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UNSTEADY FREE AND FORCED CONVECTION IN VERTICAL ANNULAR AND ANNULAR SECTOR TUBES

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ABSTRACT

In this investigation solutions to the problem of unsteady laminar forced and free convection in coaxial sector tubes in the presence of a constant axial temperature gradient have been developed.

The solutions admit phenomena of oscillation and resonance which are not usually present in flows in which the dissipative mechanisms of heat conduction and viscosity are important. Several numerical examples are constructed and used to discuss the "dashpot" features of the solutions.

NOMENCLATURE

x, y, z, t	Rectangular position variables and time
r, θ, z	Position variables in cylindrical coordinates
r_i, r_o	Inner and outer radii of annulus
u	Velocity
T	Temperature
T_{c0}	Reference temperature
$C_1(t)$	Axial temperature gradient
$P'_D(t)$	Axial pressure gradient
$Q(x, y, z, t)$	Heat source intensity
β	Thermal expansivity
ν	Kinematic viscosity
α	Thermal diffusivity
C_p	Specific heat
ρ	Density
$R = r/r_o$	Dimensionless radius

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$\tau = vt/r_0^2$	Dimensionless time
$U = ur_0/\nu$	Dimensionless velocity
$\Theta = \alpha(T - T_{C0})/r_0 C_1 \nu$	Dimensionless temperature
$Pr = \nu/\alpha$	Prandtl number
$Gr = \beta g C_1 r_0 / \nu^2$	Grashof number
$Ra = Gr \cdot Pr$	Rayleigh number
$E = p'_D(\tau) r_0^3 / \nu^2$	Pressure gradient parameter
$F = r_0 Q(R, \theta, \tau) / \rho \nu C_1 C_p$	Heat source intensity parameter
$C_{\mu}(\alpha_{\mu m} R) =$ $= J_{\mu}(\alpha_{\mu m} R) Y_{\mu}(\alpha_{\mu m} a) -$ $- J_{\mu}(\alpha_{\mu m} a) Y_{\mu}(\alpha_{\mu m} R)$	Cylinder function
$\alpha_{\mu m}$	Positive zeros of $C_{\mu}(\delta)$

INTRODUCTION

The theoretical study of the combined forced and free laminar convection of mass and energy in enclosed tubes and channels began early in the last decade.

Among the earliest treatments of the problem, which were confined to the analysis of the steady state, those of Ostrach^{(1,2)*} are important because they developed the specializations in the equations of motion and energy which have since been regarded as the ones appropriate to these natural and forced flows.

Subsequent theoretical work has extended this problem to a wide variety of geometries and has introduced some new techniques with which the mathematical manipulations may be more deftly managed.^{(3,4)*}

Theoretical work on the unsteady problem of free and forced convection in vertical channels belongs to this decade. Izumi⁽⁵⁾ has studied transient free convection in an infinite circular tube, the walls of which are held at a steady uniform temperature. Tao⁽⁶⁾ studied this geometry for the case of a linearly varying wall temperature and combined free and forced convection. Zieberg and Mueller⁽⁷⁾ have treated the axially variant free and forced convection problems for a parallel plate duct. These authors have also investigated the implication of assuming that the flow is fully developed.

The purposes of this paper are threefold. Firstly, some additional implications of the assumption of fully developed velocity profiles are

* For all numbered references, see end of article.

* See also references therein.

discussed. Based on this discussion an analysis of the problem of unsteady free and forced convection and a mathematical representation of the velocity and temperature fields are developed. It is shown that there are physical mechanisms present when the temperature is a linear function of the axial coordinate which lead to "dashpot" features normally associated with the linear theory of vibrations.

Secondly, these "dashpot" features which have also been noted in Refs. 5, 6 and 7 are examined in detail and the effects of the Prandtl and Grashof number on the physical system are evaluated.

Thirdly, under certain conditions the solutions admit the possibility of resonant distortion. This phenomena is discussed and the resonant frequencies are determined for a case in which the pressure gradient is a periodic function of the time.

