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Stability of Frictionally-Heated Flow

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Extended results relative to the existence of a critical stress (a finite shear stress or pressure gradient above which fully-developed steady solutions do not exist) in Couette and Poiseuille motions are reported. The results apply to liquids under general thermal boundary conditions including a conduction–convection balance at the boundaries. A bound which gives a close *a priori* estimate of the value of the critical stress is developed. Stability characteristics of frictionally-heated Couette flow in the inviscid limit are specified. Below the critical stress the steady solutions are double-valued. The stress parameter first increases, then decreases, with increasing temperature. To the previously reported temperature instability on the second branch can be added a corresponding (inviscid) instability of the motion. On the first branch of the double-valued solution the flow is stable despite the presence of a single vorticity maximum at the channel center.

I. INTRODUCTION

THE following facts relative to frictionally-heated Couette and Poiseuille flows with wall temperatures a prescribed constant have been established (see Ref. 1).

1. Plane Couette flow of gases and liquids must develop a point of inflection at the channel center. In Poiseuille flow of liquids, two symmetrically disposed points at which the modulus of the vorticity is maximum can develop.

2. There exists a critical-stress (pressure-gradient) parameter for liquids above which there are no steady solutions compatible with prescribed conditions. Below this critical stress there exist two (or more) solutions compatible with prescribed conditions and a single value of the stress parameter. The graph of the shear-stress parameter versus maximum temperature increases from zero to a first maximum. All of the known exact solutions have only one maximum, and the maximum temperature increases to infinity as the stress parameter is reduced to zero.

3. There is a neutral solution of zero wavenumber associated with the stability of the conduction profile. The branch of the solution on which the stress increases with the maximum temperature (and velocity) is stable. The second branch on which the stress decreases with increasing maximum temperature is unstable. Hence, for a given shear stress below the critical there are two solutions characterized by smaller and larger maximum values for temperature and velocity. The solution with the higher temperature for the given stress is unstable and if disturbed would presumably lead to turbulence or decay to the stable laminar state associated with the given stress.

The following two extensions are developed in this paper.

1. In Sec. II the critical phenomenon is shown to apply generally to all noninertial flows in which the distribution of the shear stress may be obtained as a first integral of the appropriate momentum equation. The thermal boundary condition for which the phenomenon exists is generalized to include a conductive–convective exchange of heat at the boundary. An *a priori* estimate of the critical stress parameter is developed. The estimate gives values for the stress parameter which are in good agreement with exact values obtained from known solutions.

2. In Sec. III we consider the stability problem associated with the velocity profiles generated in frictionally-heated plane Couette flow in the inviscid limit ($\mathrm{Re} \to \infty$). The stability results are similar to those previously reported for the conduction problem. In particular there are no neutral or amplified disturbances which can be associated with the first branch of the solution. On the second branch both neutral and amplified solutions exist. The results apply generally to liquids independent of a detailed specification of the viscosity-temperature variation. No conclusions are drawn relative to the Couette flow of gases. However, the profiles generated have a vorticity minimum[1] at the channel center and do not satisfy the Fjørtoft[2]–Høiland[3] condition. This suggests that frictional heating is stabilizing for Couette flow of gases.

The inviscid Orr–Sommerfeld equation which

[1] D. D. Joseph, Phys. Fluids 7, 1761 (1964).

[2] R. Fjørtoft, *Geophys. Publ.*, Oslo, 17, No. 6 (1950).

[3] E. Høiland, *Geophys. Publ.*, Oslo, 18, No. 9 (1953).

governs neutral disturbances of the Couette flow of liquids

$$v'' - \alpha^2 v - (u''/u)v = 0, \qquad v(\pm 1) = 0,$$

is of a type $[-(u''/u) > 0]$ which may possess a neutral solution with $\alpha^2 > 0$. Nevertheless, the system has no neutral solution on the first (physically more important) branch of the steady solution. That antisymmetric velocity profiles with bounded positive $-u''/u$ and an interior vorticity maximum need not be compatible with neutral or amplified solutions was demonstrated in 1945 by Lin.[4] Lin's counterexample proceeds from the examination of a sinusoidal velocity profile for which the inviscid Orr–Sommerfeld equation may be solved explicitly. This profile is actually the one generated by the linear fluidity-temperature relation. (Fluidity is the reciprocal of the dynamic viscosity.) A similar result holds, however, for a whole manifold of solutions corresponding to the physically observed variations of fluidity with temperature in liquids. A neutral solution with a zero wavenumber can be found when the stress parameter is at its maximum (critical) value. Neutral disturbances with finite-positive wavenumbers and neighboring amplified disturbances also exist for the branch of the steady solution for which the temperature (and velocity) increases as the stress parameter is reduced.

