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Lubrication of a Porous Bearing —Stokes' Solution

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Coupling of flows induced by the rotation of an infinite cylinder in an eccentric cylindrical hole in a fluid-saturated porous space is investigated in the context of a coupled boundary-value problem in which the Stokes flow outside porous regions and the Darcy flow inside porous regions are connected by continuity requirements on the pressure and normal component of velocity. The configuration is used to model the effects of a thick porous bearing. The solution simplifies considerably in the Reynolds limit of small clearance, and compact approximations for the pressure distribution and other relevant physical variables are derived. It is shown that transverse pressure gradients in the lubricant which are normally neglected in the Reynolds limit do increase, but not significantly, as a result of bearing flow. It follows that candidate Reynolds' equations may ordinarily ignore effects of transverse pressure gradients in the lubricant even when the bearing is porous. A principal effect of the porous flow on the coupled motion is a diminution of pressure differences which would develop if all solids were impermeable. Corresponding changes in the shear stress resultant, which is neglected relative to the pressure resultant in the impermeable Reynolds limit, can become dominant because of the diminished pressures which attend porous flow. For large eccentricity ratios, the shear resultant is negative, and the load capacity may fall to zero and even change sign.

Introduction

THE PROBLEM of finding the pressure distribution and force resultants created by the rotation of an impermeable journal in a permeable bearing is of practical engineering importance. Many bearings of this type are used in the construction of machine elements. In this paper, we treat an idealized version of this problem with the aim of isolating the physical effects of the bearing flow.

There is a number of approaches to this and related problems which take the one-dimensional equations of Reynolds as a starting point. In the case of a viscous fluid which is entrained in the gap between rotating porous rollers, it has been found—by G. I. Taylor and J. C. Miller [1][1] for zero clearance and by L. N. Tao and D. D. Joseph [2] for finite clearance—that the loss of fluid through the porous surface reduces the magnitude and shifts the position of the cross section of maximum suction. This suggests that the introduction of porous materials may be useful in applications where high suction is undesirable; e.g., where cavitation is a problem. An essential defect of the approach used by [1] and [2] is that the flow through the permeable surface is assumed proportional to the pressure. According to Darcy's law, the through-flow should be locally proportional to the transverse gradient of pressure. It is by means of the transverse gradients that the fluid in the channel between journal and bearing is sucked from or forced into the channel; since the one-dimensional treatment cannot accommodate transverse pressure gradients, it is not capable of describing the through-flow in any but ad hoc fashion.

A treatment of the related problem of the lubrication of a porous bearing which introduces the through-flow in a more natural way is due to V. Morgan and A. Cameron [3] and to A. Cameron, V. Morgan, and A. Stainsby [4].

Though these authors start from the Reynolds' equation, they do require that the through-flow at the porous surface be related to the pressure in the bearing by Darcy's law. Darcy's law implies, as a consequence of the conservation of mass, that the pressure in the porous bearing be harmonic. This condition is also approximately satisfied by Cameron, et al. The strength of their treatment is in the attempt to treat the flow as coupled. Of course, the transverse variation of pressure in the gap is necessarily neglected by virtue of the averaging implied by the Reynolds' equation. But, in addition to this, the solution has a very restricted range of applicability and is valid only for very narrow bearings in which axial variations of pressure dominate circumferential variations. For longer bearings (these are typical), circumferential variations are dominant. A solution of the modified Reynolds' equation proposed by Morgan and Cameron [3] which accounts for circumferential variations of pressure and

[1] Numbers in brackets designate References at end of paper.

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Nomenclature

a = radius of bipolar circle
c = clearance
d = distance between center of bipolar circle and origin
e = eccentricity
h = metric coefficient, $=\tau/(\cosh\gamma - \cos\eta)$
k = permeability
K = k/τ^2
M_0 = resultant moment about origin
M = torque on cylinder
N = index in series expansion of solution
p = pressure in free-fluid region
P = pressure in porous space
$\mathbf{Q}$ = filter velocity in porous space
Q = volume rate of flow
q = fluid velocity
u = x-component of velocity in a Cartesian coordinate
v = y-component of velocity in a Cartesian coordinate

(*Continued on next page*)

neglects the axial variations has been constructed by Shir and Joseph [5].

In this paper, we will consider the porous bearing problem from a quite different point of view. Here, we seek to find the variations of pressure and the force resultants in the context of a coupled boundary-value problem in which different flow regimes are matched at common porous boundaries. For the bearing problem, we require that the flow in the bearing satisfy Darcy's law, and that the flow in the lubricating film satisfy the low Reynolds number approximations of Stokes. For the clearances which prevail in typical lubrication problems, the Reynolds numbers are typically small enough so that Stokes' approximations are valid. The solutions in the two regimes, the bearing and the liquid film, are to be matched at the bearing surface by the requirement of continuity as applied to the normal component of stress and velocity.

The idea of using Stokes' equations for examination of journal-bearing lubrication is not new. In fact, G. Wannier [6] has shown, for the nonporous bearing, how the Reynolds-Sommerfeld solution of Reynolds' equation may be recovered from a Stokes solution taken in the limit of small clearance. Applied to the permeable bearing, this method has the obvious advantages of a fully two-dimensional treatment. In addition, it is possible to obtain a Reynolds-Sommerfeld solution as a limiting case of the Stokes solution. One may in this way obtain an appropriate Reynolds-Sommerfeld solution without a corresponding Reynolds' equation. It is this procedure which we adopt here.

For mathematical simplicity, we shall simulate a thick porous bearing with a fluid-saturated porous space with an internal hole in which a journal rotates. The harmonic pressure dies exponentially with distance from the internal hole, and the configuration can be shown to simulate the thick bearing with increasingly good accuracy as the radius ratio increases from a value of about three halves [5]. Viewed in another way, the finite-bearing solution of the Reynolds' equation used in [5] has a greater creditability by virtue of the fact that it reduces properly to the Stokes solution as the radius ratio tends to infinity.

