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Ground Flow Induced by a Moving Cylinder

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Ground flow induced by rotation and translation of a solid cylinder is investigated in the context of a coupled boundary-value problem in which the Stokes flow outside porous regions and the Darcy flow inside porous regions are connected by the continuity requirements on the pressure and normal component of velocity. A principal effect of the porous flow on the coupled motion is a diminution of pressure differences which would develop if all solids were impermeable. Corresponding changes in shear stress (which are largely induced by the Couette motion) are slight, and this can markedly alter the character of force resultants. Thus classical results which give (a) zero force on a rotating cylinder near a stationary non-porous ground and (b) zero torque on a nonrotating cylinder near a moving solid wall are annulled as a consequence of ground flow.

A. INTRODUCTION

THIS analysis aims at a quantitative assessment of coupling effects as these evolve in the motions of viscous liquids around and through liquid-saturated porous materials. The formulation is developed and applied to obtain the variation in the field and force resultants as these evolve in ground flow induced by a rotating cylinder as it translates in a viscous liquid above a liquid-saturated porous ground.

We assume that the permeable materials are homogeneous and isotropic, and that flow within these materials satisfies

$$\mathbf{Q} = (k/\mu)(\nabla P - \mathbf{F}), \qquad (1)$$

where k, μ, P, and $\mathbf{F}$ are the permeability, dynamic viscosity, pressure, and vector body force, respectively. The vector $\mathbf{Q}$ is a volume flow rate per unit cross-sectional area. As such it represents the filter velocity rather than the true velocity of the fluid in the pores.

To find the velocity and pressure fields in a permeable medium, it is necessary to solve a Poisson equation for the pressure,

$$\nabla^2 P = \nabla \cdot \mathbf{F}, \qquad (2)$$

which is obtained from the Darcy law (1) by application of the principle of conservation of mass (div $\mathbf{Q} = 0$). The appropriate boundary conditions fix the value of the pressure or the normal derivative of the pressure, i.e., the normal velocity, or a linear combination of these two on the boundaries. In problems which involve flow between materials of different permeability, boundary pressures and velocities may be unknown. In this case the boundary conditions are supplied by the requirement that both pressure and the normal component of velocity are continuous across surfaces separating regions of different permeability. As in the hydrodynamics of ideal fluids, no requirements are imposed on the tangential components of velocity. This condition is consistent with the nature of the Darcy law which postulates that changes in the empirically equivalent velocity $\mathbf{Q}$ do not depend on the mechanisms of shear. Obviously the governing equation (2) is not of sufficiently high order to accommodate conditions on the tangential component of velocity, and a discontinuity of this component of velocity across the surface separating regions of different permeability is to be expected.

When the porous boundary separates a region of permeable material from a region where such material is absent, it is necessary, if the governing differential system is not to be underdeterminate, to specify the behavior of the viscous liquid immediately outside the porous region.

In the case of a viscous flow, the order of the equations governing the external motion is raised by two though the equations governing the internal motion are unchanged. It is therefore necessary to impose *some* requirement on the tangential component of velocity at the interface. When much of what would nominally be an impermeable surface is really hole entrance, the appropriate boundary condition is not easy to anticipate. However, when the permeability is small relative to some typical gross area of the porous body, it would seem valid to retain the no-slip condition relative to the wall. Since the permeability is slight in most natural materials this condition is not severely restrictive.

Collecting the conditions consistent with the formalism, we require that: (a) The external field satisfy the Navier–Stokes equations; (b) the internal field satisfy Poisson's equation (2) for the pressure; (c) the normal component of stress and velocity be continuous at the boundary; (d) the relative tangential velocity vanish on the exterior of a porous wall.

In our present effort we make the low-Reynolds-number (Stokes) approximations to the viscous equations. These approximations are particularly convenient in the coupled problem. Not only are the governing equations linear, but both the external and internal pressure fields are harmonic, a fact which considerably lightens the labor involved in satisfying the matching conditions. For Stokes' classical problem (streaming of a viscous liquid past a porous sphere), the above formulation is self consistent, and a uniform approximation to the total velocity field can be obtained.[1,2] The formalism is not restricted to any particular specialization of the Navier–Stokes equations, and at least one example of a coupled flow using the full Navier–Stokes equations (rotation of a porous disk in a visious liquid) has been constructed.[3]

The problems we consider here may be represented geometrically by two nonconcentric circles of a bipolar family (Fig. 1). For these there are available classical results[4] which apply in Stokes' regime (Re $\rightarrow$ 0) when all boundary surfaces are impermeable. It would seem that when two stationary cylinders rotate in an infinite viscous liquid, corresponding solutions cannot be made to yield a uniform approximation to the total velocity distribution.[5] The problem may, however, be solved uniquely subject to no-slip conditions and to the condition that the velocity at infinity is constant (of unspecified magnitude). As Jefferey[4] remarks, this situation resembles an analogous situation which develops in the application of Stokes' equations to two dimensional streaming motions past a circular cylinder. In this latter problem the velocity diverges logarithmically with distance, and it is not possible to determine constants which fix the drag. Despite this logarithmic divergence, the velocity field is well represented as long as the viscous terms of full Navier–Stokes equations dominate the inertial terms. This situation persists far enough from the cylinder, and the uniform stream is nearly attained before the Stokes approximations fail.[6]

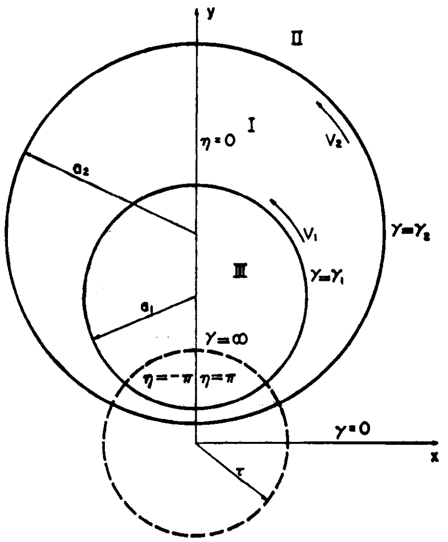

FIG. 1. Geometry of flow between nonconcentric cylinder. Region I: Free fluid region; $\nabla^4\psi = 0$, $\nabla^2\psi + ip/\mu = f(z)$. Region II: Fluid saturated porous matrix; $\mathbf{Q} = -(k/\mu)\nabla P$, $\nabla^2 P = 0$. Region III: Solid cylinder.

For bounded flows between cylinders, problems of the above type cannot arise even if the solids are permeable, because there is no far field in the free fluid. The intermediate case in which the cylinder rotates above an impermeable ground translating parallel to itself also develops without anomalies. (The velocity at infinity becomes uniform and equal to that of the ground.) However, the generalization of the ground problem to include effects of induced ground flow leads to a divergent velocity field. In this respect this solution resembles that describing two-dimensional streaming past a cylinder. But whereas the streaming problem is indeterminate and no drag is obtained, the ground flow problem is deterministic and leads to a finite drag and convergent pressure field. Since the solution evolves continuously from the impermeable case, it can scarcely be doubted that the near velocity field and pressure field as well as the force resultants are accurately represented in the limit Re $\rightarrow$ 0, $k \rightarrow 0$. For larger values of the permeability the regions in which the inertial terms dominate viscous terms encroach more and more on the near field. Thus our analysis, which neglects the inertial effects, can be expected to lead to results which become less reliable as the permeability is increased. The possible usefulness of the analysis is perhaps enhanced by the fact that the (small) permeability values which

[1] St. I. Gheorghitza, Arch. Ratl. Mech. Anal. **12**, 52 (1962).
[2] D. D. Joseph and L. N. Tao, Z. Angew. Math. Mech. **44**, 361 (1964).
[3] D. D. Joseph, Quart. J. Mech. Appl. Math. (to be published).
[4] G. Jefferey, Proc. Roy. Soc. (London) **A101**, 169 (1922).
[5] R. A. Frazer, Phil. Trans. Roy. Soc. (London) **225**, 93 (1926).
[6] I. Proudman and J. R. Pearson, J. Fluid Mech. **2**, 237 (1957).

occur in natural materials are precisely those for which the analysis can be expected to have the greatest validity.