II. THE CRITICAL STRESS

Flows which for reasons of symmetry depend on one space variable and for which inertial terms vanish identically (Couette and Poiseuille flows in straight channels, circular cylinders, and cylindrical annuli) have the property that the distribution of the shear stress may be obtained explicitly as a first integral of the appropriate equation of momentum. The energy equation in such circumstances can be formed as a (in general) nonlinear ordinary differential equation for the temperature distribution. The dissipation function for these problems $2\tau\epsilon^0 = \tau^2/\mu$ (ϵ^0 is the rate of shear strain) depends on the temperature through the dynamic viscosity μ, the distribution of stresses τ being known to within an arbitrary constant. To illustrate, consider plane Couette flow and Poiseuille flow in pipes as (representative) examples. The distribution of stresses does not depend on viscosity, and

$$\tau(\eta) = \tau^* \qquad \text{(plane Couette flow)},$$

$$\tau(\eta) = \tfrac{1}{2}P\eta \qquad \text{(pipe flow)},$$

where τ^* is a constant, η is the coordinate transverse to the pipe or channel axis, and P is the axial pressure gradient. The corresponding equations of energy are

$$\frac{d}{d\eta}\left(k\frac{d\theta}{d\eta}\right) + \frac{\tau^{*2}}{\mu(\theta)} = 0 \quad \text{(plane Couette flow)}, \tag{1}$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left(k\eta\frac{d\theta}{d\eta}\right) + \frac{P^2\eta^2}{4\mu(\theta)} = 0 \quad \text{(pipe flow)}, \tag{2}$$

where k is the conductivity and θ some temperature difference. If the system state is to be steady, an energy transfer at the boundaries balancing the heat carried from the fluid by conduction with the heat gained by the environment by convection (or radiation) will be established. This balance is expressed as a boundary condition of the third kind

$$k(d\theta/d\eta) + h\theta = 0, \tag{3}$$

where h is a heat transfer coefficient.

Given the solution to the temperature equation, $1/\mu$ is a function of known variation, and $dV/d\eta = 2\epsilon^0 = \tau(\eta)/\mu$ may be integrated to obtain the variation of velocity $V(\eta)$. For Couette flow the magnitude of the constant τ^* can then be obtained from given velocity boundary data.

We wish to call attention to the fact that this mathematical system cannot be solved for arbitrarily large values of τ^* or P, that there exist critical values for these constants above which solutions do not exist and below which solutions are multivalued. These results are given in detail for the case in which the boundary temperature is a prescribed constant in Ref. 1. We want here to (a) redevelop the results as necessary background for the stability discussion which follows in the next section and (b) extend the results to accommodate the more general boundary condition (3).

We first observe that for most Newtonian liquids the thermal conductivity k is essentially constant over ranges of temperature in which the viscosity undergoes marked variation. Hence, it is appropriate to take thermal conductivity as constant but to account for the change of viscosity in treating the flow of liquids.

The viscosity variation is conveniently described by the fluidity function

$$\varphi(\theta) = \mu_R/\mu(\theta), \tag{4}$$

where $\mu_R = \mu(0)$ is evaluated at the minimum (zero) value of θ entering into the problem. It is consistent with the nature of the variation of fluidity with temperature in liquids that the slope of $\varphi(\theta)$ is non-negative. No further specification of the thermal

[4] C. C. Lin, Quart. J. Appl. Math. 3, 218 (1945).

dependence of material properties is required for the mathematical work which follows.