GOVERNING EQUATIONS

A central assumption of this investigation is that the flow is fully developed. By "fully developed", we mean that the velocity components are functions of the transverse coordinates. If one stipulates that the transverse components of the body force have a potential, and that there should be no relative motion of the boundaries (so that the no-slip condition will require that all velocity components vanish at the walls), then it can be rigorously demonstrated⁽⁸⁾ that in unsteady flows, the transverse velocity components may be present initially but

- (1) if present they must decay in time
- (2) if not present initially they cannot develop.

Hence the assumption that the flow is fully developed in the unsteady case is equivalent to the assumption that the flow is initially fully developed. It is known that this condition, when coupled with the assumptions that

- (1) the **material properties** of the fluid are constant except as the variation of **density with temperature** modifies the buoyancy force and
- (2) effects of **viscous dissipation** on the energetics of the problem are negligible

implies the following **restrictions** on the possible variations of the gradients of temperature and pressure in the axial (field force) direction^(7, 9):

- (1) The temperature **must be a linear function** of the axial coordinate

$$T(x, y, z, t) = C_1(t)z + T_2(x, y, t).$$

- (2) The difference between the pressure gradients in the dynamic and hydrostatic cases is at most a function of the time.
- (3) An axial temperature gradient which is variable in time implies that the heat addition must vary linearly in the axial coordinate

$$Q(x, y, z, t) = \rho C_p z \frac{dC_1}{dt} - Q_2(x, y, t)$$

where Q_2 is an arbitrary function of x, y and t .

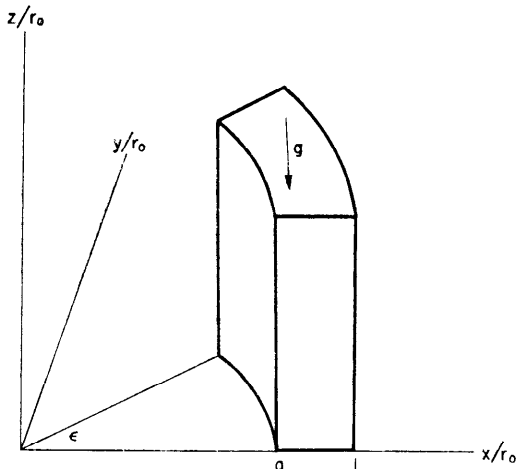


FIG. 1. Geometry of sector tube.

Under these assumptions the dynamics and energetics of the fluid in a plane perpendicular to the pipe axis are governed by

$$\left[\frac{\partial}{\partial t} - \nu \Delta^2 \right] u - g\beta(T_2 - T_{c_0}) = -p'_D(t) \tag{1}$$

$$\left[\frac{\partial}{\partial t} - \alpha \Delta^2 \right] T_2 + C_1 u = Q_2 / \rho C_p.$$

More specifically we now consider fully developed unsteady laminar flows in annular or sector tubes (Fig. 1) under the combined influence of free and forced convection.

We further specify that the axial temperature gradient C_1 is constant in time and that the velocity vanishes while the temperature assumes prescribed values on the tube walls.

The governing differential system when written in terms of the non-dimensional variables is

$$\begin{aligned}\frac{\partial U}{\partial \tau} &= \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial U}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 U}{\partial \theta^2} + Ra\Theta - E(\tau) \\ Pr \frac{\partial \Theta}{\partial \tau} &= \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \Theta}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \Theta}{\partial \theta^2} - U + F(R, \theta, \tau)\end{aligned}\quad (2)$$

$$\begin{aligned}\tau = 0 \quad U_1(R, \theta) &= U(R, \theta, 0) \\ \Theta_1(R, \theta) &= \Theta(R, \theta, 0)\end{aligned}\quad (3)$$

$$\begin{aligned}U(R, 0, \tau) = U(R, \varepsilon, \tau) &= U(i, \theta, \tau) = U(a, \theta, \tau) = 0 \\ \Theta(1, \theta, \tau) &= \Theta_1(\theta, \tau) \\ \tau > 0 \quad \Theta(a, \theta, \tau) &= \Theta_a(\theta, \tau) \\ \Theta(R, 0, \tau) &= \Theta_0(R, \tau) \\ \Theta(R, \varepsilon, \tau) &= \Theta_i(R, \tau).\end{aligned}\quad (4)$$

The initial conditions are not arbitrary as the flow is initially fully developed. The temperature and velocity fields at beginning time may however be obtained from the solution of the appropriate steady state problem.