In Sec. 2, we develop the equations and boundary conditions, introduce appropriate solution expansions and reduce the algebraic system which fits the solution expansions to conditions specified at the boundary. In Sec. 3, we develop expressions for force and moment resultants. In Sec. 4, the effects of ground flow are analyzed. Formulas are derived to represent the variation of field quantities and force resultants. These formulas are exact for both large and small permeabilities and are reasonable for the intermediate values. The ground flow is particularly effective in reducing large pressure gradients which would otherwise develop. Even when the pressure gradients are slight, interesting changes may be introduced by the presence of porous members. For example, it is a known result of classical theory that a cylinder rotating in the neighborhood of a stationary ground induces no force by virtue of its rotation.[5,7] The pressure forces equilibrate the shear forces. The classical result does not hold if the ground is permeable, for the pressure is reduced by virtue of the ground flow while the shear stresses, which are dominated by the Couette motion, are essentially unaltered. The cylinder cannot therefore remain in equilibrium without the addition of an external force. Similar results hold for the torque on a nonrotating cylinder in the neighborhood of a porous ground.

B. COUPLED FLOW BETWEEN NONCONCENTRIC CYLINDERS

We consider the flow between a solid cylinder (III) and a cylindrical cavity in an infinite porous matrix (II). The low Reynolds number approximations of Stokes govern region I, and the equations of Darcy apply in II. The effect of a rotating cylinder near a moving porous ground is obtained when $\gamma_2 = 0$. The cylinders are represented by bipolar circles (Fig. 1).

We introduce Lagrange's stream function for the fluid in region I. In the Stokes approximation the pressure is harmonic and conjugate to the Laplacian of the stream function. Thus,

$$\nabla^2\psi + ip/\mu = f(z) \tag{3}$$

is an analytic function of the complex variable $z = x + iy$. The x and y components of velocity are $-\partial\psi/\partial y$ and $\partial\psi/\partial x$, respectively.

[7] G. Wannier, Quart. J. Appl. Math. 8, 1 (1950).

We do not consider the rotation of the porous space, so that $V_2 = 0$ when $\gamma_2 \neq 0$.

The governing equations are

$$\left.\begin{aligned}\nabla^4\psi &= 0,\\ \nabla^2\psi + ip/\mu &= f(z);\end{aligned}\right\} \quad \text{(in I)} \tag{4}$$

$$\left.\begin{aligned}\mathbf{Q} &= -(k/\mu)\nabla P,\\ \nabla^2 P &= 0.\end{aligned}\right\} \quad \text{(in II)}. \tag{5, 6}$$

[Lower case letters (q, p) give velocity and pressure in I. Upper case letters (Q, P) give these quantities in II.]

The boundary conditions are as follows:

(1) The pressure is a single-valued function of the bipolar circular variable η, where

$$\zeta = \gamma + i\eta = \log\frac{z + i\tau}{z - i\tau} \tag{7}$$

and τ is one half the distance between pole points.

(2) Pressure is continuous across porous boundaries,

$$P(\gamma_2, \eta) = p(\gamma_2, \eta).^{8}$$

(3) The normal component of velocity vanishes at the solid boundary

$$q_\gamma(\gamma_1, \eta) = -\frac{1}{h}\frac{\partial\psi}{\partial\eta}(\gamma_1, \eta) = 0 \tag{8}$$

and is continuous across the porous boundary

$$hq_\gamma(\gamma_2, \eta) = \frac{\partial\psi}{\partial\eta}(\gamma_2, \eta) = \frac{k}{\mu}\frac{\partial P}{\partial\gamma}(\gamma_2, \eta), \tag{9}$$

where $h = \tau/(\cosh\gamma - \cos\eta)$ is a metric coefficient.

(4) The fluid does not slip at the porous boundary;

$$q_\eta(\gamma_1, \eta) = \frac{1}{h}\frac{\partial\psi}{\partial\gamma}(\gamma_1, \eta) = -V_1, \tag{10}$$

$$q_\eta(\gamma_2, \eta) = \frac{1}{h}\frac{\partial\psi}{\partial\gamma}(\gamma_2, \eta) = -V_2. \tag{11}$$

It develops that it is most convenient to express the solutions in real bipolar coordinates. The general solution for the biharmonic equation in these coordinates may be constructed by methods suggested by Jefferey.[9]

Elements of the complex theory of the biharmonic Eq. (7) are also conveniently applied to this problem, particularly in relation to the evaluation of the force

[8] For the porous ground the condition of a uniform tangential velocity requires that the viscous part of the normal stress vanish identically. In any case this part of the normal stress is proportional to the Darcy coefficient (k) and would, for natural materials, introduce a negligibly small correction.

[9] G. Jefferey, Phil. Trans. Roy. Soc. (London) **A221**, 265 (1921).

and torque on the journal. For this reason, it is useful to write solutions in complex as well as in real forms.

The relevant general solutions are listed below:

(1) The real stream function (which is symmetric in x) takes the form

$$\psi = \frac{\tau}{C_\gamma - \cos\eta} \sum_{N=0} \Phi_N(\gamma) \cos N\eta, \qquad (12)$$

where

$$C_{N\gamma} = \cosh N\gamma, \qquad S_{N\gamma} = \sinh N\gamma,$$

$$\Phi_0(\gamma) = \varphi_{01}C_\gamma + \varphi_{02}S_\gamma + \varphi_{03}\gamma C_\gamma + \varphi_{04}\gamma S_\gamma,$$

$$\Phi_1(\gamma) = \varphi_{11} + \varphi_{12}\gamma + \varphi_{13}C_{2\gamma} + \varphi_{14}S_{2\gamma},$$

$$\Phi_N(\gamma) = \varphi_{N1}C_{N-1,\gamma} + \varphi_{N2}S_{N-1,\gamma} + \varphi_{N3}C_{N+1,\gamma} + \varphi_{N4}S_{N+1,\gamma}.$$

(2) The stream function may also be obtained as the real part of the function $\Gamma(\zeta, \bar{\zeta})$, where

$$\Gamma(\zeta, \bar{\zeta})/\tau = -\varphi_{12}\zeta + \tfrac{1}{2}(\coth \bar{\zeta}/2 + \coth \zeta/2)\{\varphi_{04}\zeta + 2\varphi_{02} + \textstyle\sum_1^\infty (\varphi_{N3} - \varphi_{N1})S_{N\zeta} + (\varphi_{N4} - \varphi_{N2})C_{N\zeta}\} + \tfrac{1}{2}(\coth \zeta/2 \coth \bar{\zeta}/2 + 1)\{2\varphi_{01} + (\varphi_{03} + \varphi_{12})\zeta + \textstyle\sum_1^\infty (\varphi_{N2} + \varphi_{N4})S_{N\zeta} + (\varphi_{N1} + \varphi_{N3})C_{N\zeta}\}. \qquad (13)$$