We now introduce dimensionless variables and accentuate the role of the stress parameters λ and λ_P. Let:

$$x = \eta/L, \qquad \psi = \theta(d\varphi/d\theta)(0), \tag{5}$$

$$\lambda = \frac{\tau^{*2}L^2}{k\mu_R(d\varphi/d\theta)(0)}, \qquad \lambda_P = \frac{P^2L^4}{4k\mu_R(d\varphi/d\theta)(0)},$$

where L is the radius of the pipe or the channel half-height. With this change of variables, Eqs. (1) and (2) may be rewritten as

$$(d^2\psi/dx^2) + \lambda\varphi(\psi) = 0 \quad \text{(plane Couette flow)}, \tag{6}$$

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d\psi}{dx}\right) + \lambda_P x^2\varphi(\psi) = 0 \qquad \text{(pipe flow)}. \tag{7}$$

These equations and appropriate boundary conditions govern the distribution of temperature generated by frictional heating. Similar equations are easily derived for the other types of Poiseuille and Couette flow and for combinations of these. It is shown in Ref. 1 that when (a) the boundary temperature is a prescribed constant and (b) the requirement that the positive heat source generate positive temperatures is imposed so that

$$\psi(1) = (d\psi/dx)(0) = 0, \tag{8}$$

then there exist maximum values of λ and λ_P above which there are no positive solutions and below which solutions are multivalued. In general solutions exist for all positive values of $\psi_M = \psi(0)$ but only for a bounded set of λ.

The preceding remarks are also valid under far more general conditions than those specified above or in Ref. 1. To show this we consider the problem

$$(d/dx)[p(x)(d\psi/dx)] + \lambda f(x)[\psi + G(\psi)] = 0, \tag{9}$$

$$\left.\begin{aligned}\beta(a)(d\psi/dx)(a) + \gamma(a)\psi(a) &= 0,\\ \beta(b)(d\psi/dx)(b) + \gamma(b)\psi(b) &= 0,\end{aligned}\right\} \tag{10}$$

where

$$\varphi(\psi) = \psi + G(\psi), \qquad G(\psi) > 1, \qquad G(0) = 1,$$
$$p(x) > 0, \qquad f(x) > 0 \quad \text{in} \quad a < x < b.$$

The system (9), (10) is sufficiently general to include all standard combinations of Couette and Poiseuille flow. This system is to be compared with the homogeneous, self-adjoint, linear system

$$(d/dx)[p(x)(d\hat{\psi}/dx)] + \Lambda f(x)\hat{\psi} = 0, \tag{11}$$

$$\left.\begin{aligned}\beta(a)(d\hat{\psi}/dx)(a) + \gamma(a)\hat{\psi}(a) &= 0,\\ \beta(b)(d\hat{\psi}/dx)(b) + \gamma(b)\hat{\psi}(b) &= 0.\end{aligned}\right\} \tag{12}$$

The result which follows from this comparison is that the values of $\lambda > 0$ for which (9) and (10) have positive solutions ψ are bounded above by the number

$$\Lambda_0 \operatorname*{Max}_{y\geq 0} \frac{y}{\varphi(y)}$$

and by the function of $\psi_M = \operatorname{Max} \psi$

$$\Lambda_0\psi_M/(\psi_M + 1).$$

Thus, positive solutions of the problem (9) and (10) can exist only for values of $\lambda > 0$ satisfying the composite inequality (see Fig. 1).

$$\lambda(\psi_M) \leq UB(\psi_M) = \Lambda_0 \operatorname{Min}\begin{cases}\operatorname*{Max}_{y\geq 0} \dfrac{y}{\varphi(y)}\\ \psi_M/(\psi_M + 1).\end{cases} \tag{13}$$

Here $\Lambda_0 > 0$ and $\hat{\psi}_0$ are the least eigenvalue and positive eigenfunction, respectively, of the system (11), (12).

