_s} = \frac{\lambda-1}{N_s}\widetilde{RA},$$

or

$$\frac{d\widetilde{RA}}{dN_s} = \frac{\widetilde{RA}}{N_s+1}\left(\frac{\lambda-1}{N_s}-1\right). \tag{48}$$

Of course $\lambda = \lambda(N_s)$ and is not known explicitly. Nevertheless,

$$d\log\widetilde{RA} = \frac{(\lambda-1)}{N_s}d\log(N_s+1)-d\log(N_s+1)$$

may be integrated from the known point $(\widetilde{RA},N_s) = (1708,0)$ to obtain

$$\log\left\{\frac{\widetilde{RA}(N_s+1)}{1708}\right\} = \int_0^{N_s}\frac{(\lambda-1)\,dN_s}{N_s(N_s+1)}. \tag{49}$$

Equation (41) follows easily from (49) under the assumption that $\lambda-1>0$, as is strongly suggested by (47) and (44).

Equation (49) is an exact result. It implies not only the estimate (41) but also the exact limits

$$\left(\frac{d\log \widetilde{RA}}{d\log N_s}\right)_{N_s=0} = 0, \quad \left(\frac{d\log \widetilde{RA}}{d\log N_s}\right)_{N_s\to\infty} = -1.$$

In figure 2 we have compared the estimate (41) with the exact solution, obtained numerically by the Runge–Kutta–Gill method (Harris & Reid 1964, Sparrow 1964). This method is applied to the equations

$$(D^2-k^2)\phi - \frac{\frac{1}{2}R_\lambda}{\sqrt{\lambda}}\left(\lambda\frac{d\psi}{dz}-1\right)\omega = 0, \tag{50}$$

$$(D^2-k^2)^2\omega + \tfrac{1}{2}k^2\frac{R_\lambda}{\sqrt{\lambda}}\left(\lambda\frac{d\psi}{dz}-1\right)\phi = 0, \tag{51}$$

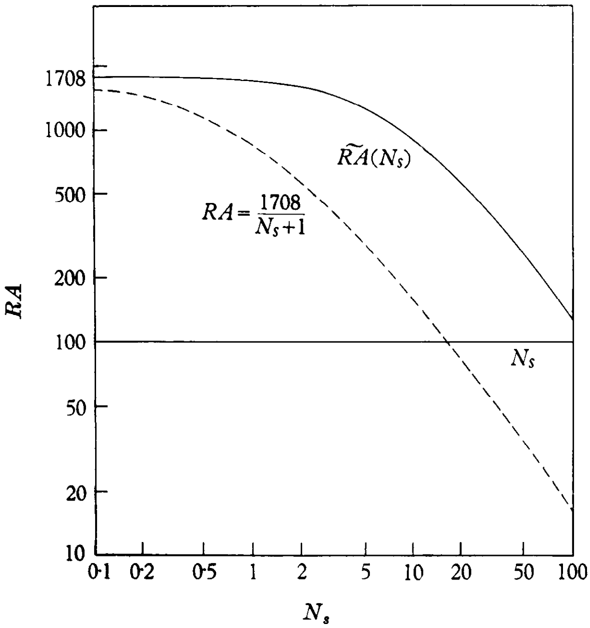

FIGURE 2. The optimum stability boundary compared with an *a priori* estimate.

where ω and ϕ are amplitudes and k the overall wave-number of the (periodic) normal velocity and temperature disturbances. Equations (50) and (51) follow easily upon substitution of a normal mode proportional to $\exp(ik_x x + ik_y y)$ into (39) and (40). They are to be solved for the conditions

$$\omega = D\omega = \phi = 0 \quad \text{at} \quad z = 0, 1. \tag{52}$$

The problem defined by (50), (51) and (52) is a classic eigenvalue problem. We here regard R_λ as the eigenvalue. In general, the values of R_λ for which non-trivial solutions exist depend on the other parameters so that

$$R_\lambda = R_\lambda(\lambda, k, N_s).$$

For a fixed value of N_s and any fixed $\lambda > 0$, the flow is stable provided that

$$\sqrt{Ra} < \widetilde{R}_\lambda = \min_{k \geq 0} . R_\lambda. \tag{53}$$

The best value for λ is that which gives the best stability limit, i.e. the largest RA. Hence,

$$\sqrt{\widetilde{Ra}} = \max_{\lambda > 0} . \min_{k \geq 0} . R_\lambda. \tag{54}$$