(3) The external pressure is given as μ times the imaginary part of

$$4\left|\frac{d\zeta}{dz}\right|^2 \frac{\partial^2\Gamma(\zeta, \bar{\zeta})}{\partial\zeta\,\partial\bar{\zeta}},$$

which is evaluated as

$$\frac{p}{\mu} = \frac{2}{\tau}\operatorname{Im}\Big\{\varphi_{04}(1 - C_\zeta) + 2\varphi_{01} + \varphi_{11} + (\varphi_{03} + \varphi_{12})(\zeta - S_\zeta) + \sum_{N=1}^{\infty}\{S_{N\zeta}[(1 - N)(\varphi_{N2} + \varphi_{N-1,4}) + (N + 1)(\varphi_{N4} + \varphi_{N+1,2})] + C_{N\zeta}[(1 - N)(\varphi_{N1} + \varphi_{N-1,3}) + (N + 1)(\varphi_{N3} + \varphi_{N+1,1})]\}\Big\}$$

$$= \frac{2}{\tau}\Big\{-\varphi_{04}S_\gamma \sin\eta + (\varphi_{03} + \varphi_{12})(\eta - C_\gamma \sin\eta) + \sum_{N=1}^{\infty}\{(1 - N)(\varphi_{N2} + \varphi_{N-1,4})C_{N\gamma} + (N + 1)(\varphi_{N4} + \varphi_{N+1,2})C_{N\gamma} + (1 - N)(\varphi_{N1} + \varphi_{N-1,3})S_{N\gamma} + (N + 1)(\varphi_{N3} + \varphi_{N+1,1})S_{N\gamma}\}\sin N\eta\Big\}. \qquad (14)$$

(4) Harmonic functions P which are finite at ∞ are

$$\frac{P}{\mu} = \frac{2}{\tau}\sum_{N=1}^{\infty} A_N^{(1)} e^{-N(\gamma - \gamma_1)} \sin N\eta. \qquad (15)$$

Application of boundary conditions and techniques for evaluating the coefficients of the general solution.

If the pressure is to be single valued it follows from (13) that

$$\varphi_{12} = -\varphi_{03}.$$

The other boundary conditions applied routinely lead to six term recursion relations (see Appendix IA) for the φ_{MN}. These relations may be simplified considerably (Appendix IA) and lead finally to ($i = 1, 2$)

$$A_1^{(i)} = -\varphi_{04}S_{\gamma_i} + 2(\varphi_{14} + \varphi_{22})C_{\gamma_i} + 2(\varphi_{13} + \varphi_{21})S_{\gamma_i}, \qquad (16)$$

$$A_N^{(i)} = \{(1 - N)(\varphi_{N2} + \varphi_{N-1,4}) + (N + 1)(\varphi_{N4} + \varphi_{N+1,2})\}C_{N\gamma_i} + \{(1 - N)(\varphi_{N1} + \varphi_{N-1,3}) + (N + 1)(\varphi_{N3} + \varphi_{N+1,1})\}S_{N\gamma_i}, \qquad (17)$$

$$\varphi_{12} + 2\varphi_{13}S_{2\gamma_i} + 2\varphi_{14}C_{2\gamma_i} = E_1^{(i)} = (-1)^{i-1}2K_iA_1^{(i)}S_{\gamma_i}, \qquad (18)$$

$$\varphi_{02} + \varphi_{04}(\gamma_i + S_{\gamma_i}C_{\gamma_i}) - C_{\gamma_i}^2\varphi_{12} = E_2^{(i)} = -C_{\gamma_i}V_i + (-1)^{i-1}K_iA_1^{(i)}S_{\gamma_i}, \qquad (19)$$

$$\varphi_{00} - \varphi_{04}S_{\gamma_i}^2 + \varphi_{12}S_{\gamma_i}C_{\gamma_i} + \varphi_{13}C_{2\gamma_i} + \varphi_{14}S_{2\gamma_i} = E_3^{(i)} = S_{\gamma_i}V_i - (-1)^{i-1}K_i(A_2^{(i)} - C_{\gamma_i}A_1^{(i)}), \qquad (20)$$

$$\varphi_{N1}C_{N-1,\gamma_i} + \varphi_{N2}S_{N-1,\gamma_i} + \varphi_{N3}C_{N+1,\gamma_i} + \varphi_{N4}S_{N+1,\gamma_i} = E_{4N}^{(i)} = -(-1)^{i-1}K_i(-2C_{\gamma_i}A_N^{(i)} + A_{N-1}^{(i)} + A_{N+1}^{(i)}), \qquad (21)$$

$$(N - 1)\{\varphi_{N1}S_{N-1,\gamma_i} + \varphi_{N2}C_{N-1,\gamma_i}\} + (N + 1)\{\varphi_{N3}S_{N+1,\gamma_i} + \varphi_{N4}C_{N+1,\gamma_i}\} = E_{5N}^{(i)} = 2(-1)^{i-1}K_iS_{\gamma_i}A_N^{(i)} \qquad (22)$$

$$K_i = \delta_{i2}k/\tau^2.$$

It will be observed that φ_{01} and φ_{11} appear only in the combination $\varphi_{00} = \varphi_{01} + \varphi_{11}$. This leaves an undetermined constant which can be used to assign a zero streamline.

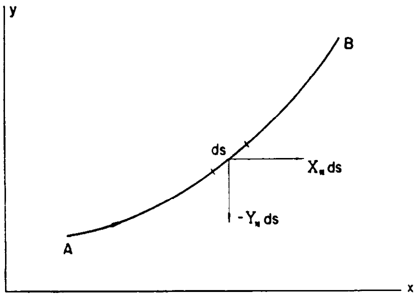

FIG. 2. Forces on arc AB.

To obtain the φ_{NM} it is necessary to solve an infinite number of the five-term recursion relations (18)–(22). Either the method of truncation or of successive approximation or a combination of these may be used to solve these equations. With $K = 0$ one recovers the exact solution given by Jefferey.[4] Successive approximations from this zeroth order solution may be easily constructed as a power series in the permeability parameter K. However, this method converges only for very small values of K and many of the interesting differences between the permeable and impermeable cases are obscured. Truncation proves to be a much more successful method. The coefficients φ_{NM} $(N > N')$ are set to zero and the $4N' + 2$ linear equations are solved for the $4N' + 2$ unknown φ_{NM}. It develops, as will be seen in the sequel, that the true solution is very adequately represented with N' quite small (less than 10). This fact has the fortunate consequence that compact formulas which give the variation of important quantities with position and parameters may be derived.

C. FORCE AND TORQUE ON THE CYLINDER

Here it is convenient to adapt the theory of complex functions which has been successfully used to study problems in plane elasticity to our fluid problem. As is well known, the solution of the biharmonic equation can be constructed from the analytic functions f and g. Thus

$$2\psi = \bar{z}f(z) + z\bar{f}(\bar{z}) + g(z) + \bar{g}(\bar{z}), \tag{23}$$

where

$$u = -\partial\psi/\partial y, \qquad v = \partial\psi/\partial x$$

are rectangular components of velocity in a Cartesian coordinate system. The stresses which are given by

$$\sigma_{xx} = -2\mu\frac{\partial^2\psi}{\partial x\,\partial y} - p,$$

$$\sigma_{yy} = 2\mu\frac{\partial^2\psi}{\partial x\,\partial y} - p,$$

$$\sigma_{xy} = \mu\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right),$$

may be used to find the force $(X_N ds,\ Y_N ds)$ acting on an element ds of the arc AB (Fig. 2).