To prove (13) we note that

$$0 = \left[p\left(\hat{\psi}_0\frac{d\psi}{dx} - \psi\frac{d\hat{\psi}_0}{dx}\right)\right]_a^b$$
$$= (\Lambda_0 - \lambda)\int_a^b f\hat{\psi}_0\psi\,dx - \lambda\int_a^b f\hat{\psi}_0 G(\psi)\,dx. \tag{14}$$

This is reduced by using the boundary conditions (10) and (12) to

$$\frac{\lambda}{\Lambda_0} = \begin{cases}\dfrac{\int_a^b [f\hat{\psi}_0\psi\varphi(\psi)/\varphi(\psi)]\,dx}{\int_a^b f\hat{\psi}_0\varphi(\psi)\,dx} \leq \operatorname*{Max}_{y\geq 0}\dfrac{y}{\varphi(y)},\\[2ex] \left\{1 + \dfrac{\int_a^b f\hat{\psi}_0 G(\psi)\,dx}{\int_a^b f\hat{\psi}_0\psi\,dx}\right\}^{-1} \leq \dfrac{\psi_M}{\psi_M + 1},\end{cases} \tag{15}$$

proving (14).

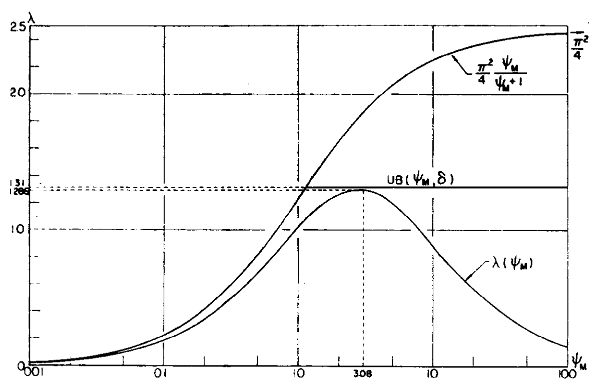

Fig. 1. Comparison of exact solution for $\lambda(\psi_M)$ with the bound (15) for $\varphi(\psi) = 1 + \psi + \delta\psi^2$, $\delta = 0.195$.

Complications introduced by the boundary conditions and the particular functional form of p and f are absorbed in the determination of the parameter Λ_0 from the linear eigenvalue problem (11) and (12). This result can be easily extended to partial differential equations driven by a nonlinear source.[5]

Finally, we should like to compare (15) with some exact solutions[1] of (6), (7), and (8). We first consider solutions of (6) and (8). For this problem:

(a) With $\varphi = 1 + \psi$ (linear approximation),

$$\lambda(\psi_M) = \Lambda_0 \psi_M / (\psi_M + 1),$$

and $\lambda(\psi_M)$ is monotone with a maximum $\lambda = \Lambda_0$ as $\psi_M \to \infty$. The first eigenvalue of the homogeneous linear system is an upper bound on the linear nonhomogeneous system.

(b) With $\varphi = e^{\psi}$ (Couette flow of oil), the bound (15) gives

$$\lambda_{\mathrm{Max}} \leq \mathrm{Max}\; UB(\psi_M) = \pi^2/4e = 0.91,$$

comparing with the exact value

$$\lambda_{\mathrm{Max}} = 0.893.$$

(c) With $\varphi = 1 + \psi + \delta\psi^2$ (Couette flow of water), the bound (15) gives

$$\lambda_{\mathrm{Max}}(\delta) \leq \underset{\psi_M \geq 0}{\mathrm{Max}}\; UB(\psi_M \delta) = \frac{\pi^2}{4[2(\delta)^{\frac{1}{2}} + 1]},$$

which is compared with the exact solution[1] in Fig. 2. With the wall temperature fixed at freezing, $\delta = 0.195$ and $1.286 = \lambda_{\mathrm{Max}}(0.195) \leq 1.31$.

