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by the Method of Energy*

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A. Introduction

The linear theory of hydrodynamic stability suffers from the defect that one cannot, in principle, make judgements regarding the growth potential of finite disturbances. Thus, one cannot say for certain that a given flow will remain stable if disturbed under conditions judged favorable by linear theory. This and other questions about the effects of finite disturbances are in the province of nonlinear theory (see SEGEL [1] for the most recent review).

The oldest method of stability analysis which can accommodate finite disturbances is the method of energy. This method can be traced to the work of REYNOLDS [2] and ORR [3]. Older references are summarized by BATEMAN *et al.* [4]. In the modern formulation of the energy method (SERRIN [5], JOSEPH [6]) one seeks the conditions under which the energy of a difference motion (which includes small perturbations as a subclass) will surely decrease. The results of these investigations are the delineation of regions of certain stability of the governing hydrodynamic system.

In this paper we concretize and extend our earlier formulation [6] of energy theory appropriate to fluid systems governed by the nonlinear Boussinesq equations. The class of flows accommodated by the method is enlarged (B), and criteria establishing the existence of a region of universal stability in a Rayleigh-Reynolds number plane are briefly reviewed (C). The general theory is then extended to obtain an improved criterion for stability, comparable in rigor with that giving universal stability (D), and a general description of the optimum stability boundary (E).

In particular, an explicit representation of the slope of this boundary is derived; the Reynolds number is shown to be a monotone-decreasing function of the Rayleigh number for small Rayleigh numbers. Sufficient conditions for the nonexistence of subcritical instabilities are also developed, and it is shown that rigid rotation cannot destabilize that class of motions which is subcritically stable (F).

The general theory is applied to the problem of a plane Couette flow heated from below (G). The stability boundary for this problem is obtained explicitly ($Re^2 + Ra = 1708$). In the zero Rayleigh number limit this result replaces the celebrated criterion of ORR, which is shown to be too high by somewhat more than a factor of two.

The paper concludes (H) with an examination and brief critique of the energy method.

B. Energy Identities for the Difference Motion

The essential elements of the energy method as this is applied to Boussinesq fluids evolve from deductions made from the energy identities*

$$\frac{dK}{dt} = \frac{d}{dt} \int \frac{u^2}{2} = - \int \{ \mathbf{u} \cdot \mathbf{D} \cdot \mathbf{u} + \alpha \mathbf{g} \cdot \mathbf{u} \vartheta + 2\nu \mathbf{d} : \mathbf{d} \} \tag{1}$$

and

$$\frac{d\Theta}{dt} = \frac{d}{dt} \int \frac{\vartheta^2}{2} = - \int \{ \vartheta \mathbf{u} \cdot \nabla T + \kappa \nabla \vartheta \cdot \nabla \vartheta \} - \kappa \oint \sigma \vartheta^2. \tag{2}$$

Here $\mathcal{V} = \mathcal{V}(t)$ is a region of space occupied by a basic fluid motion; $\mathbf{u} = \mathbf{V}^* - \mathbf{V}$ and $\vartheta = T^* - T$ are, respectively, the differences of velocity and temperature between the disturbed (starred) and undisturbed (unstarred) motion; $(\mathbf{D})_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i})$, $\mathbf{d} = \mathbf{D}^* - \mathbf{D}$ and ∇T are, respectively, the strain-rate tensor of the basic and difference motion and the gradient of the temperature of the basic fluid motion; $\kappa(t)$, $\alpha(t)$, $\nu(t)$ and $\mathbf{g}(\mathbf{r}, t)$ are, respectively, the thermometric coefficient, the coefficient of thermal expansion, the kinematic viscosity and the field force (typically gravity) vector.

Consistent with the requirement that the two flows satisfy the same conditions at the (closed) boundary \mathcal{S} of \mathcal{V} are

$$\mathbf{u} = 0 \quad (\text{rigid surface, velocity } \mathbf{V} \text{ prescribed}), \tag{3}$$

or

$$(\mathbf{N} \cdot \mathbf{d}) \times \mathbf{N} = 0, \quad \mathbf{u} \cdot \mathbf{N} = 0 \quad (\text{free surface, normal velocity } \mathbf{V} \cdot \mathbf{N} \text{ prescribed}), \tag{4}$$

and a radiation condition

$$\frac{\partial \vartheta}{\partial N} + \sigma \vartheta = 0 \tag{5}$$

for the temperature. Here \mathbf{N} is the outward normal to \mathcal{S} , $\sigma(\mathbf{r}, t) \geq 0$ is a piecewise continuous function of position (Nusselt number), $\mathbf{N} \cdot \mathbf{D}$ is proportional to the viscous part of the surface tractions which are assumed entirely normal. A mixture of these conditions may prevail on subelements of \mathcal{S} .

Equations (1) and (2) follow from the integration of suitably-multiplied differential equations governing the difference motion. The equations for the difference motion are formed by subtracting the Boussinesq equations** for the basic (unstarred) flow,

$$\begin{aligned} \frac{d\mathbf{V}}{dt} &= -\nabla \frac{p}{\rho_0} + \{1 - \alpha(T - T_0)\} \mathbf{g} + 2\nu \nabla \cdot \mathbf{D}, \\ \frac{dT}{dt} &= \kappa \nabla^2 T + Q(\mathbf{r}, t), \end{aligned} \tag{6}$$

$$\nabla \cdot \mathbf{V} = 0,$$

* In writing integrals we shall omit infinitesimal volume elements; moreover, all integrals are understood to be extended over the entire region \mathcal{V} , except for integrals over \mathcal{S} , the boundary of \mathcal{V} , which are indicated by a circle drawn through the integral sign.

** Approximations which lead to the nonlinear Boussinesq equations are discussed in detail by CHANDRASEKHAR [7], SPIEGEL and VERONIS [8] and by MIHALJAN [9].

from the same equations for the disturbed (starred) flow. Here $T_0(t)$ and $Q(\mathbf{r}, t)$ are, respectively, a prescribed reference temperature and a prescribed heat-source function.

We call attention to a small difference between this and earlier formulations [5, 6]. In this formulation the mechanical dissipation is given by $2\nu \int \mathbf{d} : \mathbf{d}$; in earlier formulations this is expressed as $\nu \int \nabla \mathbf{u} : \nabla \mathbf{u}$. The term

$$\int \mathbf{d} : \mathbf{d}$$

of equation (1) arises from integration of

$$\mathbf{u} \cdot \{ \nabla \cdot (\mathbf{D}^* - \mathbf{D}) \} = \mathbf{u} \cdot (\nabla \cdot \mathbf{d}) = \nabla \cdot (\mathbf{u} \cdot \mathbf{d}) - \mathbf{d} : \mathbf{d} \tag{7}$$

over \mathcal{V} . We note that

$$\int \nabla \cdot (\mathbf{u} \cdot \mathbf{d}) = \oint \mathbf{u} \cdot \mathbf{d} \cdot \mathbf{N}. \tag{8}$$

On rigid sub-elements of \mathcal{S} , $\mathbf{u} = 0$. On free sub-elements of \mathcal{S} , \mathbf{u} is entirely tangential, and $\mathbf{d} \cdot \mathbf{N}$ is entirely normal. In both cases the integrals (8) vanish.

The more general formulation is related to earlier formulations by the identity

$$\int \nabla \mathbf{u} : \nabla \mathbf{u} = 2 \int \mathbf{d} : \mathbf{d} - \oint \{ (\mathbf{u} \cdot \nabla) \mathbf{u} \} \cdot \mathbf{N}. \tag{9}$$

The surface integral vanishes on free, planar, subelements of \mathcal{S} and, of course, on subelements of \mathcal{S} on which $\mathbf{u} = 0$. If the entire boundary is a combination of subelements of this kind, then the old and new formulations coincide.

Subsequent deductions about the stability of the difference motion are extracted from the energy identities (1) and (2), the boundary constraints (3), (4), and (5) and the kinematic constraint $\text{div } \mathbf{u} = 0$. The local nonlinear conservation equations do not play a direct role in further construction of the theory.