These forces are given by

$$X_N\,ds = \sigma_{xx}\,dy - \sigma_{xy}\,dx, \qquad Y_N\,ds = \sigma_{xy}\,dy - \sigma_{yy}\,dx.$$

Manipulation of the above equations leads to the following simplified expression for the force:

$$(X_N + iY_N)\,ds = 2\mu\,d\{f(z) - \overline{g'(z)} - z\overline{f'(z)}\}. \tag{24}$$

The force resultant on AB is given by the integration of Eq. (24) over AB;

$$X + iY = 2\mu[f(z) - \overline{g'(z)} - z\overline{f'(z)}]_A^B. \tag{25}$$

The resultant moment about the origin of the stress vector may be formed directly by integration;

$$M_0 = \int_{AB}(xY_N - yX_N)\,ds.$$

This may be written as

$$M_0 = 2\mu\,\mathrm{Im}\,(z\bar{z}f' + zg' - g)_A^B. \tag{26}$$

Comparison of Eqs. (23) and (13) reveals that

$$g = -\left(\tau\varphi_{12} + \frac{iz}{2}\varphi_{04}\right)\log\frac{z + i\tau}{z - i\tau} + \mathrm{SVF},$$

$$g' = -\tfrac{1}{2}i\varphi_{04}\log\frac{z + i\tau}{z - i\tau} + \mathrm{SVF},$$

$$f = \tfrac{1}{2}i\varphi_{04}\log\frac{z + i\tau}{z - i\tau} + \mathrm{SVF},$$

$$f' = \mathrm{SVF},$$

where SVF is a single valued function.

To find the force per unit length on the cylinders, one considers the behavior of Eq. (25) as a counterclockwise circuit is made around a cylinder. Thus

$$\begin{aligned} X + iY &= -2\mu\eta\varphi_{04}]_{\pi}^{-\pi} = 4\mu\pi\varphi_{04}, \\ X &= 4\mu\pi\varphi_{04}, \qquad Y = 0. \end{aligned} \tag{27}$$

The resultant moment of forces about the acting on the cylinder is given by Eq. (26). One finds that

$$M_0 = 2\mu\,\mathrm{Im}\,\tau\varphi_{12}\log\frac{z + i\tau}{z - i\tau}\Big]_{\pi}^{-\pi} = -4\pi\mu\tau\varphi_{12}.$$

To obtain the torque referred to cylinder center, one uses the parallel axis theorem to obtain

$$M = -4\pi\mu\tau(\varphi_{12} - \coth\gamma_1\varphi_{04}). \tag{28}$$

Formulas for the pressure (X_p) and stress deviator (X_s) resultants are not of great practical interest but do reveal important aspects of the physical effects of the porous flow (Appendix IB).

$$X_p = 2\pi\mu\Big\{\varphi_{04}(1 - e^{-2\gamma_1}) - \Big(\sum_{N=1}^{\infty} N[(N+1)(\varphi_{N3} + \varphi_{N+1,1}) + (1-N)(\varphi_{N1} + \varphi_{N-1,3})](1 - e^{-2N\gamma_1}) + N[(N+1)(\varphi_{N4} + \varphi_{N+1,2}) + (1-N)(\varphi_{N2} + \varphi_{N-1,4})](1 + e^{-2N\gamma_1})\Big)\Big\}; \quad (29)$$

$$X_s = 2\pi\mu\Big\{\varphi_{04}(1 + e^{-2\gamma_1}) - \Big(\sum_{N=1}^{\infty} N[(N+1)(\varphi_{N3} + \varphi_{N+1,1}) + (1-N)(\varphi_{N1} + \varphi_{N-1,3})](1 + e^{-2N\gamma_1}) + N[(N+1)(\varphi_{N4} + \varphi_{N+1,2}) + (1-N)(\varphi_{N2} + \varphi_{N-1,4})](1 - e^{-2N\gamma_1})\Big)\Big\}. \quad (30)$$

D. EFFECTS OF GROUND FLOW

With $\gamma_2 = 0$ we obtain a description of flow induced by rotating an impermeable cylinder near a porous half space which translates parallel to itself.

An approximate solution of the algebraic system which is asmptotically exact in the limits of both large and small permeability may be obtained by truncation at $N' = 3$, $\varphi_{NM} = 0$ $(N > 3)$. The result follows.

$$\varphi_{32} = KG_{32}(\gamma_1 S_{\gamma_1} V_1 + S_{\gamma_1}^2 V_2), \quad (31)$$

$$G_{32} = -(T_{\gamma_1}^3 - 10K)/Dg_{32},$$

$$D = 3K(\gamma_1 + S_{\gamma_1}C_{\gamma_1}) + 2\gamma_1 S_{\gamma_1}^2 T_{\gamma_1},$$

$$g_{32} = K(75K + 25T_{\gamma_1}^3/2) + 2S_{\gamma_1}^3(2C_{\gamma_1}^2 + 1)(T_{\gamma_1}^3 + 10K)/C_{\gamma_1}(2C_{\gamma_1}^2 + 1) - (\gamma_1 + S_{\gamma_1}C_{\gamma_1})\{210K^3 + 3T_{\gamma_1}^3 K^2 + 48S_{\gamma_1}^3(2C_{\gamma_1}^2 + 1)K^2/C_{\gamma_1}(2C_{2\gamma_1}^2 + 1)\}/D,$$

$$\varphi_{34} = -\tfrac{1}{2}\varphi_{32}, \quad (32)$$

$$\varphi_{33} = \varphi_{32}(3S_{\gamma_1}^2 + S_{3\gamma_1}^2)/(3S_{2\gamma_1} + S_{6\gamma_1}), \quad (33)$$

$$\varphi_{31} = 2\varphi_{32}(3S_{\gamma_1}^2 - S_{3\gamma_1}^2)/(3S_{2\gamma_1} + S_{6\gamma_1}), \quad (34)$$

$$\varphi_{22} = K(\gamma_1 S_{\gamma_1} V_1 + S_{\gamma_1}^2 V_2)G_{22}, \quad (35)$$

$$G_{22} = \frac{\{6 + K[15D - 18K(\gamma_1 + S_{\gamma_1}C_{\gamma_1})]G_{32}\}}{(T_{\gamma_1}^3 + 10K)D - 24K^2(\gamma_1 + S_{\gamma_1}C_{\gamma_1})},$$

$$\varphi_{24} = -\tfrac{1}{3}\varphi_{22}, \quad (36)$$

$$\varphi_{23} = 2\varphi_{22}(2S_{\gamma_1}^2 + S_{2\gamma_1}^2)/(2S_{2\gamma_1} + S_{4\gamma_1}), \quad (37)$$

$$\varphi_{21} = 2\varphi_{22}(2S_{\gamma_1}^2 - S_{2\gamma_1}^2)/(2S_{2\gamma_1} + S_{4\gamma_1}), \quad (38)$$

$$D\varphi_{14} = (\gamma_1 S_{\gamma_1} V_1 + S_{\gamma_1}^2 V_2)\{K^2(\gamma_1 + S_{\gamma_1}C_{\gamma_1})\cdot(3G_{32} - 4G_{22}) - 1\}, \quad (39)$$

$$D\varphi_{04} = 3K\{V_2 - C_{\gamma_1}V_1 - 2S_{\gamma_1}^2 K(\gamma_1 S_{\gamma_1} V_1 + S_{\gamma_1}^2 V_2)\cdot(G_{32} - \tfrac{4}{3}G_{22})\} + 2S_{\gamma_1}^3 V_2/C_{\gamma_1}, \quad (40)$$

$$\varphi_{13} = -S_{\gamma_1}\varphi_{14}/C_{\gamma_1}, \quad (41)$$

$$\varphi_{12} = -2\varphi_{14}, \quad (42)$$

$$\varphi_{02} = -C_{\gamma_1}V_1 - (\gamma_1 + S_{\gamma_1}C_{\gamma_1})\varphi_{04} - 2S_{\gamma_1}^2\varphi_{14}, \quad (43)$$

$$\varphi_{00} = S_{\gamma_1}^2\varphi_{04} + S_{\gamma_1}C_{2\gamma_1}\varphi_{14}/C_{\gamma_1} + S_{\gamma_1}V_1. \quad (44)$$

In Figs. 5–7 we have compared the approximate solution with convergent machine calculations. The comparisons are representative.