(d) For the problem (7) and (8), $\Lambda_0 = (4.81)^2$, which is determined as the first positive root of $J_0(\Lambda_0^{\frac{1}{2}}/2) = 0$. From the exact solution[6] and (15), we have

$$\lambda_{\mathrm{Max}} = 8.0 \leq (4.81)^2/e \simeq 8.5.$$

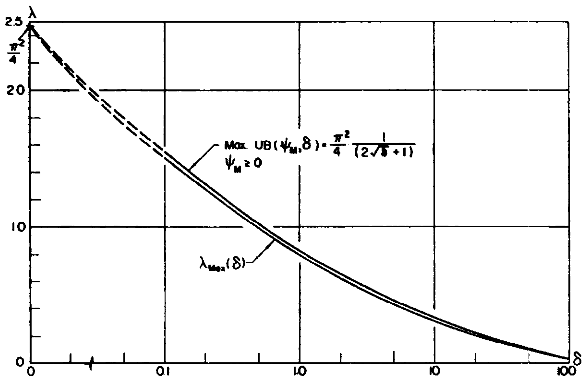

FIG. 2. Comparison of exact solution with the bound $UB(\psi_M, \delta)$ for $\varphi(\psi) = 1 + \psi + \delta\psi^2$ and variable δ.

[5] D. D. Joseph, *Quart. J. Appl. Math.* (in press).

[6] E. A. Kearsley, Trans. Soc. Rheol. **6**, 253 (1962).

TABLE I. Values of parameters for $\lambda = \lambda_{\mathrm{Max}}$ (Couette flow).

	Water	Oil
k (g cm/sec^2 °C)	58 100	12 100
μ_R(g/cm sec)	0.01779	4.26
ν_R(cm^2/sec)	0.01779	4.0
θ_{Max}(°C)	91.5	18.7
$L\tau_{\mathrm{Max}}$*(dyn/cm)	196	853
V_0(cm/sec)	38 000	4535

The graph of $\lambda(\psi_M)$ in Fig. 1 is representative of all the exact solutions. λ first increases as the maximum temperature ψ_M is increased. As ψ_M is further increased, λ passes through a maximum and then decreases. When $\lambda = \lambda_{\mathrm{Max}}$, the shear stress is maximum. The maximum temperature and velocity are finite but not maximum (see Table I). Mathematically there are steady solutions with temperatures and velocities greater than those which previal when $\lambda = \lambda_{\mathrm{Max}}$. But these solutions are unstable in the following sense[1]: The diffusion equation which governs perturbations of temperature when velocity perturbations are suppressed has a neutrally stable solution with a zero wavenumber when $\lambda = \lambda_{\mathrm{Max}}$ and $\psi_M = \psi_M(\lambda_{\mathrm{Max}})$. There are also unstable solutions (which grow exponentially in time) compatible with steady solutions for which $\psi_M > \psi_M(\lambda_{\mathrm{Max}})$. Thus, the second branch of the doubled valued solution is unstable.

The instability which we have described above is somewhat unusual in that it is associated with the normally stable process of diffusion. The critical feature here is the nonlinear temperature-dependent source. If this source emits heat at too great a rate, the medium cannot conduct this heat away at a rate sufficiently rapid to establish steady conditions. It is mathematically possible to attain high steady temperatures for low rates of shear, but these temperatures are unstable. These remarks also apply to conducting solids with nonlinear heat sources,[7] and the phenomenon is not restricted to fluids. For this reason we interpret the destiny of the thermal instability as follows: The system if disturbed on the second (high temperature) branch will merely reject the excess heat and assume the stable temperatures (and velocities) compatible with the given rate of shear. Thus, this instability, we assert, need not lead to turbulence or to secondary laminar motions. The destiny of the velocity perturbations is a matter for separate investigation. It is to this investigation that we now turn.

[7] D. D. Joseph, Intern. J. Heat Mass Transfer, **8** 281 (1965).

III. STABILITY OF FRICTIONALLY-HEATED PLANE COUETTE FLOW

We consider the inviscid stability of a plane Couette flow in a channel of height $2L$. The top plate moves relative to the bottom with a speed $2V_0$. The coordinate origin is located at the channel center where the velocity vanishes. The top and bottom plate have the same prescribed temperature which can be conveniently chosen as a reference temperature so that the temperature difference ψ vanishes on the boundaries. The dimensionless steady velocity ($u = V\mu_R/L\tau^*$) and temperature [see Eq. (5)] are obtained as solutions to the differential system

