REPRINTED FROM

Quarterly Journal of Mechanics and Applied Mathematics

VOLUME XVIII PART 3

AUGUST 1965

NOTE ON STEADY FLOW INDUCED BY ROTATION OF A NATURALLY PERMEABLE DISC

By DANIEL D. JOSEPH
(*Institute of Technology, University of Minnesota*)

CLARENDON PRESS · OXFORD

Subscription price (for 4 numbers) 75s. post free

NOTE ON STEADY FLOW INDUCED BY ROTATION OF A NATURALLY PERMEABLE DISK

By DANIEL D. JOSEPH
(*Institute of Technology, University of Minnesota*)

[Received 1 July 1964. Revise 12 January 1965]

SUMMARY

Coupled flow induced by the steady rotation of a fluid saturated, naturally permeable and infinite disk is compared with the flow induced by the rotation of an otherwise impermeable disk over which a uniform suction has been prescribed. The coupled problem also accommodates a uniform suction velocity. The similarity reduction of von Karman (**1**) and the numerical results of Stuart (**3**) thus carry over directly. The suction parameter introduced by Stuart is here identified with a permeability parameter which depends on the Darcy coefficient and other given data.

1. Introduction

VON KARMAN (**1**) was the first to show that the Navier-Stokes equations governing the steady rotation of a rotating infinite disk could be reduced to a set of coupled ordinary differential equations. Batchelor (**2**) showed that von Karman's reduction could be retained when the fluid is sucked uniformly from the flow field. Stuart (**3**) integrated Karman's equations and gave a quantitative description of the effect of the suction. In this note we examine the flow when the disk is permeable and the flow in and exterior to the disk is coupled at common boundaries.

A naturally permeable body differs from one for which mass removal is prescribed in that flow conditions at the porous surface cannot be determined *a priori* but depend on the nature of the field flow within the porous matrix as well as the flow exterior to it. Mathematically such flows should be posed as coupled boundary value problems. Certain simple cases exist for which the coupled problem is not much more difficult than the corresponding uncoupled problem (**4, 5**) e.g. streaming motion past a porous sphere. This is also the case for the permeable disk. When the disk is impermeable, the governing equations accommodate a similarity (Karman's) solution. The principal feature of this solution is that the velocity component normal to planes parallel to the plate surface ($z > 0$) is uniform on these planes. If the disk is permeable, homogeneous and isotropic (so that the internal flow is governed by Darcy's law) a solution of the coupled problem which assumes the similarity solution for $z > 0$ can be found. Hence, Stuart's problem and this one are equivalent except that the suction velocity which may be arbitrarily prescribed in Stuart's problem is dependent on the permeability and other data in the coupled problem.

2. Mathematical formulation

For statistically homogeneous and isotropic porous materials one may write

$$\mathbf{Q} = -\frac{k}{\mu}(\nabla\Pi - \mathbf{F}) \tag{1}$$

where k, μ, and Π are the permeability, dynamic viscosity and pressure, respectively. The vector $\mathbf{Q}$ is a volume flow rate per unit cross-sectional area. As such it represents the filter velocity rather than the true velocity of the fluid in the pores. The vector body force $\mathbf{F}$ is the force per unit volume felt by a fluid particle in a frame of reference which is fixed in the porous matrix.

To find the velocity and pressure fields in a permeable medium, it is necessary to solve a Poisson equation for the pressure

$$\nabla^2\Pi = \operatorname{div} \mathbf{F} \tag{2}$$

which is obtained from the Darcy law (1) by application of the principle of conservation of mass ($\operatorname{div} \mathbf{Q} = 0$). The appropriate boundary conditions fix the value of the pressure or the normal derivative of the pressure, i.e. the normal velocity, or a linear combination of these two on the boundaries. In problems which involve flow between materials of different permeability, boundary pressures and velocities may be unknown. In this case the boundary conditions are supplied by the requirement that the normal component of stress and velocity are continuous across surfaces separating regions of different permeability. As in the hydrodynamics of ideal fluids, no requirements are imposed on the tangential components of velocity. This condition is consistent with the nature of the Darcy law which postulates that changes in the empirically equivalent velocity $\mathbf{Q}$ do not depend on the mechanisms of shear. Obviously, the governing equation (2) is not of sufficiently high order to accommodate conditions on the tangential component of velocity, and a discontinuity of this component of velocity across the surface separating regions of different permeability is to be expected.