The classical solutions may be obtained from Eqs. (31)–(44) by setting $K = 0$. For these solutions the velocity components at infinity ($\eta = 0$, $\gamma = 0$ in that order) are $q_\gamma = 0$, $q_\eta = -V_2$. If $-V_2$ is everywhere added to the x component of velocity, the rotating cylinder moves to the left, and the fluid at infinity and at the ground are at rest.

There is a special case in which Eqs. (31)–(44) are exact for all K. If $\gamma_1 S_{\gamma_1} V_1 + S_{\gamma_1}^2 V_2 = 0$ all coefficients vanish except $\varphi_{04} = -V_1/S_{\gamma_1}$. The stream function is given by $\psi = \tau V_1 S_\gamma (\gamma_1 - \gamma)/(C_\gamma - \cos\eta)$ and the pressure by $p = 2\mu V_1 S_\gamma \sin\eta/\tau S_{\gamma_1}$. The pressure is constant at the porous boundary, and hence, throughout the porous matrix. The solution is not interesting because there is no exchange of fluid. It does show, however, that the impermeable and permeable solutions tend to coincide when the velocity of the ground and cylinder have an opposing sense.

Equations (31)–(44) are also exact in the physically meaningless limit $K \to \infty$. In this limit the pressure in the porous matrix vanishes like $1/K$, but filter velocity (**Q**) and the force resultants remain finite. The pressure at infinity in the free liquid

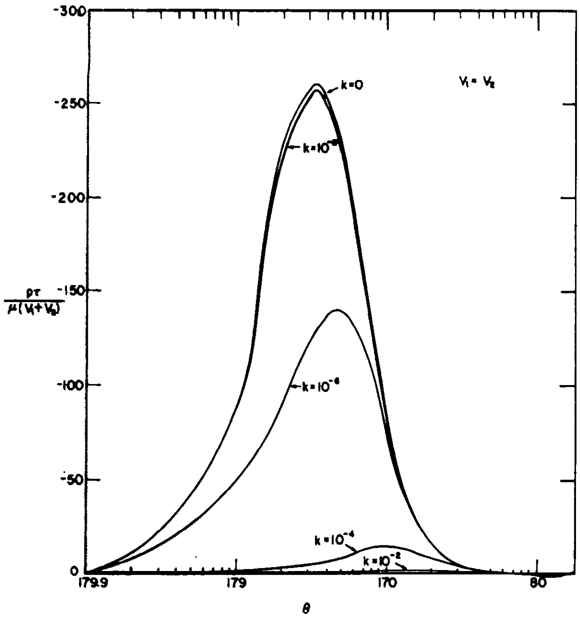

FIG. 3. Pressure distribution on cylinder ($\gamma_1 = 0.1$).

also vanishes. The stream function is given in rectangular coordinates by

$$\psi = \text{const} + \varphi_{02}y + \frac{\varphi_{00}(x^2 + y^2)}{2\tau} + \tfrac{1}{2}y\varphi_{04} \log \frac{x^2 + (y + \tau)^2}{x^2 + (y - \tau)^2},$$

$$\varphi_{02} = -V_2,$$

$$\varphi_{04} = (V_2 - C_{\gamma_1}V_1)/(\gamma_1 + S_{\gamma_1}C_{\gamma_1}),$$

$$\varphi_{00} = S_{\gamma_1}(\gamma_1 V_1 + S_{\gamma_1}V_2)/(\gamma_1 + S_{\gamma_1}C_{\gamma_1}).$$

The x component of velocity tends to infinity with y. This singular behavior is not confined to the limit $K \to \infty$, but is true for all $K \neq 0$ even though permeable and impermeable solutions are indistinguishable for small K.

Figures 3–12 give the variation of field quantities and force resultants. For all cases $a_1 = 1 = \tau/S_{\gamma_1}$. From these figures we observe:

(1) The ground flow greatly reduces the pressure maxima and shifts these toward the top of the cylinder (Fig. 3). This effect may be clearly observed in the streamline sketches (Figs. 9–12). The stagnation points which lie close to the gap migrate toward the top of the cylinder as K is increased.

(2) The force resultants are sensitive to pressure changes induced by the ground flow (Figs. 4–5). The torque is less sensitive to pressure changes and is largely dominated by the shearing motion (Figs. 6–7).

(3) More ground flow for a given permeability is induced by decreasing the gap size (smaller γ_1). With $\gamma_1 = 0.1$ and $a_1 = 1$ in. the permeability k is in.^{2} and the gap size is 5×10^{-3} in. From Fig. 4 we see that the ground flow has little effect on the force for permeabilities less than 10^{-8} in.^{2} For permeabilities of the order of 10^{-5} in.^{2} the ground flow has already appreciably reduced the drag.

(4) The pressure force resultants vanish like $1/K$ for large K (Fig. 8). The force resultants in this

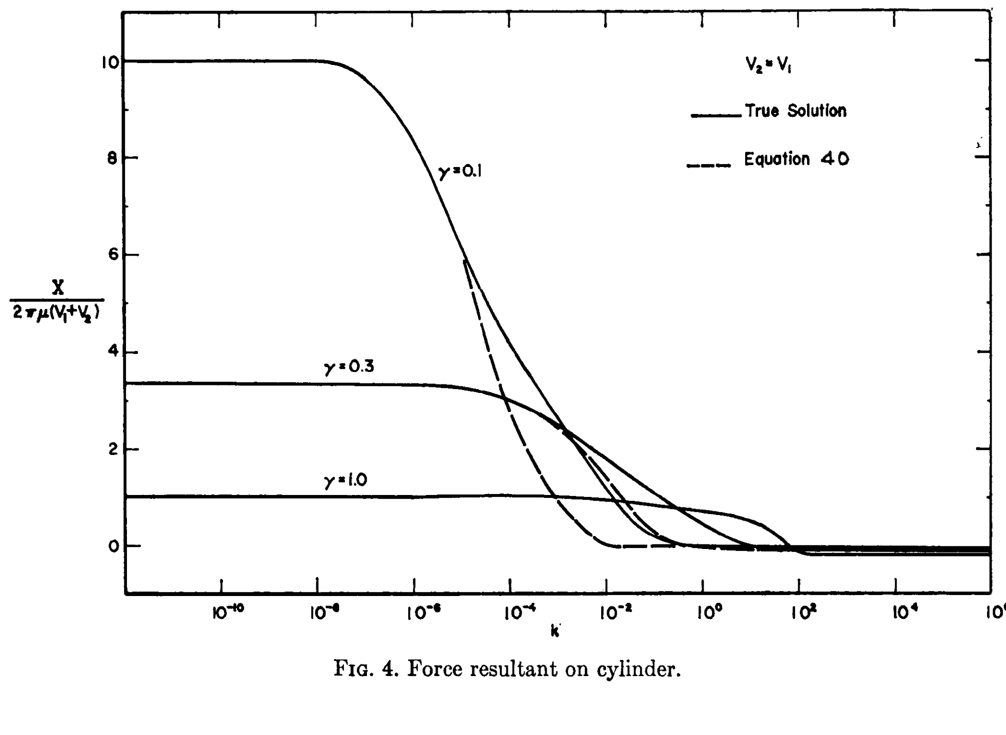

FIG. 4. Force resultant on cylinder.