Two lines of deduction which start from the energy identities are possible. The first of these leads to a criterion for universal stability. The universal criterion does not depend on details of the motion or geometry of the basic flow. When satisfied the criterion guarantees asymptotic stability in the sense of an exponential decay of disturbance of any magnitude. The region of certain stability can be extended by a sharper criterion which makes more efficient use of details of the basic flow. This leads to a second line of deduction which we call "the problem of the optimum stability boundary".

C. Universal Stability

A first avenue of analysis leads to the delineation of a region (Fig. 1) in the neighborhood of the origin in which all "Boussinesq" flows which can be contained in a sphere of diameter d and for which the boundary temperature and velocity are prescribed are certainly stable. This result can be regarded as a rigorous proof of stability of all kinds of "Boussinesq" flows for sufficiently small values of the stability parameters, *i.e.*, a Reynolds number (Re) and a Rayleigh number

(Ra), and can be expressed as an inequality (see [6] for details)

$$\frac{\text{Re}}{m} \sqrt{K} + \frac{\gamma \sqrt{\text{Ra}}}{\beta} \sqrt{\Theta} \leq \left(\frac{\text{Re}}{m} \sqrt{K_0} + \frac{\gamma \sqrt{\text{Ra}}}{\beta} \sqrt{\Theta_0} \right) \exp \{ -\xi (3\pi^2 \gamma - \sqrt{\text{Ra}}) t \}, \quad (10)$$

where

$$\xi = \frac{\nu \gamma}{d^2} \quad \text{when } \text{Pr} \gamma^2 \leq 1,$$

$$\xi = \frac{\kappa}{\gamma d^2} \quad \text{when } \text{Pr} \gamma^2 \geq 1,$$

and provided that

$$0 \leq \text{Ra} < 3\pi^2 (\delta - \text{Re}) = 9\pi^4 \gamma^2. \quad (11)$$

Here K_0 and Θ_0 are initial disturbance values, $-m$ is the least characteristic value of D , $\beta = \text{Max} |\nabla T|$ and g, ν, κ are maximum values in the time interval $[0, t]$. The dimensionless numbers are defined by $\text{Pr} = \nu/\kappa$, $\text{Ra} = \alpha \beta g d^4/\nu \kappa$, $\text{Re} = d^2 m/\nu$ and $\delta \cong 80$ is the least positive root of $\text{Tan} \sqrt{\delta/2} = \sqrt{\delta/2}$. If $0 \leq \text{Ra} < 3\pi^2 (\delta - \text{Re})$ for all t , then $K \rightarrow 0$ and $\Theta \rightarrow 0$ as $t \rightarrow \infty$, and the flow is stable.

To obtain (10) one estimates the various terms of the R.H.S. of equations (1) and (2) with the object of bounding these with respect to K, Θ and multiples of these. This program then yields ordinary differential inequalities which, after some manipulation, may be combined and integrated to produce (10). Estimates of this kind can be constructed for parallelepipeds, cylinders and channels as well as spheres (see [6] for references).

The limits of stability given by estimates of this kind do not depend on specific details of the geometrical configuration or on the distribution of the basic field variables. This generality is obtained at the expense of rather more conservative estimates than can be obtained by using all of the available information. It is, of course, desirable to obtain the best possible estimate of the limits of stability. We shall now show how the problem of extension of the region of stability can be formulated and solved.

D. Problem of the Optimum Stability Boundary

First we accentuate the role of the Rayleigh number by introducing the variables

$$\tau = \frac{\nu}{d^2} t, \quad v = u \sqrt{\frac{\nu \beta}{\alpha g \kappa}}, \quad \varepsilon = \frac{D}{m}, \quad \nabla \psi = \frac{\nabla T}{\beta},$$

$$e = d \sqrt{\frac{d^2 \nu \beta}{\alpha g \kappa}}, \quad g = g f, \quad h = \sigma d, \quad |f| = 1,$$

where the dimensionless numbers Pr , Ra and Re are defined as before, and the length scale is normalized with d . The difference velocity and temperature then

necessarily satisfy energy identities obtained from (1) and (2),

$$\frac{dK}{d\tau} = \frac{d}{d\tau} \int \frac{v^2}{2} = - \int \{ \text{Re } \mathbf{v} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v} + \sqrt{\text{Ra}} \mathbf{f} \cdot \mathbf{v} \vartheta + 2 \mathbf{e} : \mathbf{e} \}, \quad (12)$$

$$\text{Pr} \frac{d\Theta}{d\tau} = \text{Pr} \frac{d}{d\tau} \int \frac{\vartheta^2}{2} = - \int \{ \sqrt{\text{Ra}} \nabla \psi \cdot \mathbf{v} \vartheta + \nabla \vartheta \cdot \nabla \vartheta \} - \oint h \vartheta^2, \quad (13)$$

the divergence constraint

$$\text{div } \mathbf{V} = 0 \quad \text{in } \mathcal{V}, \quad (14)$$

and

$$\mathbf{v} = 0 \quad (\text{rigid surface}), \quad (15)$$

$$(\boldsymbol{\varepsilon} \cdot \mathbf{N}) \times \mathbf{N} = 0, \quad \mathbf{v} \cdot \mathbf{N} = 0 \quad (\text{free surface}), \quad (16)$$

$$\frac{\partial \vartheta}{\partial N} + h \vartheta = 0, \quad (17)$$

on \mathcal{S}^* .

Deductions about the stability of the difference motion are possible by the introduction of a stability criterion. We first introduce a coupling parameter λ and define an "energy"

$$E_\lambda = K + \lambda \text{Pr } \Theta.$$

The requirement that this energy be positive is equivalent to the restriction that $\lambda > 0$. We next simplify the problem by introducing another positive parameter μ ($0 \leq \mu \leq \infty$) by the relation $\text{Re} = \mu \sqrt{\text{Ra}}$. We regard μ as preassigned and use it to eliminate explicit dependence on the Reynolds number. Introduce the notation

$$\begin{aligned} I_1 &= \int (\mu \mathbf{v} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v} + \mathbf{f} \cdot \mathbf{v} \vartheta), & I_2 &= \int \nabla \psi \cdot \mathbf{v} \vartheta, \\ D &= 2 \int \mathbf{e} : \mathbf{e}, & \mathcal{D} &= \int \nabla \vartheta \cdot \nabla \vartheta + \oint h \vartheta^2, \\ I_\lambda &= I_1 + \lambda I_2, & D_\lambda &= D + \lambda \mathcal{D}; \end{aligned}$$

and form the inequality

$$\frac{dE_\lambda/d\tau}{D_\lambda} = -1 + \sqrt{\text{Ra}} \left(\frac{-I_\lambda}{D_\lambda} \right) \leq -1 + \sqrt{\text{Ra}} \text{Max} \left(\frac{-I_\lambda}{D_\lambda} \right) = -1 + \frac{\sqrt{\text{Ra}}}{\rho},$$

or

$$\frac{dE_\lambda}{d\tau} \leq - \left(1 - \frac{\sqrt{\text{Ra}}}{\rho} \right) D_\lambda, \quad (18)$$

* Equations (12) and (13) are appropriate to situations in which the material properties κ, ν and the reference temperature T_0 are constant in time. This slight restriction can be easily removed by setting

$$\int \mathbf{e} : \mathbf{e} \rightarrow \frac{\nu(\tau)}{\nu} \int \mathbf{e} : \mathbf{e}, \quad \int \nabla \vartheta \cdot \nabla \vartheta + \oint h \vartheta^2 = \frac{\kappa(\tau)}{\kappa} \{ \int \nabla \vartheta \cdot \nabla \vartheta + \oint h \vartheta^2 \}.$$

Since the numbers $\nu(\tau)/\nu$ and $\kappa(\tau)/\kappa$ are bounded away from 0, no essential differences are introduced by redefining the dissipation integrals in this way. Even in the simpler case, however, the quantities $f, h, \nabla \psi, \boldsymbol{\varepsilon}$ and, through these, the values $\rho(\lambda, \mu)$ (equation (18)) may depend on time.

where

$$\rho^{-1} = \rho^{-1}(\lambda, \mu) = \text{Max} \left\{ -\frac{I_\lambda}{D_\lambda} \right\}^*$$

From the inequality (18) follows

Theorem 1. *Let the inequalities*

$$\begin{aligned} \frac{a^2}{2} \int v^2 &\leq D(\mathbf{v}, \mathbf{v}), \\ \frac{\text{Pr } b^2}{2} \int \vartheta^2 &\leq \mathcal{D}(\vartheta, \vartheta) \end{aligned}$$

with $a^2 > 0$, and $b^2 > 0$, hold. Then, if for any fixed values $\lambda > 0$ and $\mu \geq 0$, $\text{Ra} < \rho(\lambda, \mu)$ in the time interval $[0, \tau]$, we have

$$E_\lambda(\tau) \leq E_\lambda(0) \exp \left\{ -\left(1 - \frac{\sqrt{\text{Ra}}}{\rho}\right) \xi^2 \tau \right\} \tag{19}$$

where $E_\lambda(0)$ is the initial energy of the difference motion and $\xi^2 = \text{Min}(a^2, b^2)$. If $\sqrt{\text{Ra}} < \rho$ for all τ , then $E_\lambda \rightarrow 0$, and the flow is asymptotically stable in the mean**.