It is assumed that conditions of full hydrodynamic lubrication prevail; that the permeable bearing is homogeneous and isotropic; and that the flow within it satisfies Darcy's law

$$\mathbf{Q} = -\frac{k}{\mu} \nabla P \quad (1)$$

where k, μ, and P are the permeability, dynamic viscosity, and pressure, respectively. The vector $\mathbf{Q}$ is a volume-flow rate per unit cross-sectional area. As such, it represents the filter velocity rather than the true velocity of the fluid in the pores. In the bearing

$$\nabla^2 P = 0 \quad (2)$$

as follows from mass conservation, $\nabla \cdot \mathbf{Q} = 0$.

One further assumption specifying some condition on the tangential component of velocity at the exterior of the bearing surface is required. Why this is so is made clear from the following considerations.

Consider a general flow in a layered media satisfying Darcy's law in each of the separate layers. In these problems, which involve flow between materials of different permeability, boundary pressures and velocities are unknown and the boundary conditions are supplied by the requirement that both normal stress and the normal component of velocity are continuous across surfaces separating regions of different permeability. As in the hydrodynamics of ideal fluids, no requirements are imposed on the tangential components of velocity. This condition is consistent with the nature of the Darcy law, which postulates that changes in the empirically equivalent velocity $\mathbf{Q}$ do not depend on the mechanisms of shear. Obviously, the governing equation (2) is not of sufficiently high order to accommodate conditions on the tangential component of velocity, and a discontinuity of this component of velocity across the surface separating regions of different permeability is to be expected.

When the porous boundary separates a region of permeable material from a region where such material is absent, it is necessary, if the governing differential system is not to be underdeterminate, to specify the behavior of the viscous liquid immediately outside the porous region.

In the case of a viscous flow, the order of the equations governing the external motion is raised by two, although the equations governing the internal motion are unchanged. It is therefore necessary to impose *some* requirement on the tangential component of velocity at the interface. When much of what would nominally be an impermeable surface is really hole entrance, the appropriate boundary condition is not easy to anticipate. However, when the permeability is small relative to some typical gross area of the porous body, it would seem valid to retain the no-slip condition relative to the wall. Since the permeability is slight in most natural materials, this condition is not severely restrictive.

In our present effort, we make the low Reynolds number (Stokes) approximations to the viscous equations. These approximations are particularly convenient in the coupled problem. Not only are the governing equations linear but both the external and internal pressure fields are harmonic, a fact which considerably lightens the labor involved in satisfying the matching conditions. For Stokes' classical problem (streaming of a viscous liquid past a porous sphere), the foregoing formulation is self-consistent, and a uniform approximation to the total velocity field can be obtained [7, 8]. A more difficult matching problem is involved in an attempt to assess the effects of the ground flow induced by the motion of a cylinder near the ground [9]. This latter problem is very closely related to the lubrication problem considered in this paper, and the physical effects of the ground flow model those of the bearing flow.

The foregoing formulation is not restricted to a linear representation (Stokes' approximation) of thé external flow, and at least one example of an external flow governed by the nonlinear Navier-Stokes equations (rotation of a porous disk in a viscous liquid) has been constructed [10].

The lubrication problem considered here may be represented geometrically by nonconcentric circles of a bipolar family, Fig. 1. For these, there are available classical results which apply in Stokes' regime (Re $\rightarrow \infty$) when all boundary surfaces are impermeable. Although the general Stokes solution for the permeable case cannot be obtained in closed form, a very accurate compact approximation, equation (41), which is valid in the limit of small clearance can be derived. Moreover, in the practical range of permeabilities, this solution has almost the same circum-

Nomenclature

V = velocity of cylinder or porous space
X = x-component of force on cylinder
Y = y-component of force on cylinder
X_s = shear-stress resultant on cylinder
X_p = pressure-stress resultant on cylinder
$S_{\gamma i}$ = $\sinh(\gamma_i)$
$C_{\gamma i}$ = $\cosh(\gamma_i)$
α = $N - 1$
β = $N + 1$
γ, η = bipolar coordinates
ζ = $\gamma + i\eta$
θ = angle measured counterclockwise from top of polar circle
μ = dynamic viscosity
σ = stresses
τ = one-half the distance between pole points in bipolar coordinate
Φ_N = coefficient in series expansion of solution
ϕ_{MN} = constant coefficient in series expansion of solution
ψ = stream function

ferential variation as the known impermeable solution to which it reduces for vanishing permeability.

The solutions show that the principal physical effects introduced by the bearing flow are a diminution of pressure differences which would develop if journal and bearing were impermeable. The positions of maximum and minimum pressure are shifted away from the minimum gap. Total forces are altered not only by diminution of the pressure resultant but also by the relative insensitivity of the Couette-induced shear resultant to changes in pressure. This shift in the relative contributions of the pressure and shear resultants, which also operates on the induced groundflow problem [9], can have an interesting effect on the load capacity of the bearing. For impermeable bearings, it is usual to identify the load capacity with the bearing capacity. This procedure, strictly speaking, is not valid, because there is a net contribution due to shear. However, the shear resultant is smaller than the pressure resultant by a factor proportional to the clearance and ordinarily can be neglected [11]. The shear-stress resultant is given by

$$X_s = 4\pi\mu a_1 V\{2 - (4 - \epsilon^2)/(2 + \epsilon^2)\alpha\}/e$$

where e is the eccentricity, $\epsilon = e/c$ is the eccentricity ratio, and $c = a_2 - a_1$ is the clearance. It may be readily verified that X_s changes sign at $\epsilon = 0.721$ and is negative for larger values of ϵ. When the bearing is permeable, the pressure resultant may be greatly reduced. The shear-stress resultant varies slowly in magnitude and turns negative for values of the eccentric ratio, Fig. 2, which get smaller as k is increased. With the pressure greatly reduced, the shear-stress resultant may become dominant and, for large eccentric ratios, the bearing capacity can in principle vanish and even change sign. It follows that a large load-carrying service is not to be expected from a porous bearing.