The field of minimum values $\widetilde{R}_\lambda$ is generated by the Runge–Kutta–Gill method. This procedure is fairly standard and is briefly discussed in part 2 of this paper

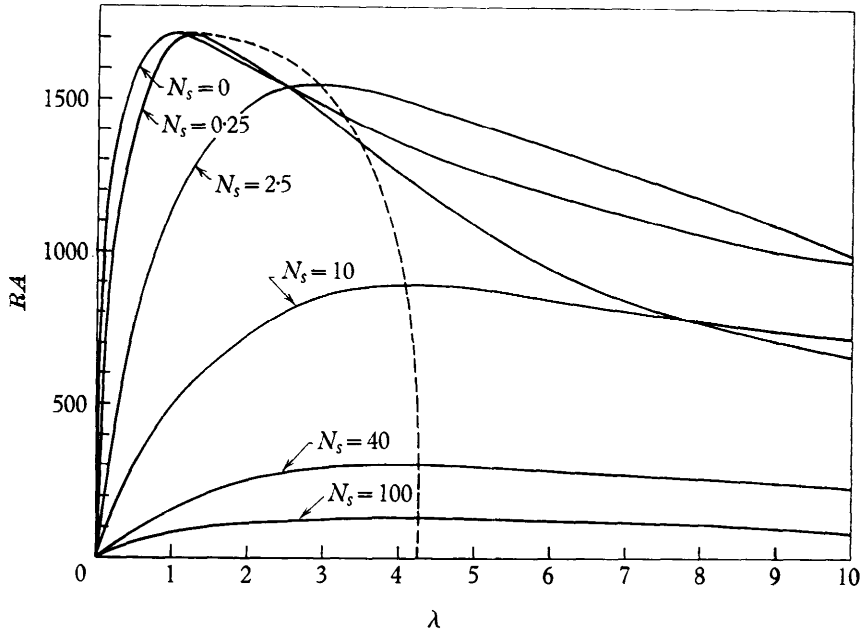

FIGURE 3. The optimum stability boundary as the loci of the best value of the coupling parameter λ. ——, $\widetilde{R}^2_\lambda(\lambda, N_s)/(N_s+1)$; ----, $RA(\lambda, N_s)$, the locus of the maxima of the curves giving $\widetilde{R}^2_\lambda/(N_s+1)$.

and in the cited references. The critical Rayleigh number is extracted from this field by numerical searching for the maxima required by (54). Figures 3 and 4 give the result of this search.

Before turning to a description of the results, we should like to remark upon the usefulness of the equation for the best λ in approximating the best value for an initial guess at a solution for (54). From (47) we find

$$\frac{1}{\lambda} = 1 - \frac{N_s}{N_s+1}\frac{(\omega\phi)^\dagger}{(\omega\phi)^\ddagger}, \tag{55}$$

where $(\omega\phi)^\dagger$ and $(\omega\phi)^\ddagger$ are mean values as defined by the first mean-value theorem of integral calculus. Alternately, if $\omega\phi$ is assumed to be one-signed on $(0, 1)$,

$$\frac{1}{\lambda} = 1 - \frac{2N_s}{N_s+1}\bar{z}, \tag{56}$$

where $\bar{z}$ is a mean value ($0 < \bar{z} < 1$). When $N_s = 0$, $\lambda = 1$. As $N_s \to \infty$, λ tends to a limiting value independent of N_s. The result, which is suggested by (55) and (50), is borne out by the numerical results. These show that for $N_s > 10$, $\bar{z} \simeq \frac{3}{8}$ and $\lambda \simeq 4{\cdot}2$.