Proof. Let the assumed inequalities hold. Then

$$E_\lambda = \frac{1}{2} \int (v^2 + \lambda \text{Pr } \vartheta^2) \leq a^{-2} D(\mathbf{v}, \mathbf{v}) + \lambda b^{-2} \mathcal{D}(\vartheta, \vartheta) \leq \xi^{-2} D_\lambda,$$

which may be combined with (18) to produce

$$\frac{dE_\lambda}{d\tau} \leq -\left(1 - \frac{\sqrt{\text{Ra}}}{\rho}\right) D_\lambda \leq -\xi^2 \left(1 - \frac{\sqrt{\text{Ra}}}{\rho}\right) E_\lambda.$$

This last inequality is then integrated on $[0, \tau]$ proving (19) and the theorem.

The hypotheses of Theorem 1 are not very restrictive. It is clear that $a^2/2$ is the smallest of the eigenvalues associated with the vector Helmholtz equation for \mathbf{v} and the conditions (14), (15), and (16). The quantity $\text{Pr } b^2/2$ is similarly identified as the least eigenvalue of the scalar Helmholtz equation for ϑ and the condition (17). Under nearly all situations encountered in applications, the existence of a positive, least eigenvalue can be assumed and, in many instances, proved (see [6] for references).

Roughly speaking, then, stability is guaranteed if $\sqrt{\text{Ra}} < \rho(\lambda, \mu)$. This leads naturally to the formulation of a maximum problem for the number $1/\rho$. This number is to be sought as the maximum value of the expression

$$\frac{-I_\lambda}{D_\lambda} = \frac{-I_1(\mathbf{v}, \vartheta) - \lambda I_2(\mathbf{v}, \vartheta)}{D(\mathbf{v}, \mathbf{v}) + \lambda \mathcal{D}(\vartheta, \vartheta)} \tag{20}$$

* In the notation of [6], $\rho(\lambda, \mu) = \tilde{R}_\lambda(\mu)$.

** Theorem 1 stems from the recognition that the requirement that the energy decrease monotonically is not sufficient to guarantee asymptotic stability in the mean. It was J. SERRIN who first suggested that it might be possible to obtain this more precise basis for the maximum problem. Theorem 1 grows out of this and a subsequent conversation and is a joint result. We note that if the hypotheses of Theorem 1 are satisfied, one may also state a uniqueness theorem for steady flows (cf. SERRIN [5], JOSEPH [6]).

over a field of twice-continuously-differentiable functions ϑ and (solenoidal) v satisfying (15), (16), and (17).

This maximum problem generates a field of values $\rho(\lambda, \mu)$ for each (λ, μ) parameter pair. Since for each fixed value of μ the flow is stable provided only that $\sqrt{Ra} < \rho(\lambda, \mu)$, we may select λ so as to give the best possible limit for stability. Since this best limit is clearly that for which Ra is largest, we seek the largest of the values of $\rho(\lambda, \mu)$ over λ for a fixed μ and define

$$R(\mu) = \text{Max}_{\lambda > 0} \rho(\lambda, \mu).$$

The locus of values $R(\mu)$ gives the optimum stability boundary $F(\sqrt{\tilde{Ra}}, \tilde{Re}) = 0$, parametrically through the equations $\sqrt{\tilde{Ra}} = R(\mu)$ and $\tilde{Re} = \mu R(\mu)$. The value of $\lambda = \lambda(\mu)$ which is associated with the maximum ρ , i.e.,

$$R(\mu) = \rho(\lambda(\mu), \mu),$$

is called the best value for the coupling parameter λ . If this best value is assumed finite, then it may be found as a root of the equation

$$\left(\frac{\partial \rho(\lambda, \mu)}{\partial \lambda} \right)_{\mu} = 0. \tag{21}$$

It follows that

$$\frac{dR}{d\mu} = \left(\frac{\partial \rho}{\partial \mu} \right)_{\lambda} + \left(\frac{\partial \rho}{\partial \lambda} \right)_{\mu} \frac{d\lambda}{d\mu} = \left(\frac{\partial \rho(\lambda, \mu)}{\partial \mu} \right)_{\lambda}, \tag{22}$$

and $R(\mu)$ is an envelope of the curves $\rho(\text{const.}, \mu)$ depending on the parameter $\lambda = \text{const.}$

A summary statement of the structure of the problem may be readily grasped from Fig. 1. In this figure, I is the region of universal stability. The problem of

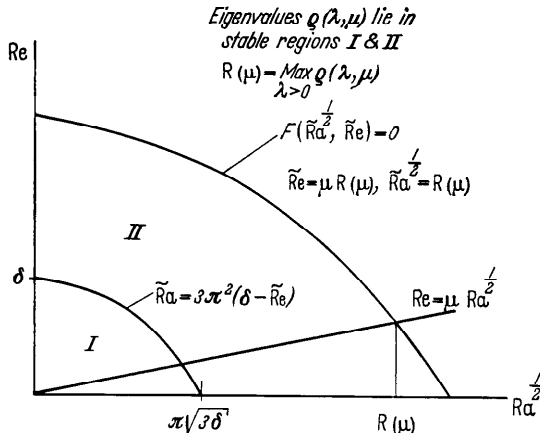


Fig. 1. Stability region and stability boundary

the optimum stability boundary is posed so as to delineate a larger region of certain stability (II) by using the details of the basic motion. The stability boundary $F(\sqrt{\tilde{Ra}}, \tilde{Re}) = 0$ is determined as follows: We first fix μ . This determines a ray

from the origin. A set of maximizing eigenvalues $1/\rho$ are then found for different λ and the fixed μ .

The λ which produces the maximum value of ρ on the given ray determines the critical value $R(\mu)$. The corresponding critical Reynolds number is given parametrically by $\mu R(\mu)$. The stability boundary $F(\sqrt{\widetilde{Ra}}, \widetilde{Re})=0$ is generated as μ takes on allowed values in the first quadrant.

E. Solution of the Problem of the Optimum Stability Boundary

1. The Maximum Problem

The maximum problem (20) is easily formulated in the framework of variational calculus. We require that

$$-\{I_1(v, \vartheta) + \lambda I_2(v, \vartheta)\} = \text{Max} = \frac{1}{\rho(\lambda, \mu)} \tag{23}$$

hold for a class of twice-continuously-differentiable functions* v and ϑ satisfying (14)–(17) and the normalizing condition

$$D(v, v) + \lambda \mathcal{D}(\vartheta, \vartheta) = 1. \tag{24}$$

Lagrange multipliers R_λ and $P(x, y, z, t)$ are then introduced, and (14)–(17), (23), and (24) are reformulated in a system of partial differential equations by requiring

$$\delta \left\{ I_1(v, \vartheta) + \lambda I_2(v, \vartheta) - \frac{2P}{R_\lambda} \nabla \cdot v + \frac{1}{R_\lambda} (D(v, v) + \lambda \mathcal{D}(\vartheta, \vartheta)) \right\} = 0. \tag{25}$$

The Euler-Lagrange equations corresponding to (25) are

$$\frac{R_\lambda}{2\lambda} (\lambda \nabla \psi + f) \cdot v = \nabla^2 \vartheta, \tag{26}$$

$$\mu R_\lambda v \cdot \epsilon + \frac{R_\lambda}{2} (\lambda \nabla \psi + f) \vartheta = -\nabla P + \nabla^2 v, \tag{27}$$

which are to be solved subject to (14)–(17) and (24).