The governing differential system for the full annulus is the same as for the sector except that U , Θ and their first derivatives are single valued functions of the angular variable Θ .

SOLUTIONS FOR THE SECTOR PROBLEM

Equations (2) are coupled but linear. They may be readily reduced to ordinary differential equations in the time by application of the finite Fourier and Hankel transforms.⁽¹⁰⁾ Solutions to these equations which satisfy the time dependent boundary conditions and heat-source functions may be constructed by the application of Duhamel's theorem to the coupled linear systems.^(6, 11)

Duhamel's auxiliary functions, $\varphi(R, \theta, \tau, \tau')$ and $\psi(R, \theta, \tau, \tau')$ are defined by

$$\begin{aligned}U(R, \theta, \tau) &= \frac{\partial}{\partial \tau} \int_0^{\tau} \varphi(R, \theta, \tau - \tau', \tau') d\tau' \\ \Theta(R, \theta, \tau) &= \frac{\partial}{\partial \tau} \int_0^{\tau} \psi(R, \theta, \tau - \tau', \tau') d\tau'.\end{aligned}\quad (5)$$

The auxiliary functions satisfy the same equations and boundary conditions as the function $U(R, \theta, \tau)$ and $\Theta(R, \theta, \tau)$ except that, in the auxiliary system, the role of the time τ in the expressions of the given heat-source intensity, pressure-gradient parameter and boundary conditions is taken by a fixed parameter τ' .

Initially $\varphi(R, \theta, 0, \tau') = U_i(R, \theta, 0)$ and $\psi(R, \theta, 0, \tau') = \Theta_i(R, \theta, 0)$.

We omit the detailed derivations⁽⁹⁾ and record the solutions for the channel of sector shape.

$$\begin{aligned} \begin{bmatrix} U(R, \theta, \tau) \\ \Theta(R, \theta, \tau) \end{bmatrix} &= \frac{2}{\varepsilon} \sum_{\substack{m=1 \\ n=1}}^{\infty} \frac{C_{\mu}(\alpha_{\mu m} R) \sin \mu \theta}{N_{\mu m}} \left\{ \begin{bmatrix} \varphi_{\mu m}(\tau) \\ \psi_{\mu m}(\tau) \end{bmatrix} + \right. \\ &\quad \left. + \frac{\partial}{\partial T} \int_0^{\tau} d\tau' \begin{bmatrix} f_{\mu m}(\tau - \tau', \tau') \\ g_{\mu m}(\tau - \tau', \tau') \end{bmatrix} \right\} \end{aligned} \tag{6}$$

where

$$\begin{aligned} \begin{bmatrix} f_{\mu m}(\tau, \tau') \\ g_{\mu m}(\tau, \tau') \end{bmatrix} &= \frac{e^{-\beta_{\mu m} \tau}}{Ra} \left\{ A_{\mu m}(\tau') \begin{bmatrix} Ra \\ \delta_{\mu m} + \gamma_{\mu m} \end{bmatrix} e^{\gamma_{\mu m} \tau} + \right. \\ &\quad \left. + B_{\mu m}(\tau') \begin{bmatrix} Ra \\ \delta_{\mu m} - \gamma_{\mu m} \end{bmatrix} e^{-\gamma_{\mu m} \tau} \right\} \end{aligned} \tag{7}$$

$$\begin{aligned} \begin{bmatrix} A_{\mu m}(\tau') \\ B_{\mu m}(\tau') \end{bmatrix} 2\gamma_{\mu m} &= Ra(T_{SH}^{\mu}[\Theta_i] - \psi_{\mu m}(\tau')) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \\ &\quad + (T_{SH}^{\mu}[U_i] - \varphi_{\mu m}(\tau')) \begin{bmatrix} \gamma_{\mu m} - \delta_{\mu m} \\ \gamma_{\mu m} + \delta_{\mu m} \end{bmatrix} \end{aligned} \tag{8}$$