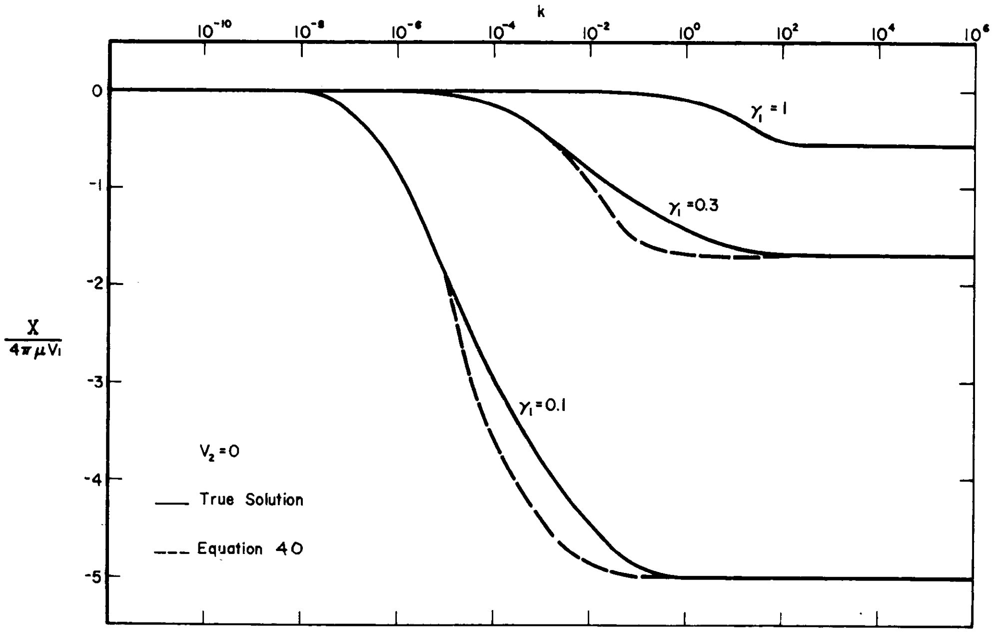

FIG. 5. Force resultant on cylinder.

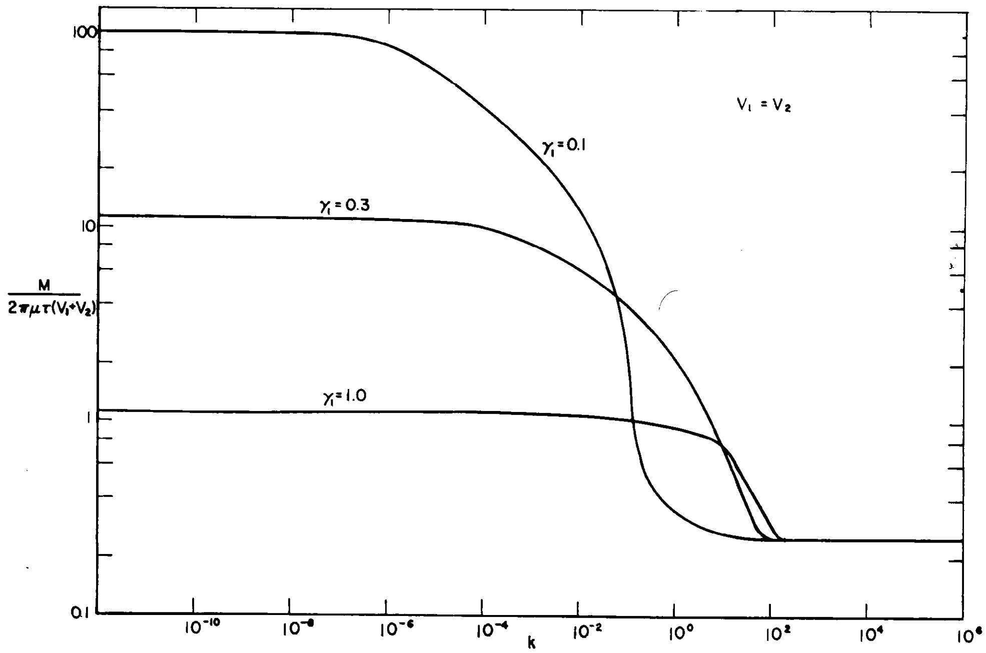

FIG. 6. Torque on cylinder.

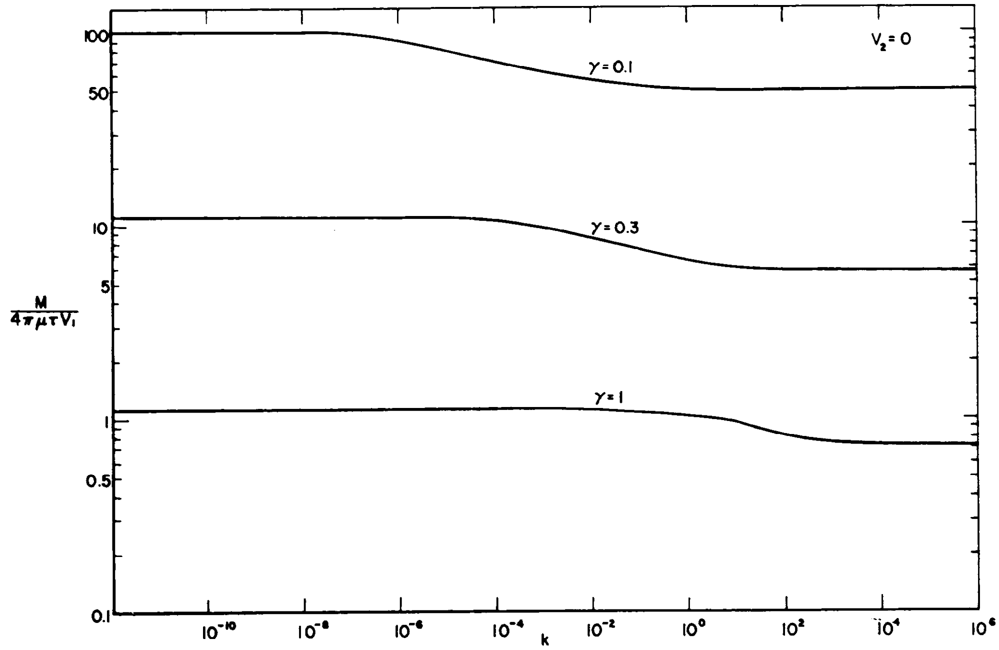

FIG. 7. Torque on cylinder.

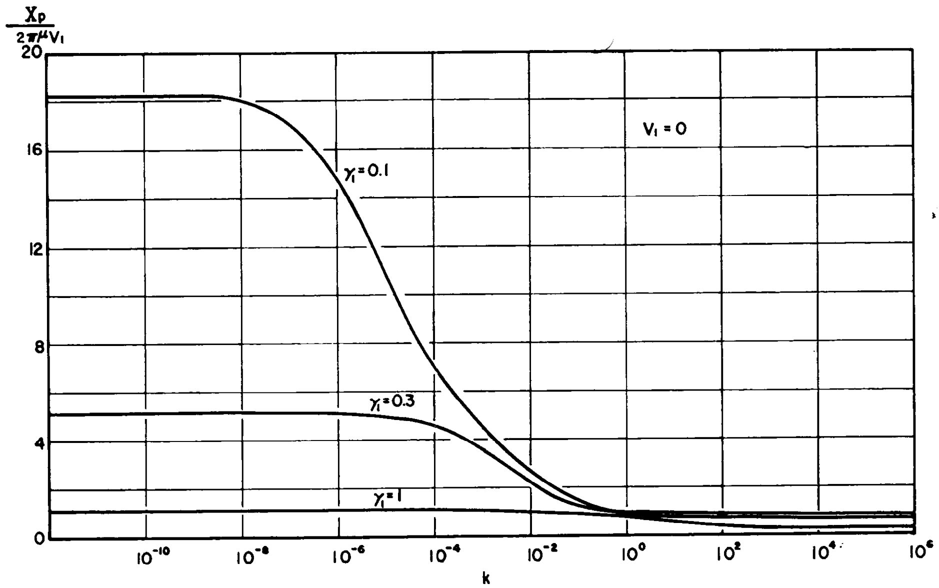

FIG. 8. Pressure force resultant on cylinder.