Now for any solution of the variational equations (14)–(17), (26), and (27), we have by suitable multiplications and the divergence theorem that

$$\frac{R_\lambda}{2} \int (\lambda \nabla \psi + f) \cdot v \vartheta + \lambda \mathcal{D}(\vartheta, \vartheta) = 0, \tag{28}$$

$$\frac{R_\lambda}{2} \int (\lambda \nabla \psi + f) \cdot v \vartheta + \mu R_\lambda \int v \cdot \epsilon \cdot v + D(v, v) = 0. \tag{29}$$

By addition and equation (24)

$$-I_1(v, \vartheta) - \lambda I_2(v, \vartheta) = \frac{1}{R_\lambda}. \tag{30}$$

* It, of course, does not automatically follow that this maximum is attained in the function space defined by the differentiability requirements and boundary conditions though this is in fact the case. It follows from this that the maximizing functions are eigenvectors of the Euler-Lagrange equations (26) and (27), and (23) can be identified with the largest of the eigenvalues $1/R_\lambda$. A fuller discussion of this point can be found in [6].

The maximum of these eigenvalues $1/R_\lambda$ coincides with the solution of (14)–(17), (23), and (24). Hence, it follows that

$$\rho(\lambda, \mu) = \text{Min } R_\lambda(\mu) \tag{31}$$

for any of the positive set of eigenvalues R_λ . Then the eigenvalue problem and the optimum stability boundary are related by

$$R(\mu) = \text{Max}_{\lambda > 0} \rho(\lambda, \mu) = \text{Max}_{\lambda > 0} (\text{Min } R_\lambda(\mu)).$$

2. Limiting Cases

In the limit $\mu \rightarrow \infty$, one obtains from (30)

$$-\int \mathbf{v} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v} = \frac{1}{\text{Re}}, \tag{32}$$

and from (26) and (27)

$$\text{Re } \mathbf{v} \cdot \boldsymbol{\varepsilon} = -\nabla P + \nabla^2 v. \tag{33}$$

These are SERRIN'S [5] results. Equation (33) has been solved by SERRIN for rotating cylinders (narrow gap). He finds a band of angular velocities for which the flow is universally stable. When the rotation of the inner and outer cylinder have the same sense, SERRIN'S region closely borders TAYLOR'S experimental limit. Somewhat less successful is ORR'S [3] earlier analysis of (33) for plane Couette flow. His value for the critical Reynolds number is far below observed experimental values. Moreover, ORR'S value is incorrect and must be replaced by a yet lower value (see Section G).

In the limit $\mu \rightarrow 0$, the method produces a much stronger result. In fact, with $\mu = 0$,

$$\nabla \psi = \mathbf{f} = -i, \quad \vartheta = \frac{\varphi}{\sqrt{\lambda}} \quad \text{and} \quad R_\lambda = \frac{2\sqrt{\lambda}}{1+\lambda} \sqrt{\text{Ra}},$$

the boundary value problem (14)–(17), (26), and (27) coincides with the linear perturbation problem. Hence, if Ra_c is the critical Rayleigh number of the linear problem, then the maximum of ρ is obtained when $\lambda = 1$ and the flow is stable to arbitrary nonlinear disturbances provided only that

$$\sqrt{\text{Ra}} < \text{Max}_{\lambda > 0} \rho(\lambda, \mu) = R(\mu) = \sqrt{\text{Ra}_c}.$$

The results of UKHOVSKII and IUDOVICH [10], HOWARD [11], SANI [12], and PLATZMAN [13] relative to the nonexistence of *steady* subcritical instabilities are in this way extended to *all* subcritical instabilities.

The condition $\lambda = 1$ is valid whenever $\nabla \psi = \mathbf{f}$. For this case with

$$\vartheta = \frac{\varphi}{\sqrt{\lambda}} \quad \text{and} \quad R_\lambda = \frac{2\sqrt{\lambda}}{\lambda+1} A,$$

equations (26) and (27) may be written as

$$A \mathbf{f} \cdot \mathbf{v} = \nabla^2 \varphi,$$

and

$$\frac{2\mu\sqrt{\lambda}}{\lambda+1} A \mathbf{v} \cdot \boldsymbol{\varepsilon} + A f \varphi = -\nabla P + \nabla^2 \mathbf{v},$$

which are to be solved subject to (14)–(17). Then for a fixed value, μ , the eigenvalues A depend on λ only through the combination $\sqrt{\lambda}/(\lambda+1)$ and

$$\rho(\lambda, \mu) = \frac{2\sqrt{\lambda}}{\lambda+1} A \left(\frac{2\mu\sqrt{\lambda}}{\lambda+1} \right).$$

The best value of the coupling parameter is found according to the prescription of (21) as a root of the equation

$$0 = \left(A + \frac{2\mu A' \sqrt{\lambda}}{\lambda+1} \right) \frac{d}{d\lambda} \left(\frac{2\sqrt{\lambda}}{\lambda+1} \right)$$

which has a solution for $\lambda=1$.

These results may be recovered as special case of Theorem 2 which is derived in the next subsection.

3. The Stability Boundary and the Best Coupling Parameter

We express the solution of this problem as

Theorem 2. *Let $\rho(\lambda, \mu)$ be a continuously-differentiable function of both its arguments. Then the best value of the coupling parameter satisfies the relation*

$$\lambda(\mu) = \frac{\int \mathbf{f} \cdot \tilde{\mathbf{v}} \tilde{\mathcal{G}}}{\int \nabla \psi \cdot \tilde{\mathbf{v}} \tilde{\mathcal{G}}}, \tag{34}$$

where v and \mathcal{G} are maximizing functions for (23) belonging to the pair $\mu, \lambda(\mu)$. Moreover,

$$R(\mu) = -\frac{\mathcal{D}(\tilde{\mathcal{G}}, \tilde{\mathcal{G}})}{\int \nabla \psi \cdot \tilde{\mathbf{v}} \tilde{\mathcal{G}}}. \tag{35}$$

To prove Theorem 2, we let $\mathbf{v} = \tilde{\mathbf{v}}$ and $\mathcal{G} = \tilde{\mathcal{G}} = \tilde{\varphi}/\sqrt{\lambda}$ be solutions of the variational problem for any fixed values of μ and λ . We consider two such solutions and label them with subscripts. By an obvious procedure, we obtain from equations (26) and (27) the four equations ($j=1, i=2$, and $j=2, i=1$)

$$\frac{1}{2} \int (\lambda_j \nabla \psi + \mathbf{f}) \cdot \mathbf{v}_j \varphi_i = -\frac{\sqrt{\lambda_j}}{R_j} \mathcal{D}(\varphi_j, \varphi_i), \tag{36}$$

$$\mu_j \sqrt{\lambda_j} \int \mathbf{v}_j \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}_i + \frac{1}{2} \int (\lambda_j \nabla \psi + \mathbf{f}) \cdot \mathbf{v}_i \varphi_j = -\frac{\sqrt{\lambda_j}}{R_j} D(\mathbf{v}_j, \mathbf{v}_i), \tag{37}$$

where $R_i = \rho(\lambda_i, \mu_i)$. A linear combination of the equations can then be made to produce

$$\begin{aligned} & (\mu_2 \sqrt{\lambda_2} - \mu_1 \sqrt{\lambda_1}) \int \mathbf{v}_1 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}_2 + \frac{\lambda_2 - \lambda_1}{2} \int \nabla \psi \cdot (\varphi_2 \mathbf{v}_1 + \varphi_1 \mathbf{v}_2) \\ & = -\left(\frac{\sqrt{\lambda_2}}{R_2} - \frac{\sqrt{\lambda_1}}{R_1} \right) \{ D(\mathbf{v}_1, \mathbf{v}_2) + \mathcal{D}(\varphi_1, \varphi_2) \}. \end{aligned}$$