$$du/dx = \varphi(\psi), \tag{16}$$

$$(d^2\psi/dx^2) + \lambda\varphi(\psi) = 0, \tag{6}$$

$$u(1) = -u(-1) = \mu_R V_0/L\tau^*, \tag{17}$$

$$\psi(1) = (d\psi/dx)(0) = 0. \tag{8}$$

The shear-stress parameter is determined from the given velocity data (Eq. 16) as a final stage of the solution. Concrete solutions of this problem for specified functions $\varphi(\psi)$ are given in Ref. 1. It is also shown in Ref. 1 that the system (6), (8), (16), and (17) implies a single vorticity maximum at the channel center for liquids [$\varphi'(\psi) > 0$] and a minimum for gases [$\varphi'(\psi) < 0$]. There are no other stationary points for the vorticity. We restrict our attention to liquids.

It may be readily verified that for two-dimensional disturbances at infinite Reynolds numbers, the inviscid Orr–Sommerfeld equation governs. Possible neutral disturbances are characterized by the requirement that the wave speed equal the basic fluid velocity at the point of maximum vorticity. The disturbance equation for the transverse perturbation velocity $\hat{v}(x, y, t)$ may then be reduced for periodic disturbances

$$\hat{v} = v(x)e^{i\alpha(y-ct)}$$

to

$$v'' - \alpha^2 v - [u''/(u - c)]v = 0, \tag{18}$$

with complex c, and this is to be solved for the condition

$$v(\pm 1) = 0. \tag{19}$$

In the neutral case the problem reduces to finding solutions for (18) and (19) with $c = 0$. $c_r = 0$ is the fluid velocity at the channel center where the vorticity is maximum.) If neutral disturbances with $\alpha^2 > 0$ and $-u''/u > 0$ exist, then neighboring amplified disturbances ($c_i > 0$) also exist.[8,9] The problem in this sense reduces to a search for solutions to

$$v'' - \alpha^2 v - (u''/u)v = 0, \tag{20}$$

and (6) with $\alpha^2 > 0$. This problem can be resolved completely.

It is perhaps of interest to develop first the analysis which leads to Lin's counterexample before developing the theory in the general case. Consider the linear relation between fluidity and temperature

$$\varphi(\psi) = 1 + \psi,$$

which applies with good accuracy when temperature differences are moderate. The solutions of (6), (8), and (16) are

$$\psi = -1 + \cos(\lambda^{\frac{1}{2}}x)/\cos\lambda^{\frac{1}{2}},$$

$$u = \sin(\lambda^{\frac{1}{2}}x)/\lambda^{\frac{1}{2}}\cos\lambda^{\frac{1}{2}}.$$

The value of λ is determined by application of the velocity boundary condition (17). In terms of the physical parameters [Eq. (5)] this is expressed as

$$\mu_R V_0 = \left[k\mu_R \frac{d\varphi}{d\theta}(0)\right]^{\frac{1}{2}} \tan\left\{\frac{\tau^* L}{\left[k\mu_R \frac{d\varphi}{d\theta}(0)\right]^{\frac{1}{2}}}\right\},$$

giving the value of the (constant) shear stress τ^*. The inviscid Orr–Sommerfeld equation (20) is

$$v'' - \alpha^2 v + \lambda v = 0,$$

and this possesses solutions satisfying (19) only if $\lambda \geq \frac{1}{4}\pi^2$. The condition $\lambda = \frac{1}{4}\pi^2$ corresponds to infinite temperatures and velocities, and the condition $\lambda > \frac{1}{4}\pi^2$ which lead to negative temperatures and velocities must be rejected on physical grounds. It follows that physically meaningful neutral and amplified solutions cannot be found despite the fact that there is a single vorticity maximum and $-u''/u = \lambda > 0$. The linear relation $\varphi = 1 + \psi$ has the special property that the critical stress parameter $\lambda = \frac{1}{4}\pi^2$ coincides with a singular behavior for temperature and velocity. This property is not a feature of the more refined descriptions of $\varphi(\psi)$. It is to these more general cases that we now turn our attention.

We first prove that in liquids

$$0 < -u''/u = \lambda\varphi'(\psi). \tag{21}$$

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[8] W. Tollmien, Nachr. Ges. Wiss Göttingen Fachgruppe II 1, 79 (1935).