Coupled Flow Between Nonconcentric Cylinders

We consider the flow between a solid cylinder (III) and a cylindrical cavity in an infinite porous matrix (II). The low Reynolds number approximations of Stokes govern region I and the equations of Darcy apply in II. The effect of a rotating cylinder near a moving porous ground is obtained when $\gamma_2 = 0$. The cylinders are represented by bipolar circles, Fig. 1.

We introduce Lagrange's stream function for the fluid in region I. In the Stokes approximation, the pressure is harmonic and conjugate to the Laplacian of the stream function. Thus

$$\nabla^2\psi + ip/\mu = f(z) \quad (3)$$

is an analytic function of the complex variable $z = x + iy$. The x and y-components of velocity are $-\partial\psi/\partial y$ and $\partial\psi/\partial x$, respectively.

We do not consider the rotation of the porous space, so that $V_2 = 0$ when $\gamma_2 \neq 0$.

The governing equations are

$$\left.\begin{aligned} \nabla^4\psi &= 0 \\ \nabla^2\psi + ip/\mu &= f(z) \end{aligned}\right\} \text{(in I)} \quad (4)$$

$$\mathbf{Q} = -\frac{k}{\mu}\nabla P \quad (5)$$

$$\text{(in II)}$$

$$\nabla^2 P = 0 \quad (6)^2$$

The boundary conditions are as follows:

1 The pressure is a single-valued function of the bipolar circular variable η, where

$$\zeta = \gamma + i\eta = \log\frac{z + i\tau}{z - i\tau}$$

and τ is one-half the distance between pole points.

[2] Lower-case letters (q, p) give velocity and pressure in I. Capital letters (Q, P) give these quantities in II.

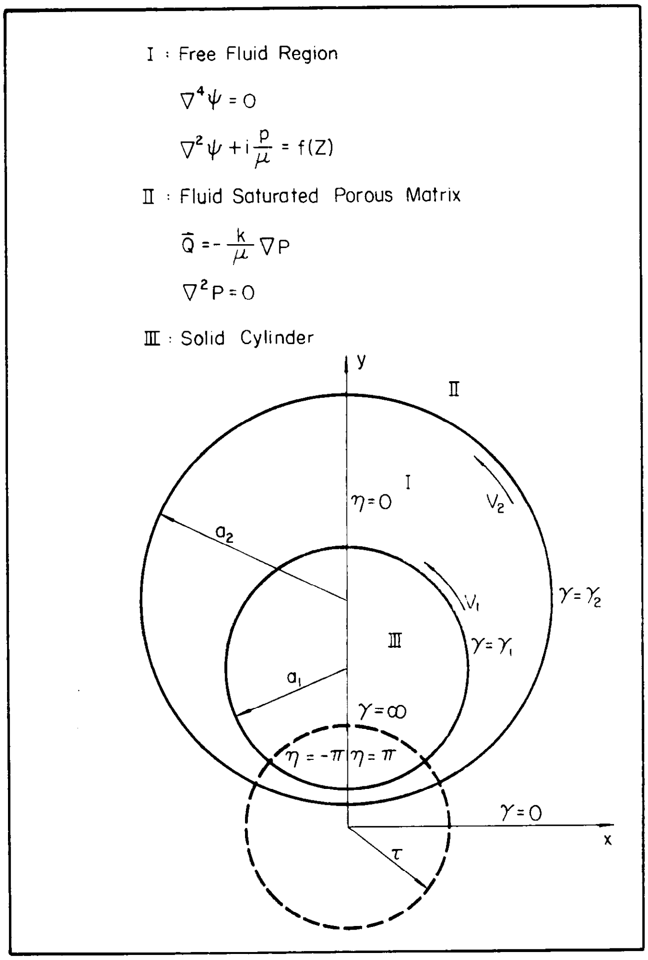

Fig. 1 Geometry of flow between nonconcentric cylinder

2 Pressure is continuous across porous boundaries[3]

$$P(\gamma_2, \eta) = p(\gamma_2, \eta)$$

3 The normal component of velocity vanishes at the solid boundary

$$q_\gamma(\gamma_1, \eta) = -\frac{1}{h}\frac{\partial\psi}{\partial\eta}(\gamma_1, \eta) = 0 \quad (8)$$

and is continuous across the porous boundary

$$hq_\gamma(\gamma_2, \eta) = -\frac{\partial\psi}{\partial\eta}(\gamma_2, \eta) = -\frac{k}{\mu}\frac{\partial P}{\partial\gamma}(\gamma_2, \eta) \quad (9)$$

where $h = \tau/(\cosh\gamma - \cos\eta)$ is a metric coefficient.

4 The fluid does not slip at the porous boundary

$$q_\eta(\gamma_1, \eta) = \frac{1}{h}\frac{\partial\psi}{\partial\gamma}(\gamma_1, \eta) = -V_1 \quad (10)$$

$$q_\eta(\gamma_2, \eta) = \frac{1}{h}\frac{\partial\psi}{\partial\gamma}(\gamma_2, \eta) = -V_2 \quad (11)$$

General solutions of equations (3)–(11) are developed in [9]. In this section, we list only those results which are used in subsequent deductions. First define

$$C_{N\gamma} = \cosh N\gamma, \qquad S_{N\gamma} = \sinh N\gamma$$

$$\Phi_0(\gamma) = \varphi_{01}C_\gamma + \varphi_{02}S_\gamma - \varphi_{12}\gamma C_\gamma + \varphi_{04}\gamma S_\gamma$$

[3] The difference between the pressure and normal stress in the exterior fluid is proportional to the Darcy coefficient (k) times the curvature of the boundary. We neglect this small difference for the bearing problem.

$$\Phi_1(\gamma) = \varphi_{11} + \varphi_{12}\gamma + \varphi_{13}C_{2\gamma} + \varphi_{14}S_{2\gamma} \tag{12}$$

$$\Phi_N(\gamma) = \varphi_{N1}C_{N-1,\gamma} + \varphi_{N2}S_{N-1,\gamma} + \varphi_{N3}C_{N+1,\gamma} + \varphi_{N4}S_{N+1,\gamma}$$