We now let the solutions coalesce and use $\varphi = \sqrt{\lambda} \vartheta$ and equation (24) to produce

$$d(\mu \sqrt{\lambda}) \int \mathbf{v} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v} + 2\lambda d \sqrt{\lambda} \int \nabla \psi \cdot \mathbf{v} \vartheta = -d \left(\frac{\sqrt{\lambda}}{R} \right). \quad (38)$$

Equation (38) is a special case of the more general equation

$$\int \mathbf{v} \cdot d(\mu \sqrt{\lambda} \boldsymbol{\varepsilon}) \cdot \mathbf{v} + \sqrt{\lambda} \int d \mathbf{G}_\lambda \cdot \mathbf{v} \vartheta = -d \left(\frac{\sqrt{\lambda}}{R} \right) - \frac{\sqrt{\lambda^3}}{R} \oint dh \vartheta^2, \quad (39)$$

which is valid when $\boldsymbol{\varepsilon}$ and $\mathbf{G}_\lambda = \lambda \nabla \psi + \mathbf{f}$ depend on parameters other than R , μ , $\sqrt{\lambda}$. This equation is obtained from (35) and (36) modified to accommodate the additional parameter dependence, that is, with $\nabla \psi$, $h\mathbf{f}$ and $\boldsymbol{\varepsilon}$ replaced with $\nabla \psi_j$, h_j , \mathbf{f}_j and $\boldsymbol{\varepsilon}_j$. The more general relation (39) has been used by JOSEPH & SHIR [14] in an application involving the effect of internal heating on convective instabilities in fluid layers.

Equation (34) may be deduced from (38). Consider first the problem of the best coupling parameter. For this we require that $\lambda(\mu)$ be a root of (20) which implies through (38) that

$$\mu \int \mathbf{v} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v} + 2\lambda \int \nabla \psi \cdot \mathbf{v} \vartheta = -\frac{1}{R(\mu)}.$$

One compares this with the maximum problem (23) to produce equation (34). Equation (35) follows easily by elimination of λ from equation (28).

Equation (34) is particularly valuable for estimating the best λ when it is not possible to determine this from *a priori* considerations (*cf.* [14, 15]). Of course the results of the preceding subsection are immediately recovered from (34) when $\mathbf{f} = \nabla \psi$.

On the other hand, with λ fixed, one obtains from (22) and (38)

$$\frac{dR(\mu)}{d\mu} = \left(\frac{\partial \rho}{\partial \mu} \right)_\lambda = R^2 \int \tilde{\mathbf{v}} \cdot \boldsymbol{\varepsilon} \cdot \tilde{\mathbf{v}}. \quad (40)$$

Since $\tilde{\mathbf{R}}e = \mu R$, we have

$$R \frac{d\tilde{\mathbf{R}}e}{dR} = \frac{1}{\int \mathbf{v} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}} + \tilde{\mathbf{R}}e. \quad (41)$$

Then with λ given by (34), rewrite (39) as

$$\tilde{\mathbf{R}}e \int \tilde{\mathbf{v}} \cdot \boldsymbol{\varepsilon} \cdot \tilde{\mathbf{v}} + 2R \int \mathbf{f} \cdot \tilde{\mathbf{v}} \tilde{\vartheta} = -1 \quad (42)$$

and (28) as

$$\int \tilde{\mathbf{v}} \cdot \mathbf{f} \tilde{\vartheta} = -\frac{\lambda}{R} \mathcal{D}(\tilde{\vartheta}, \tilde{\vartheta}). \quad (43)$$

From (41), (42), and (43) we deduce

Theorem 3. *Let the hypotheses of Theorem 2 prevail. Then the slope of the stability boundary is given by*

$$\frac{d\tilde{\mathbf{R}}e}{d\sqrt{\mathbf{R}a}} = \frac{2 \int \tilde{\mathbf{v}} \cdot \mathbf{f} \tilde{\vartheta}}{-\int \mathbf{v} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}} = \frac{2\lambda \mathcal{D}(\tilde{\vartheta}, \tilde{\vartheta})}{R \int \tilde{\mathbf{v}} \cdot \boldsymbol{\varepsilon} \cdot \tilde{\mathbf{v}}}. \quad (44)$$

If $\tilde{\mathbf{R}}e(\mu = \infty) > 0$, then this slope is negative for sufficiently large μ .

To prove the last statement of the theorem note that from (42) and (44)

$$\frac{d \tilde{Re}}{d \sqrt{\tilde{Ra}}} = \frac{-2\lambda \mathcal{D}(\tilde{\vartheta}, \tilde{\vartheta})}{\frac{R}{\tilde{Re}} \{1 + 2R \int \mathbf{f} \cdot \tilde{\mathbf{v}} \tilde{\vartheta}\}}. \tag{45}$$

Also $\tilde{Re}(\mu \rightarrow \infty) > 0$ and $\tilde{Re} = \mu R(\mu)$ implies that $R(\mu \rightarrow \infty) \rightarrow 0$. This last result and (45) prove the theorem.

Equation (44), like (34), has a practical value. As we shall show, this equation may be used to infer the exact representation for the optimum stability boundary of a plane Couette flow heated from below. For other applications see [14] and [15].

F. Subcritical Instabilities, Vorticity and Rotation

Consider the solution of the variational problem for the case of an initially quiescent fluid ($\mu = 0$). Further, let $\nabla\psi = \mathbf{f}$, so that in compliance with (34), $\lambda = 1$. For this case, we have from the variational equations (26) and (27)

$$\begin{aligned} R_1 \nabla\psi \cdot \mathbf{v} &= \nabla^2 \vartheta, \\ R_1 \mathbf{f} \vartheta &= -\nabla P + \nabla^2 \mathbf{v}, \end{aligned}$$

which are to be solved subject to (14), (15), (16), and (17). It may be readily verified that this boundary-value problem is identical to that of linear perturbation theory when the principle of exchange of stabilities is invoked and partial time derivatives are set to zero. This implies

Theorem 4. *Subcritical instabilities of an initially quiescent Boussinesq fluid cannot exist when $\nabla\psi = \mathbf{f}^*$.*

We call attention to the fact that the vorticity distribution of the basic motion enters explicitly into the energetics (equation (1)) only through the strain-rate tensor \mathbf{D} . The vorticity tensor is not explicit in the energetics. Hence, considerations which start *ab initio* from the energy identities (1) and (2) and from kinematic and boundary constraints necessarily neglect stability effects of the basic-flow vorticity. This is true of the energy method, and its scope is greatly restricted by the obvious inability of the formalism to reflect important instabilities, like those in parallel flows, which are largely a consequence of the vorticity distribution of the basic motion.

The same remarks apply to rotation of the system as a whole. This follows from the relation of rotation to vorticity. Alternately, we may note that in a system of coordinates which rotates with the angular velocity $\mathbf{\Omega}$, the Boussinesq equations are unaltered except for the addition of D'Alembert accelerations

$$-\{2\mathbf{\Omega} \times \mathbf{V} + \dot{\mathbf{\Omega}} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})\},$$

to the R.H.S. of the momentum equation. Naturally, these terms do not enter the energetics, as may be seen by forming the equations of the difference motion

* This extends Theorem 4 of JOSEPH [5]. In this extension we do not require that $\mathbf{f} = \nabla\psi = -\mathbf{i}$. Moreover, free-nonplanar boundaries are allowed, e.g., spherical shells.

and the required scalar products. It follows that equations (1) and (2) hold when \mathbf{v} is interpreted as the velocity seen by an observer rotating with the angular velocity Ω .