$$\begin{aligned} \begin{bmatrix} \varphi_{\mu m}(\tau') \\ \psi_{\mu m}(\tau') \end{bmatrix} &= \frac{T_{SH}^{\mu}[F] + T_H^{\mu}[\psi_{e0}^{\mu}] + T_S^{\mu}[\psi_a^{\mu}]}{\beta_{\mu m}^4 + Ra} \begin{bmatrix} Ra \\ \alpha_{\mu m}^2 \end{bmatrix} + \\ &\quad + \frac{E(\tau') T_{SH}^{\mu}[1]}{\alpha_{\mu m}^4 + Ra} \begin{bmatrix} \alpha_{\mu m}^2 \\ 1 \end{bmatrix} \end{aligned} \tag{9}$$

$$T_S^{\mu}[x] = \int_0^{\varepsilon} x \sin \mu \theta \, d\theta$$

$$T_H^{\mu}[x] = \int_a^1 x C_{\mu}(\alpha_{\mu m} R) R \, dR \tag{10}$$

$$T_{SH}^{\mu}[x] = \int_0^{\varepsilon} \int_0^a x C_{\mu}(\alpha_{\mu m} R) \sin \mu \theta R \, d\theta \, dR$$

$$\psi_{\epsilon}^{\mu}(R, \tau') = \frac{\mu}{R^2} \{(-1)^{n+1} \psi_{\epsilon}(R, \tau') + \psi_0(R, \tau')\} \tag{11}$$

$$\psi_{1a}^{\mu}(\theta, \tau') = \frac{2}{\pi} \left\{ \frac{J_{\mu}(\alpha_{\mu m} a)}{J_{\mu}(\alpha_{\mu m})} \psi_1(\theta, \tau') - \psi_a(\theta, \tau') \right\} \tag{12}$$

$$u = n\pi/\epsilon \tag{13}$$

$$\beta_{\mu m} = \frac{\alpha_{\mu m}^2}{2} (1 + 1/Pr) \tag{14}$$

$$\delta_{\mu m} = \frac{\alpha_{\mu m}^2}{2} (1 - 1/Pr) \tag{15}$$

$$\gamma_{\mu m} = (\delta_{\mu m}^2 - Gr)^{1/2} \tag{16}$$

$$N_{\mu m} = \frac{2}{\pi^2 \alpha_{\mu m}^2} \left[\frac{J_{\mu}^2(\alpha_{\mu m} a)}{J_{\mu}^2(\alpha_{\mu m})} - 1 \right]. \tag{17}$$

As in the steady case the pressure gradient parameter will frequently be given indirectly through a mass flow history. Hence, one may find the pressure gradient parameter from the relation

$$\bar{U}(\tau) = \frac{2}{\epsilon(1 - \alpha^2)} \int_0^{\epsilon} \int_a^1 U(R, \theta, \tau) R d\theta dR \tag{18}$$

or

$$\begin{aligned} \frac{\bar{U}(\tau) (1 - \alpha^2)}{8\epsilon} &= \sum_{m=1,3,5,\dots} \sum_{\mu=1} \frac{I_{\mu m}}{\mu N_{\mu m}} \left(\frac{\partial}{\partial \tau} \int_0^{\tau} f(\tau - \tau', \tau') d\tau' - \right. \\ &- \left. \frac{2E(\tau) I_{\mu m} \alpha_{\mu m}^2}{\mu(\alpha_{\mu m}^4 + R_a)} + \frac{R_a}{\alpha_{\mu m}^4 + R_a} [T_{SH}^{\mu}[F] + T_H^{\mu}[\psi_{s0}^{\mu}] + T_S^{\mu}[\psi_{1a}]] \right) \tag{19} \\ T_{SH}^{\mu}[1] &= \frac{2}{\mu} I_{\mu m}. \end{aligned}$$

The integral $I_{\mu m}$ can, for calculation purposes, be expressed in terms of the tabulated Lommel and associated functions.⁽¹²⁾

The mean temperature difference can be similarly defined.