The stream function for the motion of the fluid in the cavity is given by

$$\psi = \frac{\tau}{C_\gamma - \cos\eta}\sum_{N=0}\Phi_N(\gamma)\cos N\eta \tag{13}$$

and the pressure by

$$\frac{p\tau}{2\mu} = -\varphi_{04}S_\gamma \sin\eta + \sum_{N=1}\{(1-N)(\varphi_{N2} + \varphi_{N+1,4})C_{N\gamma} + (N+1)(\varphi_{N4} + \varphi_{N+1,2})C_{N\gamma} + (1-N)(\varphi_{N1} + \varphi_{N-1,3})S_{N\gamma} + (N+1)(\varphi_{N3} + \varphi_{N+1,1})S_{N\gamma}\}\sin N\eta \tag{14}$$

The harmonic pressure P in the fluid saturated porous space is given by[4]

$$\frac{P\tau}{2\mu} = \sum_{N=1} A_N^{(2)}e^{N(\gamma-\gamma_2)}\sin N\eta \tag{15}$$

The cavity and porous matrix solutions are related by boundary conditions which lead ultimately to ($i = 1, 2$):

$$A_1^{(i)} = -\varphi_{04}S_{\gamma_i} + 2(\varphi_{14} + \varphi_{22})C_{\gamma_i} + 2(\varphi_{13} + \varphi_{21})S_{\gamma_i} \tag{16}$$

$$A_N^{(i)} = \{(1-N)(\varphi_{N2} + \varphi_{N-1,4}) + (N+1)(\varphi_{N4} + \varphi_{N+1,2})\}C_{N\gamma_i} + \{(1-N)(\varphi_{N1} + \varphi_{N-1,3}) + (N+1)(\varphi_{N3} + \varphi_{N+1,1})\}S_{N\gamma} \tag{17}$$

$$E_1^{(i)} = \varphi_{12} + 2\varphi_{13}S_{2\gamma_i} + 2\varphi_{14}C_{2\gamma} = (-1)^{i-1}2K_iA_1^{(i)}S_\gamma \tag{18}$$

$$\begin{aligned} E_2^{(i)} &= \varphi_{02} + \varphi_{04}(\gamma_i + S_{\gamma_i}C_{\gamma_i}) - C_{\gamma_i}^2\varphi_{12} \\ &= -C_{\gamma_i}V_i + (-1)^{i-1}K_iA_1^{(i)}S_{\gamma_i} \end{aligned} \tag{19}$$

$$\begin{aligned} E_3^{(i)} &= \varphi_{00} - \varphi_{04}S_{\gamma_i}^2 + \varphi_{12}S_{\gamma_i}C_{\gamma_i} + \varphi_{13}C_{2\gamma_i} + \varphi_{14}S_{2\gamma} \\ &= S_{\gamma_i}V_i - (-1)^{i-1}K_i(A_2^{(i)} - C_{\gamma_i}A_1^{(i)}) \end{aligned} \tag{20}$$

$$\begin{aligned} E_{4N}^{(i)} &= \varphi_{N1}C_{N-1,\gamma_i} + \varphi_{N2}S_{N-1,\gamma_i} + \varphi_{N3}C_{N+1,\gamma_i} + \varphi_{4N}S_{N+1,\gamma} \\ &= -(-1)^{i-1}K_i(-2C_{\gamma_i}A_N^{(i)} + A_{N-1}^{(i)} + A_{N+1}^{(i)}) \end{aligned} \tag{21}$$

[4] This pressure is finite at $\gamma = -\infty$, which is a pole point in the interior of the porous material. In [9], equation (15) should be replaced with (15) above. The inappropriate (15) of [9] is a miscopy and does not change any other results in that reference.

$$\begin{aligned} E_{5N}^{(i)} &= (N-1)\{\varphi_{N1}S_{N-1,\gamma_i} + \varphi_{N2}C_{N-1,\gamma_i}\} + (N+1)\{\varphi_{N3}S_{N+1,\gamma_i} + \varphi_{N4}C_{N+1,\gamma_i}\} \\ &= 2(-1)^{i-1}K_iS_{\gamma_i}A_N^{(i)} \end{aligned} \tag{22}$$

where

$$K_1 = A_N^{(1)} = 0, \qquad K_2 = k/\tau^2$$

It will be observed that φ_{01} and φ_{11} appear only in the combination $\varphi_{00} = \varphi_{01} + \varphi_{11}$. This leaves an undetermined constant which can be used to assign a zero streamline.

A further analytical reduction of equations (16)–(22) is of value in the lubrication limit of small clearance. This reduction is carried out in the Appendix.

Interest in lubrication theory focuses on integrated force and moment resultants.

The resultant force per unit length on the rotating cylinder is given by (see [9])

$$X = 4\mu\pi\varphi_{04}, \qquad Y = 0 \tag{23}$$

The resultant torque referred to the cylinder center is

$$M = -4\mu\pi\tau(\varphi_{12} - \coth\gamma_1\varphi_{04}) \tag{24}$$

The shear-stress (more exactly, stress deviator) resultant is of some interest in this problem (see closing remarks in the Introduction) and is given by

$$\begin{aligned} X_s = 2\pi\mu\Bigg\{\varphi_{04}(1 + e^{-2\gamma_1}) - \Bigg\{\sum_{N=1}^{\infty} N[(N+1)(\varphi_{N3} + \varphi_{N+1,1}) &+ (1-N)(\varphi_{N1} + \varphi_{N-1,3})](1 + e^{-2N\gamma_1}) \\ &+ N[(N+1)(\varphi_{N4} + \varphi_{N+1,2}) + (1-N)(\varphi_{N2} + \varphi_{N-1,4})](1 - e^{-2N\gamma_1})\Bigg\}\Bigg\} \end{aligned} \tag{25}$$

Equations (12)–(25) are formally exact. All of the physical quantities of interest and their variations may be obtained once the φ_{NM} are known. In general, the recursive relations (16)–(22) are too complicated to be solved analytically, and the φ_{NM} must