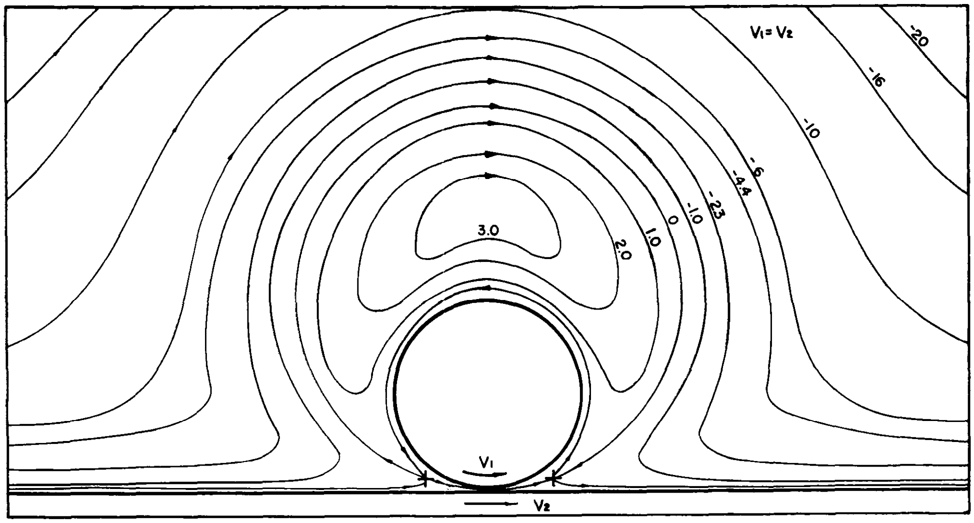

FIG. 9. Streamlines ($\gamma_1 = 0.1$, $k = 0$).

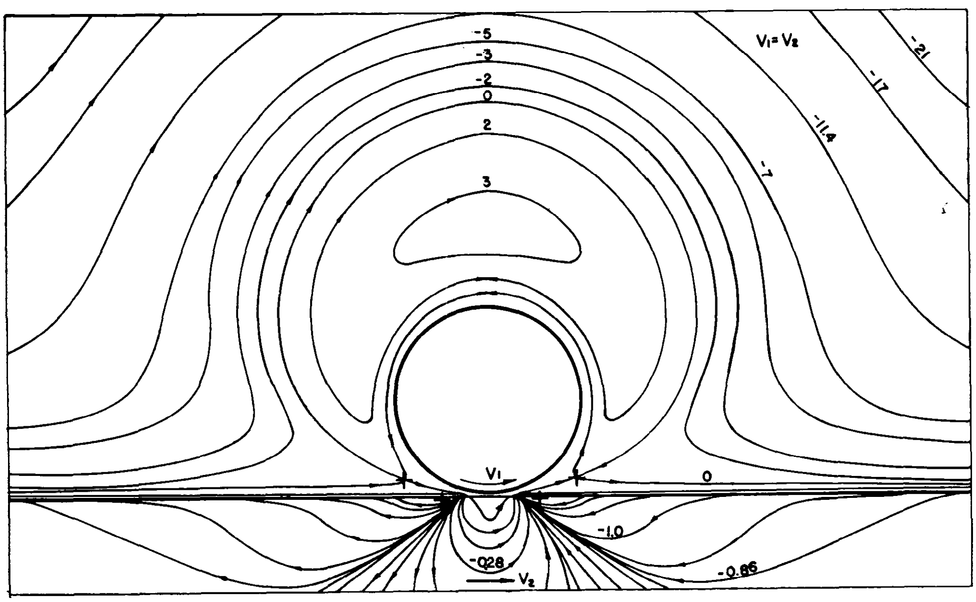

FIG. 10. Streamlines ($\gamma_1 = 0.1$, $k = 10^{-4}$).

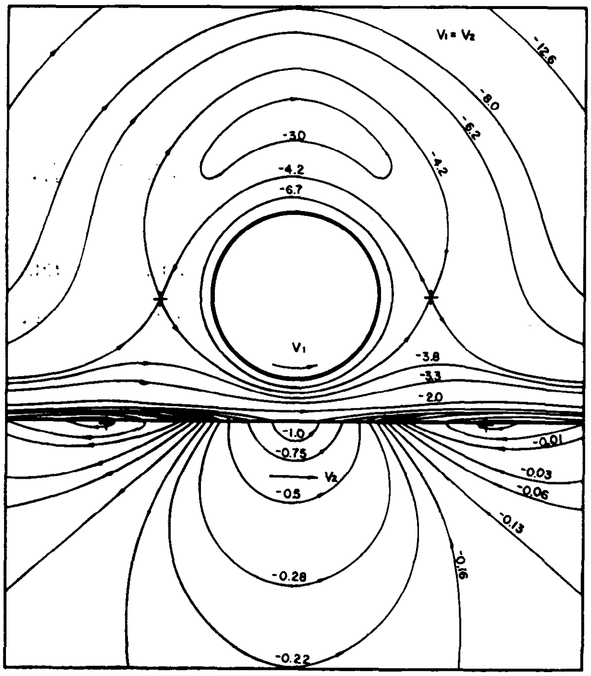

FIG. 11. Streamlines ($\gamma_1 = 1$, $K = 10^{-4}$).

limit tend to assume the value of the shear stress resultants. The classical result that a cylinder induces no drag by virtue of its own rotation,[4] i.e., $V_1 \neq 0$, $V_2 = 0$, is annulled by the ground flow (Fig. 5). A corresponding result which asserts that a moving wall exerts no torque on a stationary cylinder[5] is similarly annulled. Both conclusions follow easily from Eqs. (39), (40), and (42).

APPENDIX IA: DERIVATION OF EQS. (16)-(31).

From Eqs. (7), (14), and (15), we obtain ($i = 1, 2$)

$$A_1^{(i)} = -\varphi_{04} S_{\gamma_i} + 2(\varphi_{14} + \varphi_{22}) C_{\gamma_i} + 2(\varphi_{13} + \varphi_{21}) S_{\gamma_i}, \tag{A1}$$

$$A_N^{(i)} = \{(1 - N)(\varphi_{N2} + \varphi_{N-1,4}) + (N + 1)(\varphi_{N4} + \varphi_{N+1,2})\} C_{N\gamma_i} + \{(1 - N)(\varphi_{N1} + \varphi_{N-1,3}) + (N + 1)(\varphi_{N3} + \varphi_{N+1,1})\} S_{N\gamma_i}. \tag{A2}$$

From Eqs. (8), (9), (12), and (15),

$$\tfrac{3}{2}\Phi_2(\gamma_i) - C_{\gamma_i}\Phi_1(\gamma_i) - \Phi_0(\gamma_i) = -(-1)^{i-1} 2K_i \{(C_{\gamma_i}^2 + \tfrac{1}{4}) A_1^{(i)} - 2C_{\gamma_i} A_2^{(i)} + \tfrac{3}{4} A_3^{(i)}\}, \tag{A3}$$

$$\Phi_3(\gamma_i) - C_{\gamma_i}\Phi_2(\gamma_i) = (-1)^i K_i \{2(C_{\gamma_i}^2 + \tfrac{1}{2}) A_2^{(i)} - C_{\gamma_i}(3A_3^{(i)} + A_1^{(i)}) + A_4^{(i)}\}, \tag{A4}$$

$$\frac{N+2}{2}\Phi_{N+1}(\gamma_i) + \frac{N-2}{2}\Phi_{N-1}(\gamma_i) - NC_{\gamma_i}\Phi_N(\gamma_i) = 2(-1)^i K_i \{N A_N^{(i)} (C_{\gamma_i}^2 + \tfrac{1}{2}) - C_{\gamma_i}[(N+1) A_{N+1}^{(i)} + (N-1) A_{N-1}^{(i)}] + \frac{N+2}{4} A_{N+2}^{(i)} + \frac{N-2}{4} A_{N-2}^{(i)}. \tag{A5}$$