When the porous boundary separates a region of permeable material from a region where such material is absent, it is necessary, if the governing differential system is not to be underdeterminate, to specify the behaviour of the viscous liquid immediately outside the porous region.

In the case of a viscous flow the order of the equations governing the external motion is raised by two though the equations governing the internal motion are unchanged. It is therefore necessary to impose *some* requirement on the tangential component of velocity at the interface. When much of what would nominally be an impermeable surface is really hole entrance, the appropriate boundary condition is not easy to anticipate. However, when the permeability is small relative to some typical gross area of the porous body, it would seem valid to retain the no slip condition relative to the wall. Since the permeability is slight in most natural materials this condition is not severely restrictive.

Collecting the conditions consistent with the formalism, we have:

1. The external field satisfies the Navier-Stokes equations.
2. The internal field satisfies Poisson's equation (2) for the pressure.
3. The normal components of stress and velocity are continuous at the boundary.
4. The relative tangential velocity vanishes on the exterior of a porous wall.

The passage from one medium to another will be marked by discontinuities in the tangential component of velocity and in the derivatives of the normal components of stress and velocity. Zero permeability restores the ordinary boundary conditions corresponding to impermeable media, i.e. the vanishing of the velocity components.

3. Rotation of a porous disk in a viscous fluid

Consider the steady laminar motion induced by the rotation of a porous disk of infinite radius and thickness h. Locate the origin of cylindrical coordinates at the top of the plate so that the angular velocity ($\boldsymbol{\omega}$) and the direction of z increasing coincide. The governing Navier-Stokes equation, written in cylindrical coordinates (**3**), may be reduced upon introduction of the similarity variables

$$
\begin{aligned}
u &= r\omega F(\zeta) && \text{(radial velocity)}\\
v &= r\omega G(\zeta) && \text{(tangential velocity)}\\
w &= (\nu\omega)^{\frac{1}{2}} H(\zeta) && \text{(axial velocity)}\\
p &= \rho\nu\omega P(\zeta)+(\rho/2)(\Omega r)^2 && \text{(pressure)}\\
\zeta &= (\omega/\nu)^{\frac{1}{2}} z
\end{aligned}
$$

to the set of ordinary differential equations:

$$
\left.
\begin{aligned}
&F^2-G^2+HF' = -(\Omega/\omega)^2+F'',\\
&2FG+HG' = G'', \qquad 2F+H' = 0,\dagger \qquad -HH'-2F' = P'.
\end{aligned}
\right\} \tag{3}
$$

† This equation implies that at the disk the normal stress and pressure in the free fluid are identical; i.e. $\dfrac{\partial \omega}{\partial z} = 0$.

The boundary conditions for uniform suction through the rotating disk (Stuart's problem) are

(i) There is no slip at the top side of the disk

$$F(0) = 0, \qquad G(0) = 1. \tag{4}$$

(ii) Suction velocity is prescribed

$$H(0) = -a = \text{const.} \tag{5}$$

(iii) Radial component of velocity vanishes at infinity

$$F(\infty) = 0. \tag{6}$$

(iv) The fluid at infinity rotates like a rigid solid

$$G(\infty) = \Omega/\omega. \tag{7}$$

As Batchelor (**2**) has remarked, the term $(\Omega r)^2$ in the pressure must be retained if the fluid far from the disk rotates as a rigid solid. This term then balances the centrifugal force corresponding to the non-zero circumferential rigid body rotation. The similarity solutions also accommodate the problem of flow induced by the simultaneous rotation (at different speeds) of two parallel disks of different permeabilities. If the second disk is a distance d from the first and it rotates with angular velocity Ω then (6) and (7) above are replaced with $F(d) = 0$, $G(d) = \Omega/\omega$. In addition $H(d) = -b$, a new suction parameter, is introduced.

When the plate is naturally permeable, suction parameters a (and b) cannot be prescribed *a priori*. In the present analysis we are adopting the classical similarity solution for $z \geqslant 0$ which implies that the normal velocity at the plate (or plates) is a constant (to be determined).