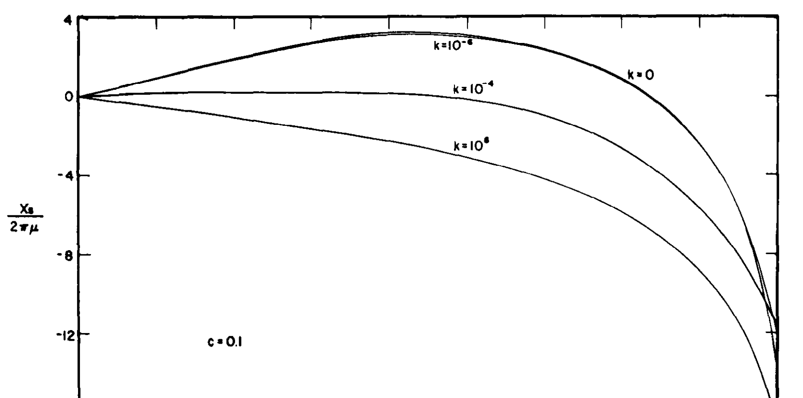

Fig. 2 Variation of shear-stress resultant with eccentricity and permeability. Shows graphs of equation (25) [or (38)]. For sufficiently large eccentricities (ϵ), this resultant opposes the sense of pressure resultant—much more so as permeability (k) is increased. For large ϵ and k, this resultant and pressure resultant are comparable.

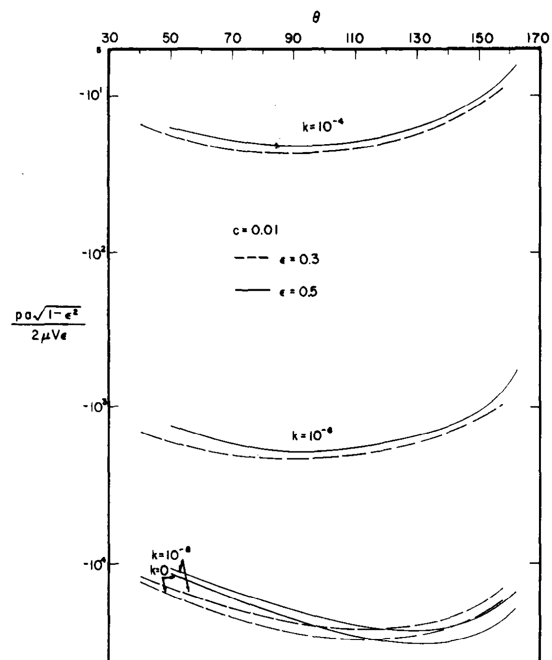

Fig. 3 Pressure variation in lubricant. Shows graphs of equation (14) [or (36)]. Pressure decreases rapidly as permeability (k) increases. Pressure minimum also shifts away from position of minimum gap (θ = 180 deg) as permeability increases.

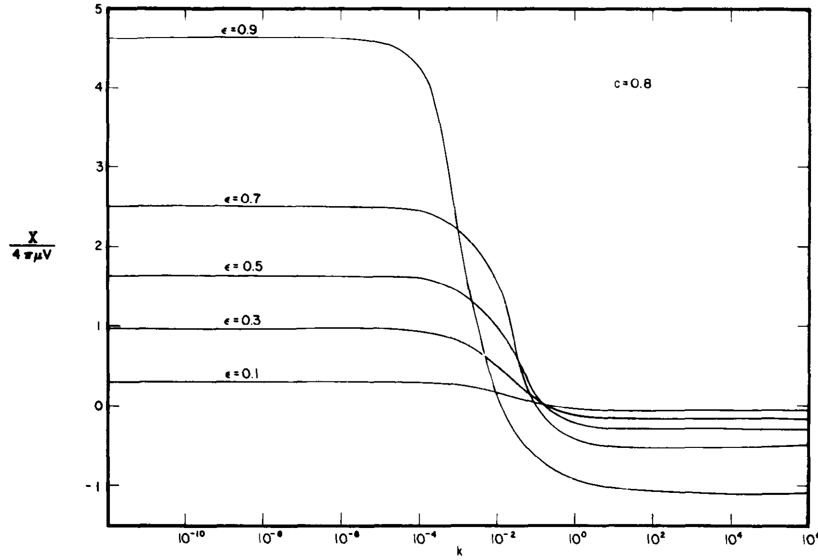

Fig. 4 Total force on journal. Shows graphs of equation (23). Total force on journal is more sensitive to changes in permeability when eccentricity is larger. For sufficiently large permeabilities, total force changes signs.

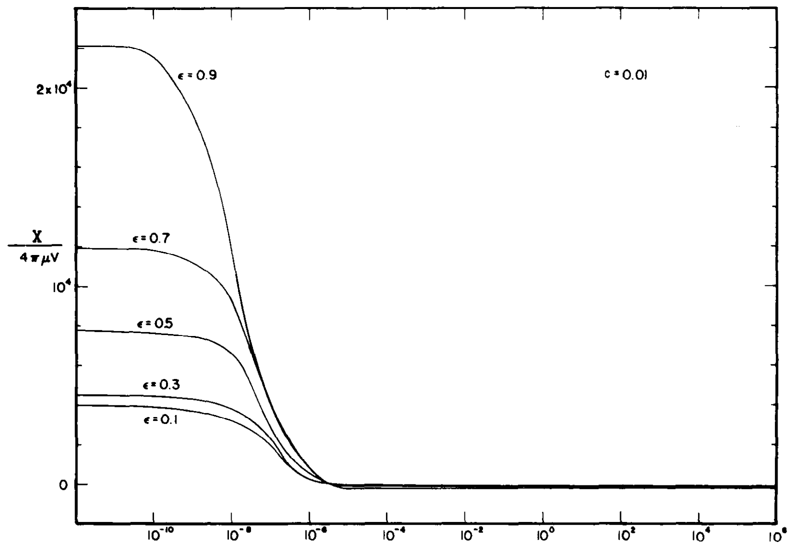

Fig. 5 Total force on journal. Shows graphs of equation (23) [or (37)] for a lubrication clearance ($c = d = 0.01$). Qualitatively, the situation is similar to that described under Fig. 4, except that, for smaller clearances, force is sensitive to smaller values of permeability.

be obtained numerically. This procedure has been followed in obtaining the results which are summarized in Figs. 2–8 and in the concluding section of this paper.