The local heat transfer at the walls may be described by a Nusselt number defined by $N_a = h\lambda/k$ which may be written as

$$N_a = - \frac{q''}{\Delta T} \frac{\lambda}{k} = \frac{1}{\Delta T} \frac{\partial \Theta}{\partial N} \tag{20}$$

where λ is a reference length, $n/\lambda = N$ is the distance in the direction normal (outward) to the boundary of the channel and ΔT is a suitable

temperature difference. An average Nusselt number may be obtained by averaging the local Nusselt number over the boundary of the cross-section. Employing the divergence theorem, one finds that

$$N_u = \frac{1}{C\Delta T} \int_A \nabla^2 \Theta \, dA = \frac{T_h}{\Delta T} \left\{ U + Pr \frac{\partial \bar{\Theta}}{\partial \tau} \right\}$$

where r_h is the hydraulic radius, defined as the ratio of the cross-sectional area to the circumferential length.

SOLUTIONS FOR COAXIAL CYLINDERS

This problem is quite closely related to the sector problem. In order to satisfy the condition that U and Θ and their first derivatives are single valued functions of θ one replaces the finite sine transform with both finite sine and cosine transforms. Again we only record the solution.

$$\begin{aligned} \left[\begin{matrix} U(R, \theta, \tau) \\ \Theta(R, \theta, \tau) \end{matrix} \right] &= \sum_{m=1}^{\infty} \frac{C_0(\alpha_{0m}R)}{2\pi N_{0m}} \left\{ \begin{matrix} \varphi_{com}(\tau) \\ \psi_{com}(\tau) \end{matrix} \right\} - \frac{\partial}{\partial \tau} \int_0^{\tau} d\tau' \left\{ \begin{matrix} f_{com}(\tau - \tau', \tau') \\ g_{com}(\tau - \tau', \tau') \end{matrix} \right\} + \\ &+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{C_{\mu}(\alpha_{nm}R)}{\pi N_{nm}} \left(\cos n\theta \left\{ \begin{matrix} \varphi_{cnm}(\tau) \\ \psi_{cnm}(\tau) \end{matrix} \right\} + \frac{\partial}{\partial \tau} \int_0^{\tau} d\tau' \left\{ \begin{matrix} f_{cnm}(\tau - \tau', \tau') \\ g_{cnm}(\tau - \tau', \tau') \end{matrix} \right\} \right) + \\ &+ \sin n\theta \left\{ \begin{matrix} \varphi_{snm}(\tau) \\ \psi_{snm}(\tau) \end{matrix} \right\} + \frac{\partial}{\partial \tau} \int_0^{\tau} d\tau' \left\{ \begin{matrix} f_{snm}(\tau - \tau', \tau') \\ g_{snm}(\tau - \tau', \tau') \end{matrix} \right\} \end{aligned} \quad (21)$$

where

$$\left[\begin{matrix} f_{smn}(\tau, \tau') \\ f_{cnm}(\tau, \tau') \end{matrix} \right] = e^{-\beta_{nm}\tau} \left\{ \begin{matrix} A_{smn}(\tau') \\ A_{cnm}(\tau') \end{matrix} \right\} e^{i\gamma_{nm}\tau} + \left\{ \begin{matrix} B_{smn}(\tau') \\ B_{cnm}(\tau') \end{matrix} \right\} e^{-\gamma_{nm}\tau} \quad (22)$$

$$\left[\begin{matrix} g_{smn}(\tau, \tau') \\ g_{cnm}(\tau, \tau') \end{matrix} \right] = \frac{e^{-\beta_{nm}(\tau)}}{R_a} \left\{ \begin{matrix} A_{smn}(\tau') \\ A_{cnm}(\tau') \end{matrix} \right\} (\delta_{nm} + \gamma_{nm}) e^{\gamma_{nm}\tau} + \left\{ \begin{matrix} B_{smn}(\tau') \\ B_{cnm}(\tau') \end{matrix} \right\} (\delta_{nm} - \gamma_{nm}) e^{-\gamma_{nm}\tau} \quad (23)$$