When the disk is statistically isotropic and the flow in it governed by Darcy's law, we have from equation (2)

$$\mathbf{Q} = -\frac{k}{\mu}\{\nabla\Pi + 2\rho(\boldsymbol{\omega}\wedge\mathbf{Q}) + \rho\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{R})\}, \tag{8}$$

where

$$2\rho(\boldsymbol{\omega}\wedge\mathbf{Q}) \quad \text{and} \quad \rho\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r})$$

are the D'Alembert coriolis and central forces respectively. In cylindrical polar coordinates (r, θ, z), $R = (r^2+z^2)^{\frac{1}{2}}$ the three component equations of (8) are

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_z \end{bmatrix} = -\frac{k}{\mu}\begin{bmatrix} \dfrac{\partial \Pi}{\partial r} - 2\rho\omega Q_\theta - \rho\omega^2 r \\ \dfrac{1}{r}\dfrac{\partial \Pi}{\partial \theta} + 2\rho\omega Q_r \\ \dfrac{\partial \Pi}{\partial z} \end{bmatrix}. \tag{9}$$

Axial symmetry is required both from the symmetry of the physical configuration and from the character of the assumed exterior solution. Hence by application of the principle of mass conservation ($\nabla . \mathbf{Q} = 0$) to the appropriate rearrangement of (9) with $\partial/\partial\theta = 0$ we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Pi}{\partial r}\right)+(1+4k^2\omega^2/\nu^2)\frac{\partial^2 \Pi}{\partial z^2} = 2\rho\omega^2 \tag{10}$$

This equation is essentially a Poisson equation for the pressure with the z coordinate reduced by a factor $(1+4k^2\omega^2/\nu^2)^{-\frac{1}{2}}$, which arises as a result of the coriolis accelerations. The exterior and interior flows are coupled by the continuity of pressure

$$\Pi(r, 0) = \rho\Omega^2 r^2/2+\rho\nu\omega P(0) \tag{11}$$

and the continuity of the normal component of velocity

$$\frac{k}{\mu}\frac{\partial \Pi}{\partial z}(r, 0) = a(\nu\omega)^{\frac{1}{2}}. \tag{12}$$

It follows from (11) and (12) that the solution of (10) appropriate to the infinite disk is in the form of a quadratic polynomial

$$\frac{\Pi(r, z)}{\rho} = \frac{(\omega^2-\Omega^2)z^2}{1+4k^2\omega^2/\nu^2}+(\Omega r^2)/2+\nu\omega P(0)+a\nu(\nu\omega)^{\frac{1}{2}}z/k. \tag{13}$$

The suction parameter (a) cannot be prescribed as in the uncoupled problem but is related to other flow parameters by the boundary condition on the underside of the disk.

(i) The disk has an impermeable underside, ($\partial\Pi(r, -h)/\partial z) = 0$

$$a = 2kh\left(\frac{\omega}{\nu}\right)^{\frac{3}{2}}\left(1-\frac{\Omega^2}{\omega^2}\right)\Big/(1+4k^2\omega^2/\nu^2). \tag{14}$$

(ii) A disk of thickness $2h$ rotates in an unbounded liquid. This case is identical to (i) as $z = -h$ is a plane of symmetry.

(iii) A constant pressure $\Pi(r, -h) = \rho\nu\omega\Pi_h$ is prescribed on the underside of the disk. The similarity solution cannot accommodate this boundary condition unless there is no rigid body rotation. With $\Omega = 0$

$$a = \frac{k}{h}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}(P(0)-\Pi_h)+hk\left(\frac{\omega}{\nu}\right)^{\frac{3}{2}}\Big/(1+4k^2\omega^2/\nu^2). \tag{15}$$

It follows that for the conditions specified in (i), (ii), and (iii) the coupled problem and Stuart's problem are indentical except that the suction parameter (a) is no longer arbitrary but is related to given data by (14) or (15).