Important simplifications are, however, possible in the limit of our greatest interest—the lubrication (Reynolds-Sommerfeld) limit of small clearance. It is to these simplifications that we now turn.

Reynolds-Sommerfeld Limit

It is possible and useful to go to the lubrication limit of small clearance in the Stokes solution of the preceding section. This procedure obviates the need for a Reynolds' equation valid in the lubrication limit. One obtains, in this way, the solution of the idealized porous-bearing lubrication problem without commitment to the appropriate Reynolds' equation. Proposed Reynolds' equations should then have the property that their solutions resemble the Stokes solution when the bearing ring thickness is large. This is the procedure followed in [5] and provides one useful test for proposed approximations. We also note that the simplifications introduced in the limit of small clearance are such that it is possible to write down compact closed expressions which explicitly exhibit the interplay of the small clearance, permeability, and eccentricity.

To obtain the Reynolds-Sommerfeld limit, we develop equations given in the Appendix in powers of $[\gamma]$. The lowest powers of $[\gamma]$ are retained. For higher-order coefficients ($N \geq 2$), we obtain from equations (47)–(51) the relations

$$\Delta_N = \tfrac{4}{3}N^2(N^2 - 1)[\gamma]^4 \qquad (26)$$

$$\varphi_{N3} = -S_{N+1,\gamma_1}\varphi_{N4}/C_{N+1,\gamma_1} \qquad (27)$$

$$\varphi_{N2} = -(N + 1)C_{N-1,\gamma_1}\varphi_{N4}/(N - 1)C_{N+1,\gamma_1} \qquad (28)$$

$$\varphi_{N1} = (N + 1)S_{N-1,\gamma_1}\varphi_{N4}/(N - 1)C_{N+1,\gamma_1} \qquad (29)$$

$$\Delta_N\varphi_{N4} = -4N(N - 1)[\gamma]C_{N+1,\gamma_1}E_{4N}^{(2)} \qquad (30)$$

where $E_{4N}^{(2)}$ is defined by equation (21) and evaluated with γ_1 replacing γ_2. Equation (30) may be expanded and equations (26)–(29) used to obtain a five-term recursion relation between

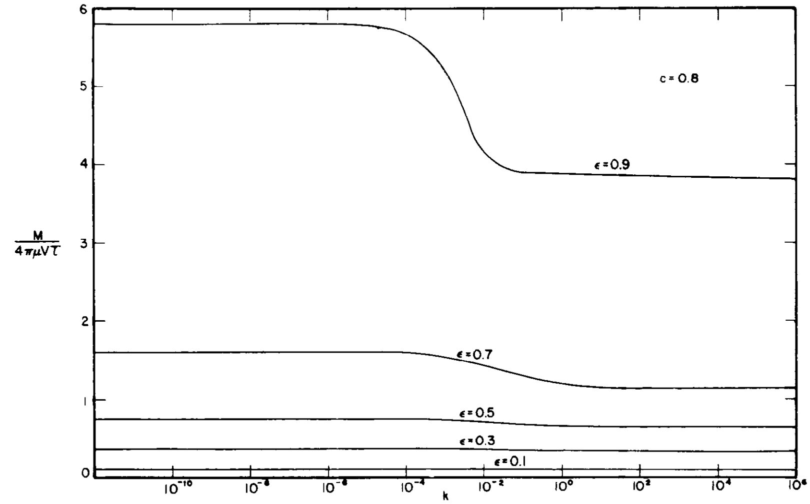

Fig. 6 Torque on journal Shows graphs of equation (24). Torque decreased, but not significantly, for large permeability. Torque is largely determined by relative motion of bearing and journal, and this does not change as a result of bearing flow.

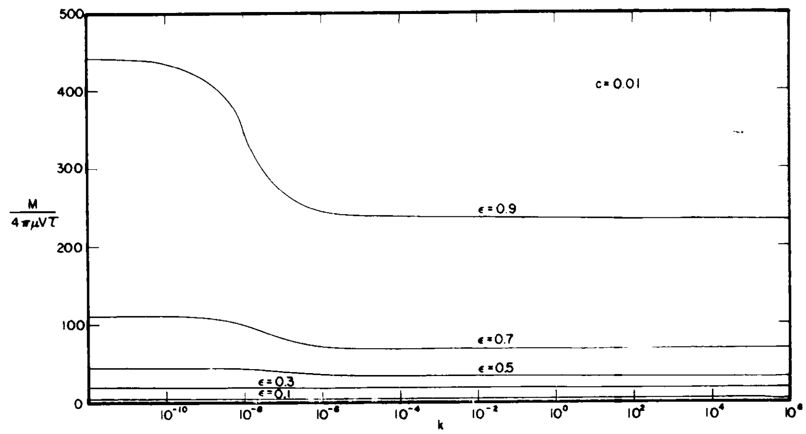

Fig. 7 Torque on journal. Shows graphs of equation (24) [or (39)] for a lubrication clearance ($c = d = 0.01$). Comments under Fig. 6 apply to this figure.