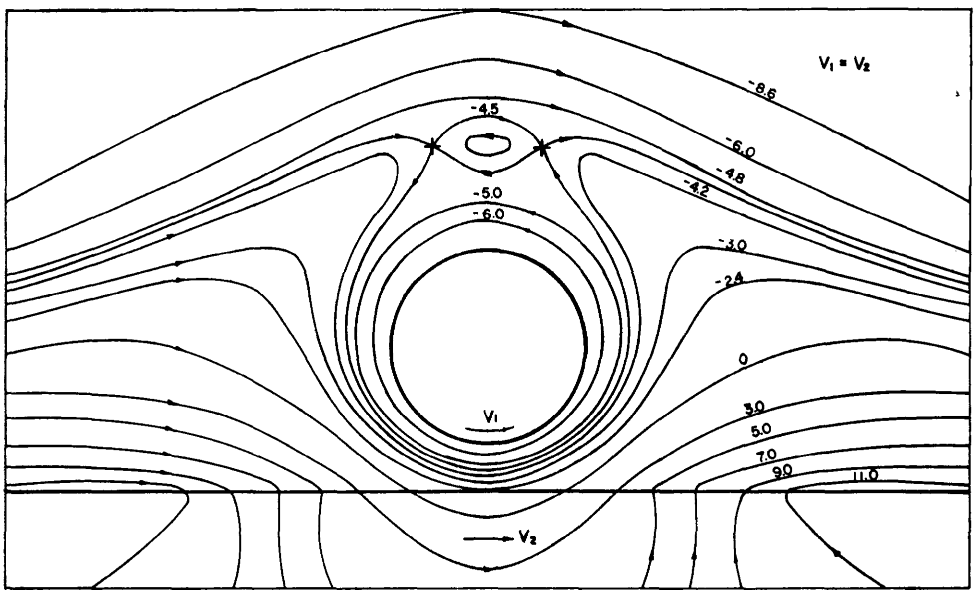

FIG. 12. Streamlines ($\gamma_1 = 1$, $k = 1$).

From Eqs. (10)–(12),

$$\dot{\Phi}(\gamma) = d\Phi/d\gamma,$$

$$S_{\gamma_i}\Phi_0(\gamma_i) - C_{\gamma_i}\dot{\Phi}_0(\gamma_i) + \tfrac{1}{2}\dot{\Phi}_1(\gamma_i) = V_i C_{\gamma_1}, \qquad \text{(A6)}$$

$$S_{\gamma_i}\Phi_1(\gamma_i) + \dot{\Phi}(\gamma_i) + \mathring{\Phi}_2(\gamma_i)/2 - C_{\gamma_i}\dot{\Phi}_1(\gamma_i) = -V_i, \qquad \text{(A7)}$$

$$S_{\gamma_i}\Phi_N(\gamma_i) - C_{\gamma_i}\mathring{\Phi}_N(\gamma_i) + \tfrac{1}{2}[\mathring{\Phi}_{N-1}(\gamma_i) + \dot{\Phi}_{N+1}(\gamma_i)] = 0. \qquad \text{(A8)}$$

It may be verified that the above equations are identically satisfied with

$$\Phi_0(\gamma_i) + C_{\gamma_i}\Phi_1(\gamma_i) = K_i(-1)^i\{C_{\gamma_i}A_2^{(i)} + (1 - 2C_{\gamma_i}^2)A_1^{(i)}\}, \qquad \text{(A9)}$$

$$\Phi_N(\gamma_i) = (-1)^i K_i\{-2C_{\gamma_i}A_N^{(i)} + A_{N-1}^{(i)} + A_{N+1}^{(i)}\}, \qquad \text{(A10)}$$

$$\dot{\Phi}_0(\gamma_i) + S_{\gamma_i}\mathring{\Phi}_1(\gamma_i) = -V_i + (-1)^i K_i(S_{\gamma_i}A_2^{(i)} - S_{2\gamma_1}A_1^{(i)}), \qquad \text{(A11)}$$

$$S_{\gamma_i}\Phi_0(\gamma_i) - C_{\gamma_i}\mathring{\Phi}_0(\gamma_i) = C_{\gamma_i}V_i + (-1)^i K_i S_{\gamma_i}A_1^{(i)}, \qquad \text{(A12)}$$

$$\mathring{\Phi}_1(\gamma_i) = 2(-1)^{i-1}K_i S_{\gamma_i}A_1^{(i)}, \qquad \text{(A13)}$$

$$\mathring{\Phi}_N(\gamma_i) = 2(-1)^{i-1}K_i S_{\gamma_i}A_N^{(i)}. \qquad \text{(A14)}$$

One of the Eqs. (A9), (A11)–(A13) is redundant.

Equations (18)–(22) are obtained by unfolding

$$\mathring{\Phi}_1(\gamma_i) = 2(-1)^{i-1}K_i A_1^{(i)},$$

$$S_{\gamma_i}\Phi_0(\gamma_i) - C_{\gamma_i}\mathring{\Phi}_0(\gamma_i) = C_{\gamma_i}V_i + (-1)^i K_i S_{\gamma_i}A_1^{(i)},$$

$$-S_{\gamma_i}\dot{\Phi}_0(\gamma_i) + C_{\gamma_i}\Phi_0(\gamma_i) + \Phi_1(\gamma_i) = S_{\gamma_i}V_i + (-1)^i K_i(A_2^{(i)} - C_{\gamma_i}A_1^{(i)}),$$

$$\Phi_N(\gamma_i) = (-1)^{i-1}K_i \cdot\{-2C_{\gamma_i}A_N^{(i)} + A_{N-1}^{(i)} + A_{N+1}^{(i)}\},$$

$$\mathring{\Phi}_N(\gamma_i) = 2(-1)^{i-1}K_i S_{\gamma_i}A_N^{(i)}.$$

APPENDIX IB: DERIVATION OF EQS. (29), (30).

Summing forces in x direction (See Fig. 2), we have

$$\begin{aligned} X_N\,ds &= \sigma_{xx}\,dy - \sigma_{xy}\,dx \\ &= -\left(2\mu\frac{\partial^2\psi}{\partial x\,\partial y} + p\right)dy - \mu\left(2\frac{\partial^2\psi}{\partial x^2} - \nabla^2\psi\right)dx \\ &= -2\mu\, d\left(\frac{\partial\psi}{\partial x}\right) + \mu\nabla^2\psi\,dx - p\,dy, \end{aligned}$$

$$\begin{aligned} X = X_s + X_p &= \mu\oint\nabla^2\psi\,dx - \oint p\,dy \\ &= \tau\mu\oint\frac{\nabla^2\psi(C_{\gamma_1}\cos\eta - 1)}{(C_{\gamma_1} - \cos\eta)^2}\,d\eta + \tau\oint\frac{pS_{\gamma_1}\sin\eta}{(C_{\gamma_1} - \cos\eta)^2}\,d\eta. \end{aligned}$$

Evaluation of the above integrals yields Eqs. (29) and (30).

ACKNOWLEDGMENTS

We are indebted to C. C. Shir of the University of Minnesota for his work on the calculations.

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