An important result may be trivially deduced from these considerations. Let us suppose that the energy problem has been solved and a critical Rayleigh number \tilde{Ra} obtained. Suppose the rotating system is unstable at some Rayleigh number Ra^Ω . Since the rotation does not change \tilde{Ra} , we have $Ra^\Omega \geq \tilde{Ra}$. In the absence of rotation, the flow is definitely unstable when $Ra \geq Ra_c$ where Ra_c is the critical value of linear theory. In general

$$Ra^\Omega \geq \tilde{Ra} \leq Ra_c.$$

When the inequality $\tilde{Ra} < Ra_c$ holds, one cannot rule out the possibility that $Ra^\Omega < Ra_c$. In this case rotation would destabilize the flow. For flows which are subcritically stable (in the sense of Theorem 4), however,

$$Ra^\Omega \geq \tilde{Ra} = Ra_c,$$

from which follows

Theorem 5. *Rigid rotation cannot destabilize that class of motions (Theorem 4) which are subcritically stable.*

G. Couette Flow Heated from Below

We shall here apply the variational formulation of the energy method to the problem of a plane Couette flow heated from below. We find: If (Ra, Re) satisfy the inequality

$$Re^2 + Ra < 1708,$$

then the flow is stable (Fig. 2). For $Ra = 0$ we have stability when (with $md^2/\nu = Bd^2/2\nu$)

$$\frac{Bd^2}{2\nu} = Re < \tilde{Re} = \sqrt{1708} \approx 41.3, \tag{46}$$

where Bz is the distribution of the basic velocity and d the distance between the plates (Fig. 2). This value compares with

$$\frac{Bd^2}{2\nu} = 88.6,$$

which has been given by ORR. ORR's value is wrong and is here replaced with the lower value (46)*.

Couette flow, according to linear theory, is stable for all values of the Reynolds number. The addition of the possibility of thermal convection does not lead to any instability provided that the critical Rayleigh number of 1708 is not reached. At this critical value the flow becomes unstable to longitudinal rolls (cross-stream disturbances) at any Reynolds number**. The combination of normal

* The value (46) may be obtained directly from SERRIN's result for Couette flow between rotating cylinders. The appropriate limits fix the difference in speed of the two cylinders and the gap size and the result then follows by letting the radii tend to infinity.

** Longitudinal rolls seem to be the preferred mode for convective instability in shear flow and have been observed by BRUNT [19].

to the plate and cross-stream disturbances serves only to raise the threshold for the onset of convection (cf. JEFFEREYS [16], DEARDORFF [17], GALLAGHER & MERCER [18]).

This same combination of disturbances is most destabilizing for the energy equations even in the zero Rayleigh number limit. ORR's statement [3, p. 17]

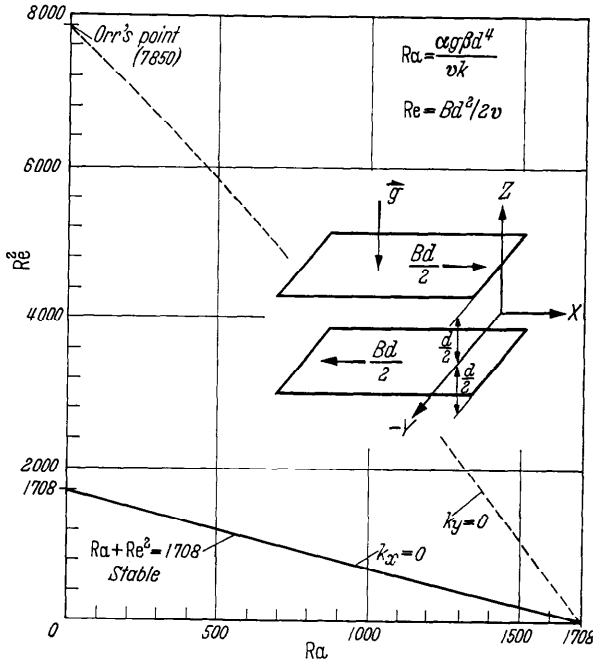


Fig. 2. Plane Couette flow heated from below

that "Analogy with other problems leads us to assume that disturbances in two dimensions will be less stable than those in three;..." is correct if modified to read "... disturbances which vary in two dimensions ...", but he has chosen the wrong two dimensions as the basis for his calculation.

The details of the basic motion can be read from Fig. 2. The coordinate origin is located at mid-channel, x is the stream axis, \hat{i} is a unit vector normal to the plate in the direction of increasing \hat{z} , and the lower plate is held at a fixed higher temperature than the upper plate.

The governing equations are (26) and (27). Here we note

$$(x, y, z) \rightarrow (\hat{x} d, \hat{y} d, \hat{z} d),$$

$$\epsilon_{13} = \epsilon_{31} = \frac{D_{13}}{m} = \frac{B/2}{B/2} = 1,$$

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_{12} = \epsilon_{21} = \epsilon_{23} = 0,$$

and

$$\nabla\psi = \mathbf{f} = -\hat{i}.$$

This last relation implies through (34) of Part I that $\lambda=1$. Equations (26) and (27) may then be written as

$$-R\omega^* = \nabla^2 \vartheta^*, \tag{47}$$

$$\mu R\omega^* = -\frac{\partial P^*}{\partial \hat{x}} + \nabla^2 u^*, \tag{48}$$

$$0 = -\frac{\partial P^*}{\partial \hat{y}} + \nabla^2 v^*, \tag{49}$$

$$\mu R u^* - R \vartheta^* = -\frac{\partial P^*}{\partial \hat{z}} + \nabla^2 \omega^*, \tag{50}$$

$$\frac{\partial u^*}{\partial \hat{x}} + \frac{\partial v^*}{\partial \hat{y}} + \frac{\partial \omega^*}{\partial \hat{z}} = 0. \tag{51}$$

Here the starred variables are the physical variables (formerly unstarred). These are to be solved subject to the conditions

$$u^*(\pm \frac{1}{2}) = v^*(\pm \frac{1}{2}) = \omega^*(\pm \frac{1}{2}) = \vartheta^*(\pm \frac{1}{2}) = 0 \tag{52}$$

and some boundedness conditions on the functions for the limit $\hat{x}^2 + \hat{y}^2 \rightarrow \infty$.

It makes no formal difference in the subsequent mathematical work whether we regard the solution as a superposition of the normal modes

$$(u^*, v^*, \omega^*, P^*, \vartheta^*) = (u, v, \omega, P, \vartheta) \exp \{i(k_x \hat{x} + k_y \hat{y})\} \tag{53}$$

or assume boundedness conditions at ∞ which assure the existence of the Fourier transforms of the functions

$$(u, v, \omega, P, \vartheta) = \iint_{-\infty}^{\infty} d\hat{x} d\hat{y} (u^*, v^*, \omega^*, P^*, \vartheta^*) e^{-i(k_x \hat{x} + k_y \hat{y})}$$

and differential equations (47)–(52) satisfied by the functions. In either case one obtains

$$-R\omega = L(\vartheta), \tag{54}$$

$$\mu R\omega = -i k_x P + L(u), \tag{55}$$

$$0 = -i k_y P + L(v), \tag{56}$$

$$\mu R u - R \vartheta = -D P + L(\omega), \tag{57}$$

$$i k_x u + i k_y v + D \omega = 0, \tag{58}$$

$$u(\pm \frac{1}{2}) = v(\pm \frac{1}{2}) = \omega(\pm \frac{1}{2}) = \vartheta(\pm \frac{1}{2}) = 0, \tag{59}$$

where

$$D = \frac{d}{d\hat{z}}, \quad k^2 = k_x^2 + k_y^2, \quad L = D^2 - k^2$$

as the system of ordinary differential equations which govern the problem.

The system may be transformed into an equivalent sixth order ordinary differential equation

$$L^3 \omega + 2i k_x \mu R L(D \omega) + R^2(k^2 + \mu^2 k_y^2) \omega = 0, \tag{60}$$

which is to be solved subject to the deduced conditions

$$\omega = D \omega = L^2(\omega) = 0 \quad \text{at} \quad \hat{z} = \pm \frac{1}{2}. \tag{61}$$

In the general case this is a formidable problem. However, as is well known, when the plates are at rest and $\mu=0$, the least eigenvalue parameter is $R^2=1708$ and the corresponding $k^2=(3.12)^2$ (see CHANDRASEKHAR [7], p. 36). Clearly, this solution is also valid when $k_x=0$ and R^2 is replaced with $R^2(1+\mu^2)$.

This simplest solution, it develops, gives the least value of R for any fixed μ . Hence,

Theorem 6. *Plane Couette flow of a Boussinesq fluid between rigid conducting plates is stable provided that*

$$Re^2 + Ra < 1708.$$

It is appropriate here to remark that the proof of Theorem 6 makes substantial use of equations relating the parameters to solutions of the Euler-Lagrange equations. These include (44) and (45) of Theorem 3 as well as (71) which is derived in the appendix and gives the first derivative of the eigenvalue $R(k_x)$. The technique used to derive these equations, we suggest, has a potential for wide application.