$\varphi_{N-2,4}$ through $\varphi_{N+2,4}$. A method of truncation can be used to express higher-order coefficients in terms of lower-order ones. In this way, higher-order coefficients are expressed in terms of φ_{14}, φ_{13}, and φ_{04}. Equations (43), (45), and (46), which are then three equations among the three unknowns, are solved, and the result is expanded in powers of $[\gamma]$. In this last operation, care must be taken not to neglect terms of order K relative to $[\gamma]$.[3] These operations are easily carried out for large N. Higher-order harmonics do not appreciably affect the value of the force resultants. This is especially true for k not large. For large k, the distribution of pressure is sensitive to higher harmonics, but the magnitude of the pressure approaches zero. The force coefficients are so well represented with $N = 2$ that the approximate and true solutions cannot be distinguished for moderate k and are within 10 percent for very large k. In the notation of lubrication theory,

$$C_{\gamma_1} = 1/\epsilon, \qquad S_{\gamma_1} = \alpha/\epsilon, \qquad \alpha = (1 - \epsilon^2)^{1/2}$$

$$d = c/a, \qquad -[\gamma] = d\alpha, \qquad \tau = a\alpha/\epsilon$$

$$K = k\epsilon^2/a^2\alpha^2 = \kappa\epsilon^2/\alpha^2$$

we obtain ($N = 2$):

$$\lambda_1 = d^3\alpha^5 + 2\kappa(3 + 2\epsilon^2) \tag{31}$$

$$\lambda = \frac{d}{\alpha\epsilon^2}\left\{\frac{d^3\alpha^5}{3}(2 + \epsilon^2) + \kappa(4 - 3\epsilon^2 + 2\epsilon^4) + 6\kappa^2\epsilon^2(3 - \epsilon^2)^2/\lambda_1\right\} \tag{32}$$

$$\varphi_{04} = \frac{V}{\epsilon\alpha^2\lambda}\{d^2\alpha^5 - \kappa(4 - \epsilon^2)/\alpha + 6\kappa^2\epsilon^2(3 - \epsilon^2)/\lambda_1\} \tag{33}$$

$$\varphi_{13} = \frac{V}{\epsilon^4\lambda}\{d^2\epsilon\alpha^3 + \kappa\epsilon\alpha(2 - \epsilon^2) - 3\kappa^2(3 - \epsilon^2)/\lambda_1\} \tag{34}$$

$$\varphi_{22} = \frac{3\kappa}{\alpha\epsilon^2\lambda_1}(\epsilon^4\varphi_{13} + \epsilon^2\alpha^2\varphi_{04}/2) \tag{35}$$

$$\frac{pa}{2\mu} = \frac{-\alpha}{(\epsilon\cos\theta + 1)^2}\{\varphi_{04}\sin\theta(\epsilon\cos\theta + 1) + \varphi_{13}\epsilon^2\sin\theta + \epsilon\alpha\varphi_{22}\sin 2\theta\} \tag{36}$$

where θ is the polar angle measured from the top of the cylinder

$$X = 4\pi\mu\varphi_{04} \tag{37}$$

$$X_S = \frac{4\pi\mu V(\alpha - 1)}{\lambda\alpha^2\epsilon^3}\left\{\frac{d^3\alpha^5}{3}(1 + 2\epsilon^2 - 3\alpha) + \kappa(4\alpha^2 + \epsilon^4) - 3\kappa^2\epsilon^2(3 - \epsilon^2)(\epsilon^2 + 1)/\alpha^2\lambda_1\right\} \tag{38}$$

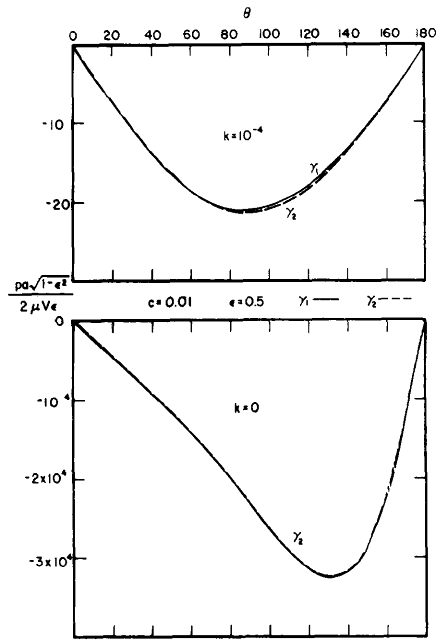

Fig. 8 Transverse pressure variation in lubricant. Shows graphs of equation (14) evaluated at journal surface (γ_1) and at bearing surface (γ_2). Difference between pressures at γ_1 and γ_2 for any fixed value of θ is a measure of sign of transverse pressure gradients (these are normally neglected in lubrication theory). It is seen that these transverse gradients are larger, but not significantly, when bearing is permeable. To neglect transverse pressure, gradients would then seem to be valid when bearing is porous.

$$M = \frac{4\pi\mu a}{\alpha\epsilon^2\lambda}\left\{\frac{d^3\alpha^4}{3} + \kappa(8 - 15\epsilon^2 + 11\epsilon^4 - 2\epsilon^6)/2\alpha - 3\kappa\epsilon^2(3 - \epsilon^2)^2/\lambda_1\right\} \quad (39)$$

For $\kappa = 0(d^3)$ the coefficients φ_{04} and φ_{13} simplify to

$$\varphi_{13} = \varphi_{04}/\epsilon^2 = Vd^2\alpha^3/\lambda\epsilon^3 \quad (40)$$

Hence, for small clearance, the load capacity of the porous bearing is reduced by the factor $\lambda(0)/\lambda$ where $\lambda(0)$ is λ evaluated with $\kappa = 0$. The pressure distribution is then given by the compact formula

$$p = \frac{p(0)\lambda(0)}{\lambda} - \frac{6\mu Vd^2\kappa(1 - 2\epsilon^4 - \epsilon^2/2 + \epsilon^6)}{a\epsilon^6\lambda\lambda_1(\epsilon\cos\theta + 1)^2} \quad (41)$$

Here, $p(0)$ is the impermeable (Reynolds-Sommerfeld) solution. A prime effect of the bearing flow is the reduction in the amplitude of the pressure (and force resultant) by the factor $\lambda(0)/\lambda$. The distribution of pressure is also altered by the additional term in (41) but this does not affect the bearing load capacity.

Discussion of Results

Figs. (2)–(11) represent the effects of the flow in the bearing on the pressure distribution and force resultants. The results are obtained from convergent machine calculations (with $a = 1$). With $d \leq 0.1$ and $0.1 \geq \epsilon \leq 0.9$, the same graphs may be obtained from equations (31)–(36).