[9] C. C. Lin, *Theory of Hydrodynamic Stability* (Cambridge University Press, London, 1955), p. 122.

From Eqs. (6) and (16), we obtain

$$(d^2\psi/dx^2) - \lambda(du/dx) = 0, \tag{22}$$

and after integration

$$d\psi/dx = \lambda u. \tag{23}$$

Now differentiate (16) and replace $d\psi/dx$ from (23) to obtain

$$d^2u/dx^2 = \varphi'(d\psi/dx) = -\lambda u\varphi'(\psi), \tag{24}$$

$$-u''/u = \lambda\varphi'(\psi), \tag{25}$$

which completes the proof.

We next review the properties[1] of the equation

$$(d^2\psi^0/dx^2) + \lambda^0\varphi + \lambda\varphi'\psi^0 = 0, \tag{26}$$

which is to be compared with the inviscid Orr–Sommerfeld equation. Here

$$\psi^0 = \partial\psi/\partial\psi_M, \qquad \psi_M = \psi(0), \qquad \lambda^0 = d\lambda/d\psi_M,$$

$$\psi^0(\pm 1) = (d\psi^0/dx)(0) = 0, \qquad \psi^0(0) = 1. \tag{27}$$

We observe that with $\lambda \to 0$ and $\psi^0(0) = 1$, (26) must generate $\psi^0 \geq 0$, $d\psi^0/dx \leq 0$ for $x \geq 0$. Moreover, (26) cannot generate $\psi^0(x)$ with an interior zero before $d\psi^0/dx$ changes sign. But

$$\frac{d\psi^0}{dx} = -\lambda^0 \int_0^x \varphi(\psi)\, dx - \lambda \int_0^x \psi^0\varphi'(\psi)\, dx \tag{28}$$

cannot vanish before λ^0 assumes a finite negative value

$$\lambda^0 = -\lambda \int_0^x \psi^0\varphi'(\psi)\, dx \Big/ \int_0^x \varphi(\psi)\, dx. \tag{29}$$

It follows that ψ^0 is nonnegative through $\lambda^0 = 0$ (critical stress) and for finite negative values of λ^0.

We now compare (26), (27) with the system

$$v'' - \alpha^2 v + \lambda\varphi'(\psi)v = 0, \tag{30}$$

$$v(\pm 1) = 0, \tag{31}$$

which governs small perturbations at infinite Reynolds numbers. That the system (30) and (31) always possesses solutions for negative α^2 follows from Sturmian theory. There exists a greatest value of $\alpha^2 = \alpha_0^2$ such that the eigenfunction ν_0 belonging to α_0 has no interior zero. The existence of a neutral solution to the stability problem is equivalent to a proof that α_0^2 is positive. By the usual method of multiplication and integration, we find that for any solution of (30) and (31)

$$\alpha^2 = -\lambda^0 \int_{-1}^{1} \varphi v\, dx \Big/ \int_{-1}^{1} \psi^0 v\, dx.$$

In particular, when $\alpha = \alpha_0$ and $v = v_0$, α_0^2 is positive if λ^0 is negative and vice versa. It follows that α_0^2 is not positive before λ^0 changes sign. For $\lambda^0 < 0(\psi^0 \geq 0)$ the system will however accommodate neutral disturbance. That neighboring unstable solutions ($c_i > 0$) exist follows directly from proofs given by Tollmien[8] and Lin.[9]

The values of the physical parameters when $\lambda = \lambda_{Max}(\lambda^0 = 0)$ are given in Table I.[10] The reference temperature for water is 0°C and for oil 15°C. Values of V_0 and θ_{Max} higher (lower) than those in Table I are unstable (stable). The stability of the low temperature branch is the physically important result. It is questionable that the high temperature solution could be obtained even in a carefully controlled experiment.

This completes our discussion of the stability of plane Couette flow. The flow is stable for temperatures below those characterizing the critical condition $\lambda = \lambda_{Max}(\lambda^0 = 0)$ and unstable above.

ACKNOWLEDGMENT

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[10] See Ref. 1 for details of the calculations leading to Table I.