The proof follows easily from the equations

$$R^2(0) = 1708 \tag{62}$$

and

$$\frac{d(\mu^2 R^2)}{dR^2} = -1. \tag{63}$$

This last equation is valid for all eigenvalues $R(k_x, k^2, \mu)$ which satisfy simultaneously the necessary condition for a minimum over k_x , i.e., $dR/dk_x=0$, and the slope equations (44) and (45). Equation (62) is a simple consequence of Theorem 4. Equation (63) is to be established.

Assuming (63), we have by integration from $\mu=0$ and (62) that

$$\mu^2 R^2 + R^2 = 1708,$$

which proves the theorem. It remains then to establish (63).

First we note that our normalization (24)

$$\int \nabla \vartheta^* \cdot \nabla \vartheta^* + \int \nabla v^* : \nabla v^* = D(\vartheta^*, \vartheta^*) + D(v^*, v^*) = 1$$

fixes the scale of the amplitude functions so that, for example

$$\int \omega^* u^* = \frac{1}{2}(\omega, \bar{u}) + \frac{1}{2}(\omega, \bar{u}),$$

$$\int \nabla \vartheta^* \cdot \nabla \vartheta^* = (D \vartheta, D \bar{\vartheta}) + k^2(\vartheta, \bar{\vartheta}),$$

where

$$(a, b) = \int_{-\frac{1}{2}}^{\frac{1}{2}} a b d\hat{z}.$$

Here the starred variables are taken as proportional to the real part of the normal mode representation and are integrated over one wave length in each of the horizontal directions. We also note that for functions which vanish together with their first derivative

$$(\omega_1, L^2 \omega_2) = (L\omega_1, L\omega_2).$$

With these preliminaries aside we establish (63). First we note that from (45)

$$\mu \frac{d(\mu R)}{dR} = \frac{d(\mu^2 R^2)}{dR^2} = \frac{-2\mu^2 D(\vartheta^*, \vartheta^*)}{1 - 2D(\vartheta^*, \vartheta^*)}. \tag{64}$$

Then we show that for those eigenvalues which are minimum over k_x (see equation (71))

$$D(\vartheta^*, \vartheta^*) = \frac{1}{2(1 + \mu^2)}, \tag{65}$$

proving (63).

Equation (65) is proved in two stages. First, five equations in six scalar products following from (54) through (59) and valid for preassigned wave numbers are derived. The condition that the R be minimum over k_x then gives a sixth equation and leads to (65) by direct elimination.

First we obtain the five equations for six scalars products. These follow from the normal mode equations plus the normalizing condition.

By subtraction of (28) from (29) we learn that

$$2\mu R \int \omega^* u^* = D(\vartheta^*, \vartheta^*) - D(v^*, v^*) = 2D(\vartheta^*, \vartheta^*) - 1 = \mu R \{(\bar{\omega}, u) + (\omega, \bar{u})\}. \tag{66}$$

From (54) and (55) we obtain

$$-(\bar{\omega}, u) - (\omega, \bar{u}) = \frac{2\mu}{R} D(\vartheta^*, \vartheta^*) + \frac{i k_x}{R} \{(\bar{\vartheta}, p) - (\vartheta, \bar{p})\}, \tag{67}$$

where we have used the relations $(\bar{u}, L\vartheta) = (\vartheta, L\bar{u})$ and $R\{(\omega, \bar{\vartheta}) + (\bar{\omega}, \vartheta)\} = 2D(\vartheta^*, \vartheta^*)$. An easy calculation following from (54) and (56) shows that

$$(\vartheta, \bar{p}) - (\bar{\vartheta}, p) = \frac{R}{i k_y} \{(\bar{\omega}, v) + (\omega, \bar{v})\} \tag{68}$$

and from (58)

$$0 = i k_x \{(\omega, \bar{u}) + (\bar{\omega}, u)\} + i k_y \{(\omega, \bar{v}) + (\bar{\omega}, v)\} + (\bar{\omega}, D\omega) - (\omega, D\bar{\omega}). \tag{69}$$

The equation

$$L^2 \omega = k^2 R \vartheta - \mu R k^2 u - i k_x \mu R D \omega,$$

which follows from eliminating p from (55), (56) and (57) is multiplied by $\bar{\omega}$, integrated by parts and added to its conjugate to obtain

$$2(L\bar{\omega}, L\omega) = 2k^2 D(\vartheta^*, \vartheta^*) - \mu R k^2 \{(\bar{\omega}, u) + (\omega, \bar{u})\} - i k_x \mu R \{(\bar{\omega}, D\omega) - (\omega, D\bar{\omega})\}, \tag{70}$$

where (54) has been used to eliminate $\bar{\omega}$ from the first term of the R. H. S.

Equations (66), (67), (68), (69) and (70) are five equations in the six unknowns $D(\vartheta^*, \vartheta^*)$, $(L\bar{\omega}, L\omega)$, real parts of $(\bar{\omega}, u)$, $(\bar{\omega}, v)$ and imaginary parts of (ϑ, p) and $(\bar{\omega}, D\omega)$. A sixth equation is obtained from the condition that R is a minimum over k_x (see the Appendix). Thus

$$0 = \frac{\{(1 + \mu^2)k^2 - \mu^2 k_x^2\}}{\mu} \frac{dR}{dk_x} = \frac{R^2 \{(1 + \mu^2)k^2 + \mu^2 k_x^2\} i \{(\bar{\omega}, D\omega) - (\omega, D\bar{\omega})\} + 2\mu k_x R(L\bar{\omega}, L\omega)}{2(L\bar{\omega}, L\omega) + \mu k_x R i \{(\bar{\omega}, D\omega) - (\omega, D\bar{\omega})\}}. \tag{71}$$

Equation (65) follows from the simultaneous solution of (66) through (71). This completes the proof of Theorem 6.

Though no further calculations are necessary to prove Theorem 6, it is perhaps of some interest resolve that eigenvalue problem explicitly. This is easily accomplished by a direct numerical solution of the algebraic system which is equivalent to the differential equation problem. This system is obtained upon substitution of the fundamental solutions

$$\omega = \sum_{j=1}^6 A_j \exp(i m_j \hat{z})$$

into (60) and (61). It is required that the six roots m_j of the equation

$$-(m^2 + k^2)^3 + 2k_x \mu R(m^2 + k^2) + R^2(k^2 + \mu^2 k_x^2) = 0$$

should determine $R(k_x, k^2, \mu)$ through the equation $D=0$, where D is the determinant whose columns are

$$\begin{aligned} &\exp\{i l_j\}, \\ &\exp\{-i l_j\}, \\ &l_j \exp\{i l_j\}, \\ &l_j \exp\{-i l_j\}, \\ &(l_j^2 + k^2)^2 \exp\{i l_j\}, \\ &(l_j^2 + k^2)^2 \exp\{-i l_j\} \end{aligned}$$

with $l_j = m_j/2$.

Representative results of this calculation are summarized in Fig. 3. Points on the curves $k = 3.12$ for which $k_x = 0$ are also points on the optimum stability boundary $R^2(1 + \mu^2) = 1708$. The curves are, of course, symmetric around $k_x = 0$ and

the inequality $R(k_x, (3.12)^2, \mu) < R(k_x, k^2, \mu)$ holds generally. A graphical representation of this inequality when $k_x=0$ can be found in CHANDRASEKHAR [7] (p.39), and elsewhere.