Cases (i) and (ii) will be more important in practice. For these cases the velocity field within the disk is

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_z \end{bmatrix} = \frac{k(\Omega^2-\omega^2)}{\nu(1+4k^2\omega^2/\nu^2)} \begin{bmatrix} -r \\ 2\dfrac{k}{\nu}r\omega \\ 2(z+h) \end{bmatrix} \tag{16}$$

The coriolis induced effects are proportional to $(k/\nu)^2$, a relatively small value in natural materials. When $\omega = \pm\Omega$ there is no internal flow. With $\omega = -\Omega$ the circumferential velocity at the plate is equal and opposite to the circumferential velocity of the rigid rotation at infinity. The radial dependence of the pressure in the plate is however proportional to the square of the angular velocity. Since the radial dependence of the pressure is the same in the plate and at infinity it does not vary anywhere in the field. In particular it is not necessary that any transverse pressure gradients develop in the plate to compensate for a difference in the magnitude of the rotational speed. When $\omega = \Omega$ there is also no internal flow as one might expect since the angular velocity of the fluid at infinity is the same as that of the disk and the whole system is a rigid body rotation. The direction of the flow within the plate depends upon whether $\Omega^2 \gtrless \omega^2$.

Physically, these solutions are meaningful only if they reasonably represent the flow induced by a disk of finite radius. If one presumes, as is done tacitly when treating the non-permeable disk, that precise conditions of the fluid motions at $r = R$ have only a local effect, then it is reasonable to expect that a finite disk will produce *edge effects* and not alter the nature of the flow in the vicinity of the origin. Of course, the precise boundary condition at the outer edge of the permeable disk could conceivably so alter the nature of the flow in the porous disk that the assumptions of a constant pressure and suction (or injection) on the disk surface would not be compatible even in the neighbourhood of the origin. This possibility, however, can be rejected for natural boundary conditions at the disk edge. These conditions will at worst produce intense alterations of the flow as *edge effects*. Consider, for example, the situation which prevails when $|\omega| > |\Omega|$ (suction rather than injection). Suppose a uniform pressure is prescribed over flat surfaces of the permeable disk (consistent with the requirements of the similarity solution). Consider then the differential equations and boundary conditions for the $\Pi - \Pi_f$ where Π is pressure in the infinite disk and Π_f the pressure in the finite disk. The governing equation for the pressure difference in the region $0 \leqslant r \leqslant R$, where R is the radius of the finite disk, is (10) made homogeneous. This is to be solved for zero pressure on the flat sides of the disk and prescribed normal gradient of the pressure difference on the disk edge. It is easily verified that if the finite disk permits the same radial outflow as the infinite disk at $r = R$, the prescribed value of the normal pressure difference gradient is zero. This implies the trivial solution and the pressure distribution in the finite disk is identical to that in the infinite disk in $0 \leqslant r \leqslant R$. Now suppose that the finite disk has an impermeable edge band. Then a non-zero normal pressure difference gradient must be prescribed at the disk edge. The problem in this case is equivalent to the steady conduction of heat in a flat disk with zero temperature prescribed on the flat faces and a non-zero heat flux on the edge. Clearly the effect of the heat flux will, in the thin disk, be confined to the edge. In the thin fluid saturated disk a rather intense blowing can develop near the edge, but this must rapidly give way (like $\exp\{-(R-r)/h\}$) to a uniform suction. Thus the nature of the solution near the origin will be largely determined by local conditions with the effects of conditions at the edge of the disk producing local edge disturbances.

On the other hand, if, as Bentwich (**6**) has recently suggested, the similarity solution itself is sensitive to conditions at $r = \infty$, then these disturbances at finite radius could conceivably alter the exterior flow even in the neighbourhood of the origin. The similarity solution, in this case, would not describe the exterior flow, and our solution could not describe the interior flow.

Assuming the validity of the similarity solution, we conclude that the rotation of a naturally permeable disk of suitable thickness is compatible with a uniform suction at the plate. This suction depends on the permeability and other given data of the problem. Special constructions for creating uniform suction in experiments and applications may in some cases, be circumvented by employing naturally porous materials.

4. Acknowledgements

This work was supported by the National Science Foundation under grant GP-3066.

REFERENCES

1. T. v. Karman, *Z. angew. Math. Mech.* (1921) 244.
2. G. K. Batchelor, *Quart. J. Mech. Appl. Math.* **4** (1951) 29.
3. J. T. Stuart, *Quart. J. Mech. Appl. Math.* **7** (1954) 446.
4. D. D. Joseph and L. N. Tao, *Z. angew. Math. Mech.* **44** (1964) 361.
5. St. I. Gheorghitza, *Arch. Rat. Mech. Anal.* **12** (1962) 52.
6. M. Bentwich, *Jour. Fluid Mech.* **18** (1964) 499.