The following observations are worth emphasis:

1 Increased flow within the porous bearing, i.e., increased permeability, leads to drastic reductions in the maximum pressure as well as a shift of this maximum away from the minimum gap, Fig. 3. This shift is less drastic than in the porous ground [9], because the difference in the curvature of the boundaries is less.

2 The total force resultant decreases strongly with permeability (Figs. 4, 5). The decrease in the torque with permeability is much less marked, Figs. 6, 7, and the same is true for the shear-stress resultant, Fig. 2. For very large permeabilities, the total load and the (small) shear resultant are equal and *negative*.

3 Transverse pressure gradients vanish much less rapidly as clearance is reduced when the bearing is permeable. This can be seen by comparison of the difference in pressures at the bearing and cylinder surfaces, Fig. 8. It would appear, however, that the assumption of no transverse variation of pressure continues to be valid for the porous bearing.

Finally, it must be stressed that results obtained in this analysis are valid under the assumption that the fluid does not slip relative to the porous solid. This is a plausible but not necessarily valid assumption. The influence of the tangential component of velocity is perhaps small, in any case, but the precise nature of this boundary flow needs further investigation, particularly from an experimental point of view.

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APPENDIX

A further reduction of (16)–(22) is possible. Regard $E_n^{(i)}$ as given and solve (18)–(20) for the six coefficients φ_{0M}, φ_{1M}. Equations (21) and (22) are solved for the four coefficients φ_{NM}. The result is as follows:

$$[x_i] = x_2 - x_1$$
$$[x_i|y_i] = x_2y_1 - x_1y_2 \quad (42)$$
$$\Delta_0 = 2S^2_{[\gamma_i]}\{2S_{\gamma_1}S_{\gamma_2}S_{[\gamma_i]} - [\gamma_i](S_{\gamma_2}^2 - S_{\gamma_1}^2)\}$$

$$\Delta_0\varphi_{04} = [E_2^{(i)}](2S_{[\gamma_i]}^2 - S_{[2\gamma_i]}[S_{2\gamma_i}]/2) + [S_{\gamma_i}^2]\{S_{[\gamma_i]}^2(E_1^{(1)} + E_1^{(2)}) - S_{[2\gamma_i]}[E_3^{(i)}]\} \quad (43)$$

$$\Delta_0\varphi_{12} = [\gamma_i + S_{2\gamma_i}/2]\{S_{[\gamma_i]}^2(E_1^{(1)} + E_1^{(2)}) - S_{[2\gamma_i]}[E_3^{(i)}]\} + [S_{\gamma_i}^2][E_2^{(i)}]S_{[2\gamma_i]} \quad (44)$$

$$2\Delta_0\varphi_{13} = [\gamma_i + S_{2\gamma_i}/2]\{[S_{2\gamma_i}/2]([E_1^{(i)}] + [C_{2\gamma_i}|E_1^{(i)}]) - 2[S_{\gamma_i}^2][E_3^{(i)}]\} - [S_{\gamma_i}^2]^2([C_{2\gamma_i}|E_1^{(i)}] - 2[E_2^{(i)}]) \quad (45)$$

$$2\Delta_0\varphi_{14} = [\gamma_i + S_{2\gamma_i}/2]\{[S_{2\gamma_i}][E_3^{(i)}] - [S_{\gamma_i}^2][E_1^{(i)}] + [E_1^{(i)}|S_{2\gamma_i}][S_{2\gamma_i}/2] - [S_{\gamma_i}^2]\{[S_{2\gamma_i}][E_2^{(i)}] + [S_{\gamma_i}^2][E_1^{(i)}|S_{2\gamma_i}]\} \quad (46)$$

$$\Delta_N = 4(S_{[N\gamma_i]}^2 - N^2S_{[\gamma_i]}^2)$$
$$\alpha = N - 1 \quad (47)$$
$$\beta = N + 1$$

$$\Delta_N\varphi_{N1} = \begin{vmatrix} \beta[E_{4N}^{(i)}C\beta\gamma_i] - [E_{5N}^{(i)}S\beta\gamma_i] & [S_{2N\gamma_i} - NS_{2\gamma_i}] \\ [E_{5N}^{(i)}C\beta\gamma_i] - \beta[E_{4N}^{(i)}S\beta\gamma_i] & [NC_{2\gamma_i} - C_{2N\gamma_i}] \end{vmatrix} \quad (48)$$

$$\Delta_N\varphi_{N2} = \begin{vmatrix} [NC_{2\gamma_i} + C_{2N\gamma_i}] & \beta[E_{4N}^{(i)}C\beta\gamma_i] - [E_{5N}^{(i)}S\beta\gamma_i] \\ -[NS_{2\gamma_i} + S_{2N\gamma_i}] & [E_{5N}^{(i)}C\beta\gamma_i] - \beta[S\beta\gamma_iE_{4N}^{(i)}] \end{vmatrix} \quad (49)$$

$$\Delta_N\varphi_{N3} = \begin{vmatrix} \alpha[C\alpha\gamma_iE_{4N}^{(i)}] - [E_{5N}^{(i)}S\alpha\gamma_i] & [NS_{2\gamma_i} - S_{2N\gamma_i}] \\ [C\alpha\gamma_iE_{5N}^{(i)}] - \alpha[E_{4N}^{(i)}S\alpha\gamma_i] & [NC_{2\gamma_i} + C_{2N\gamma_i}] \end{vmatrix} \quad (50)$$

$$\Delta_N\varphi_{N4} = \begin{vmatrix} [NC_{2\gamma_i} - C_{2N\gamma_i}] & \alpha[C\alpha\gamma_iE_{4N}^{(i)}] - [E_{5N}^{(i)}S\alpha\gamma_i] \\ [NS_{2\gamma_i} + S_{2N\gamma_i}] & [C\alpha\gamma_iE_{5N}^{(i)}] - \alpha[E_{4N}^{(i)}S\alpha\gamma_i] \end{vmatrix} \quad (51)$$

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