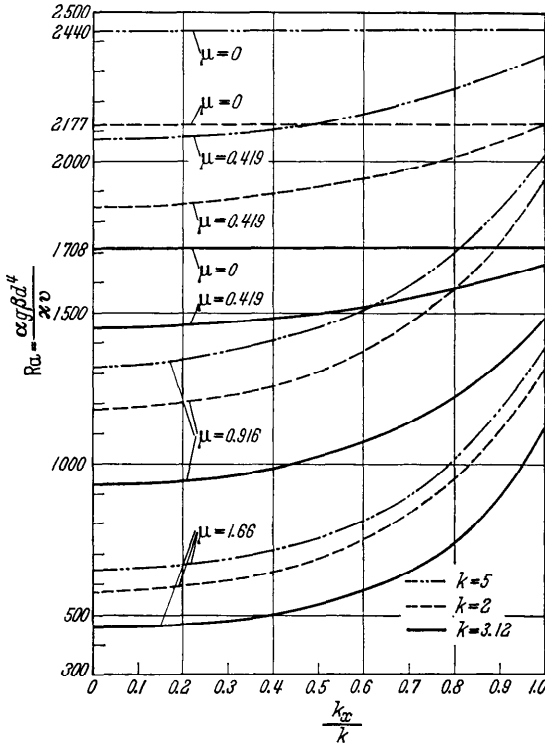


Fig. 3. Variation of Rayleigh number with wave number and wave number distribution

H. Discussion

The kernel of the energy method is a systematic procedure for review of functions admissible as solutions to the stability problem. The bases of admissibility are boundary conditions, mass conservation and continuity requirements. These latter reflect the fact that the diffusion of vorticity and heat smooths discontinuities. The functions which are admitted for review need not be possible solutions of the conservation equations (momentum and energy). It is in this sense that one may speak of dynamically inadmissible disturbances. Stability, as given by criteria of the energy method, implies that the flow is stable to all disturbances, the truly nonlinear ones as well as the spurious ones which violate local conservation requirements.

The potentials of the linear theory of stability and of the energy method are complimentary in the following sense: Linear theory gives conditions under which hydrodynamic systems are definitely unstable. It cannot with certainty conclude stability. Energy theory gives conditions under which hydrodynamic systems are definitely stable. It cannot with certainty conclude instability.

In some cases both theories are far from the real state of affairs, defined experimentally. Plane Couette flow is a case in point. Linear theory errs in giving

the flow as always stable; energy theory errs in giving a safe Reynolds number which is far too conservative.

On the other hand, there are instances in which the two theories coincide. This is true for the classical problem of a constant property "Boussinesq" fluid heated from below. Here the energy method leads to the strong result that arbitrary subcritical instabilities are not possible.

There are also cases in which the two theories, though different, lead to similar results (*cf.* [14, 15]). This is true of the Boussinesq fluid with heat sources and heated from below. Here the energy method gives critical Rayleigh numbers just slightly below the ones given by linear theory. In fact, the slightly lower values of Rayleigh number given by the energy method show that subcritical instabilities, if they occur at all, are restricted to the very narrow band of Rayleigh numbers lying between the critical values of the energy and linear theory. A similar result is true for Couette flow between rotating cylinders (SERRIN [5]) when the cylinders rotate with the same sense.

It is commonly believed that the energy method has not the potential of the linear theory for fine discriminations of the limits of stability. There are many situations for which this view is totally erroneous. Indeed, there are cases for which the two theories coincide or nearly coincide. In these cases the energy method gives sufficient conditions for stability which are physically reasonable as well as the ranges of the stability parameters for which subcritical instabilities are possible.

On the other hand, there are cases for which the criteria of the energy method are entirely too conservative, *e.g.*, plane Couette flow, and here the yield of the method is considerably less rich. It would seem that the too conservative estimates of stability are restricted to cases for which the onset of instability is characterized by propagating rather than stationary waves. Evidently, in such cases the energy method reflects a sensitivity to spurious and dynamically inadmissible disturbances. Certainly, the sufficient conditions are still valid; they are merely less useful.

The fact that the structure of mathematics of the method is rigorous and relatively simple certainly does not diminish its capacity for revealing truth. On balance it is, we believe, fair to say that the method has a potential not yet fully recognized and a domain of application which may be still further extended.

Acknowledgements. I am deeply indebted to J. SERRIN for his interest and for the many valuable suggestions which now find their place in the overall nature of this work. I wish also to thank C. C. SHIR for carrying out the numerical analysis on the computer.

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Appendix

Here we prove equation (71) of the text. First define

$$b = 2\mu k_x R, \quad (1a)$$

$$a = R^2 [k^2(1 + \mu^2) - \mu^2 k_x^2], \quad (2a)$$

$$U = L^2 \omega + i b D \omega. \quad (3a)$$

We note that U vanishes on the boundary. In this simplified notation the governing 6th order equation (60) may be written as

$$LU + a\omega = 0. \tag{4a}$$

We compare solutions of (4a) belonging to different k and R for fixed μ and k^2 . Then

$$LU_1 + a_1\omega_1 = 0,$$

$$L\bar{U}_2 + a_2\bar{\omega}_2 = 0$$

are scalar multiplied by \bar{U}_2 and U_1 , respectively, integrated and subtracted. Then as a consequence of the relation

$$(\bar{U}_2, LU_1) = (U_1, L\bar{U}_2) = -(D U_1, D \bar{U}_2) - k^2(\bar{U}_2, U_1),$$

we have

$$a_2(\bar{\omega}_2, U_1) - a_1(\bar{U}_2, \omega_1) = 0,$$

or

$$a_2(\bar{\omega}_2, L^2\omega_1 + i b_1 D \omega_1) - a_1(L^2\bar{\omega}_2 - i b_2 \bar{\omega}_2, \omega_1) = 0,$$

which may be written as

$$\delta a(L\omega_1, L\bar{\omega}_2) + i a_2 b_1(\bar{\omega}_2, D \omega_1) + i a_1 b_2(\omega_1, D \bar{\omega}_2) = 0$$

where δ is a difference operator. We add the conjugate of (5a) to (5a) to obtain

$$\delta a \{ (L\omega_1, L\bar{\omega}_2) + (L\bar{\omega}_1, L\omega_2) \} + i a_2 b_1 [(\bar{\omega}_2, D \omega_1) - (\omega_2, D \bar{\omega}_1)] - i a_1 b_2 [(\bar{\omega}_1, D \omega_2) - (\omega_1, D \bar{\omega}_2)] = 0,$$

or

$$\begin{aligned} & \delta a [(L\omega_1, L\bar{\omega}_2) + (L\bar{\omega}_1, L\omega_2)] + \\ & + i(a_1 + \delta a) b_1 [(\bar{\omega}_1 + \delta \bar{\omega}, D \omega_1) - (\omega_1 + \delta \omega, D \bar{\omega}_1)] - \\ & - i a_1 (b_1 + \delta b) [(\bar{\omega}_1, D \omega_1 + D \delta \omega) - (\omega_1, D \bar{\omega}_1 + D \delta \bar{\omega})] = 0. \end{aligned} \tag{6a}$$

Now in (6a) let solutions coalesce to obtain

$$\begin{aligned} & 2 \frac{da}{dk_x} (L\omega, L\bar{\omega}) + i \left(b \frac{da}{dk_x} - a \frac{db}{dk_x} \right) [(\bar{\omega}, D \omega) - (\omega, D \bar{\omega})] + \\ & + i a b \left[\left(\bar{\omega} + \frac{d\bar{\omega}}{dk_x}, D \omega \right) + \left(\omega, D \bar{\omega} + D \frac{d\bar{\omega}}{dk_x} \right) - \left(\omega + \frac{d\omega}{dk_x}, D \bar{\omega} \right) - \right. \\ & \left. - \left(\bar{\omega}, D \omega + D \frac{d\omega}{dk_x} \right) \right] = 0. \end{aligned}$$

The coefficient of ab above vanishes in integration leaving

$$\begin{aligned} & \left\{ 2 \left(\frac{\partial a}{\partial R} \right) (L\omega, L\bar{\omega}) + \left(b \frac{\partial a}{\partial R} - a \frac{\partial b}{\partial R} \right) i [(\bar{\omega}, D \omega) - (\omega, D \bar{\omega})] \right\} \frac{dR}{dk_x} + \\ & + 2 \frac{\partial a}{\partial k_x} (L\omega, L\bar{\omega}) + \left(b \frac{\partial a}{\partial k_x} - a \frac{\partial b}{\partial k_x} \right) i [(\bar{\omega}, D \omega) - (\omega, D \bar{\omega})] = 0. \end{aligned} \tag{7a}$$

Equation (7a) together with (1a) and (2a) prove equation (71) of the text.

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