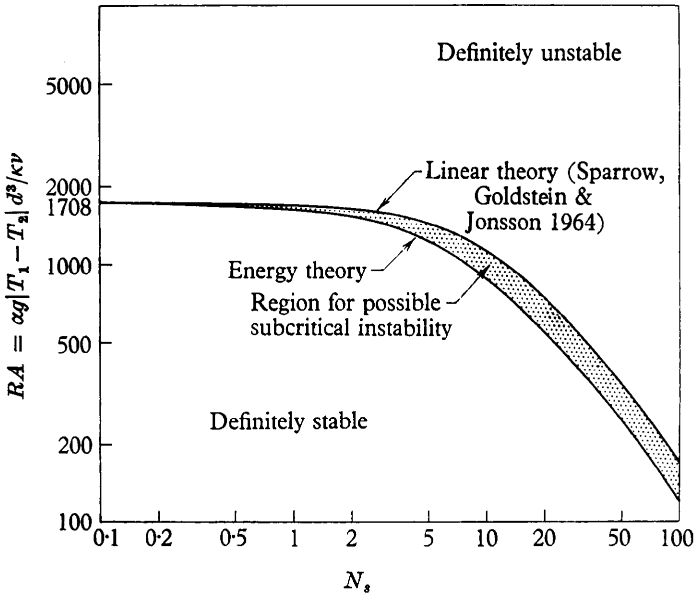

FIGURE 4. Heated from below with internal heat sources.

N_s	k	RA_c	$\widetilde{RA}$
0	3·12	1708	1708
0·25	3·12	1707	1706
2·5	3·18	1633	1545
10	3·53	1118	896
40	3·86	409	305
100	3·94	180	130

TABLE 1. Values of parameters for critical Rayleigh numbers of linear and energy theory

In figure 3 we have sketched (solid line) the variation of $\tilde{R}_\lambda^2/(N_s+1)$ with λ as the variable and N_s a parameter. The dotted line is the locus of the 'best' values of λ over the range of N_s values. This is the locus $\widetilde{RA}(N_s)$ which defines the values of RA below which arbitrary disturbances certainly decay.

In figure 4 and table 1 we have compared the locus $\widetilde{RA}_c(N_s)$ (Sparrow *et al.* 1964), which defines a boundary above which the given flow is certainly unstable, with the locus $\widetilde{RA}(N_s)$, which defines a boundary below which a given flow is certainly stable. The region between these boundaries is potentially open to subcritical instabilities.

It will be observed that the difference between $RA_c(N_s)$ and $\widetilde{RA}(N_s)$ increases monotonically from zero, when $N_s = 0$, to a finite but not large difference, as $N_s \rightarrow \infty$.

The work of parts 1 and 2 was supported in part by NASA grant (NGR-24-005-065) to the Space Science Centre of the University of Minnesota.

REFERENCES

Bateman, H., Dryden, H. & Murnaghan, F. D. 1932 *Hydrodynamics*. New York: Dover.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

Harris, D. & Reid, W. 1964 *J. Fluid Mech.* **20**, 95.

Joseph, D. D. 1965 *Arch. Rat. Mech. Anal.* **20**, 59.

Joseph, D. D. 1966 *Arch. Rat. Mech. Anal.* **22**, 163.

Orr, W. McF. 1907 *Proc. Roy. Irish Acad.* A, **27**, 69.

Reynolds, O. 1895 *Phil. Trans. Roy. Soc.* A, **186**, 123.

Sani, R. 1963 Ph.D. Thesis, University of Minnesota.

Serrin, J. 1959 *Arch. Rat. Mech. Anal.* **3**, 1.

Sparrow, E. 1964 *J. Appl. Math. Phys.* **15**, 638.

Sparrow, E., Goldstein, R. & Jonsson, V. 1964 *J. Fluid Mech.* **18**, 513.

Thomas, T. Y. 1942 *Amer. J. Math.* **64**, 754.