$$\begin{pmatrix} A_{smn}(\tau') \\ A_{cnm}(\tau') \\ B_{smn}(\tau') \\ B_{cnm}(\tau') \end{pmatrix} = \begin{pmatrix} R_a(F_{SH}^n[\Theta_i] - \psi_{smn}(\tau')) + (\varphi_{smn}(\tau') - F_{SH}^n[U_i]) (\delta_{nm} - \gamma_{nm}) \\ R_a(F_{CH}^n[\Theta_i] - \psi_{cnm}(\tau')) + (\varphi_{cnm}(\tau') - F_{CH}^n[U_i]) (\delta_{nm} - \gamma_{nm}) \\ R_a(\psi_{smn}(\tau') - F_{SH}^n[\Theta_i]) + (F_{SH}^n[U_i] - \varphi_{smn}(\tau')) (\delta_{nm} + \gamma_{nm}) \\ R_a(\psi_{cnm}(\tau') - F_{CH}^n[\Theta_i]) + (F_{CH}^n[U_i] - \varphi_{cnm}(\tau')) (\delta_{nm} + \gamma_{nm}) \end{pmatrix} \frac{1}{2\gamma_{nm}} \quad (24)$$

$$\begin{pmatrix} \varphi_{com}(\tau) \\ \psi_{com}(\tau) \end{pmatrix} = \frac{F_{CH}^0[F] + F_{CH}^0[\psi_{1a}^0]}{\alpha_{0m}^4 + R_a} \begin{pmatrix} R_a \\ \alpha_{0m}^2 \end{pmatrix} + \frac{2\pi I_{0m} E(\tau)}{\alpha_{0m}^4 + R_a} \begin{pmatrix} -\alpha_{0m}^2 \\ 1 \end{pmatrix} \quad (25)$$

$$\begin{pmatrix} \varphi_{smm}(\tau) \\ \varphi_{cmm}(\tau) \\ \psi_{smm}(\tau) \\ \psi_{cmm}(\tau) \end{pmatrix} = \begin{pmatrix} R_a F_{SH}^n \\ R_a F_{CH}^n \\ \alpha_{nm}^2 F_{SH} \\ \alpha_{nm}^2 F_{CH} \end{pmatrix} \frac{[F(R, \theta, \tau)]}{\alpha_{nm}^4 + R_a} + \begin{pmatrix} R_a F_S^n \\ R_a F_C^n \\ \alpha_{nm}^2 F_S^n \\ \alpha_{nm}^2 F_C^n \end{pmatrix} \frac{[\psi_{1a}(\theta, \tau)]}{\alpha_{nm}^4 + R_a} \quad (26)$$

$$\begin{bmatrix} F_S^n[x] \\ F_C^n[x] \end{bmatrix} = \int_{-\pi}^{\pi} x \, d\theta \begin{bmatrix} \sin n\theta \\ \cos n\theta \end{bmatrix}. \quad (27)$$

Equations (16), (14), (15) and (12) with μ replaced by n define γ_{mn} , β_{mn} , δ_{mn} and $\psi_{1a}^n(\theta, \tau)$, respectively.

When the flow has radial symmetry, the mean velocity is given by

$$\bar{U}(\tau) = \frac{2}{1 - a^2} \int_a^1 U(R) R \, dR$$

and the average Nusselt number by

$$\bar{N}_u = \frac{1}{(1 - a^2)} \Delta T \left\{ \left(\frac{\partial \theta}{\partial R} \right)_1 - a \left(\frac{\partial \theta}{\partial R} \right)_a \right\}.$$

REMARKS ON VIBRATORY AND DISSIPATIVE PHENOMENA

Equation (14) which determines whether the transients are to be “over” or “under-damped” as $\gamma_{\mu m}$ is real or imaginary suggests that the physical phenomena involved may be usefully discussed in the language of linear vibrations.

Consider the equations of motion and energy (1) for an inviscid fluid ($\nu = 0$) integrated over a straight channel of arbitrary cross-section (A). One obtains

$$\begin{aligned} \frac{d\bar{U}}{dt} - g\beta\bar{\Theta} &= P_D^1 \\ \frac{d\bar{\Theta}}{dt} + C_1 \bar{u} &= (\bar{Q} - \bar{q}'_c/r_R)/\rho C_p \end{aligned}$$

where the barred quantities are mean values and \bar{q}'_c is the average heat flux per unit area over the boundary of the cross-section. If the heat flux is specified at the wall, then this heat flux plays the role of a driving function.

The solution to this system combines the effects of the driving functions and the natural non-decaying oscillations of frequency

$(g\beta C_1)^{1/2}$. This shows that the combined effect of buoyancy and convection on the velocity and temperature excess is restoring. A positive temperature difference (by accelerating the transfer of cold fluid from below) tends to annihilate itself by convective cooling. Similarly, a temperature deficit will tend toward self-eradication by convective heating. This restoring effect, which increases with magnitude of the Grashof number, is responsible for the oscillating characteristics of the transients.

This state of affairs is, however, drastically altered if one specifies the temperature rather than the temperature gradient at the walls. In this case, wall temperature gradients are established which are compatible with the wall temperatures and the internal distribution of temperature. As a result, there is that exchange of energy between the fluid and the environment which is compatible with the altered wall conditions or altered source functions and the ability of the fluid as a thermal medium to transfer heat to and from the walls. Oscillation phenomena may also occur in this problem, but the amplitude of these oscillations must decay in time.

The addition of viscosity to the problem also leads to damping, as the effect of shear in the absence of a driving force is to reduce a fluid initially in motion to rest. The viscous effect is felt by the mean motion only through the action of shear at the wall, but the value of the shear is determined by the distribution of the velocity over the cross-section.

The effects of viscosity and thermal diffusivity are not limited to damping. There are secondary effects which are stimulated when $Pr \neq 1$. These effects are associated either with the inability of the fluid as a thermal medium to transfer heat at a rate compatible with the local change of temperature excess as determined by the damping ($Pr > 1$), or with a parallel inability of the fluid as a viscous medium to transmit the local effects of velocity changes caused by damping by the mechanism of shear. Hence, for $Pr > 1$, the velocity changes tend to overtake thermally initiated temperature changes, while for $Pr < 1$, these temperature changes outstrip the velocity changes.

This secondary effect, which increases as the Prandtl number moves away from unity, tends to displace the instantaneous configuration of velocity and temperature from the new equilibrium configuration, thus opposing the action of the free convection effects which tend to restore the system to the new configuration of equilibrium. If the quantity $|\nu - \alpha|$ is large, this effect will dominate the effect of free

convection, and the oscillations will be suppressed, If $Pr = 1$, there are no secondary effects of viscosity and conduction, and the system will oscillate around the new configuration of equilibrium. Unlike the dashpot in linear vibration the frequency of the oscillation about the equilibrium configuration is independent of the rate of damping.

The coupled secondary effects of shear and conduction cannot be understood from a discussion of the mean velocity and temperature of the fluid. These effects are essentially local and are to be observed in the distribution of velocity and temperature.

To illustrate these effects the velocity and temperature distributions over the cross-section of an annular tube have been calculated for the case of fluid brought to rest by fast removal of a temperature excess (Θ_a) from the inner tube wall. The example is selected so as to isolate the interaction of the mechanisms of shear, conduction buoyancy and convection. Since the presence or absence of driving functions will not influence the essential characteristics of these mechanisms we have put $E = F = 0$. Initially it is presumed that the fluid is driven by free convection effects emanating from a hot inner wall. Hence

$$\begin{aligned} \Theta(a, 0) &= \Theta_a & \Theta_a e^{-\omega\tau} &= \Theta(a, \tau) \\ \tau = 0 & & \tau > 0 & \\ \Theta(1, 0) &= 0 & \Theta(1, \tau) &= 0 = \Theta_1(\tau). \end{aligned}$$

The initial temperature and velocity distributions are obtained from the solution of the steady state equations subject to the boundary conditions which prevail at $\tau = 0$.

The explicit representations of the velocity and temperature fields for this problem are easily obtained from the complete solution (21). One finds that

$$\begin{aligned} \left[\begin{array}{l} U(R, \tau) \\ \Theta(R, \tau) \end{array} \right] &= -\sum \frac{2\Theta_a C_0(\alpha_m R)}{\pi N_{\alpha m}(\alpha_m + R_a)} \left(e^{-\omega\tau} \left[\begin{array}{l} R_a \\ \alpha_m^2 \end{array} \right] + \right. \\ &+ \frac{\omega}{(\beta_m - \omega)^2 - \gamma_m^2} \left\langle \frac{e^{-\beta_m\tau} \sinh \gamma_m\tau}{\gamma_m} \left[\begin{array}{l} (\beta_m\omega - \beta_m^2 - \gamma_m^2) R_a \\ \delta_m(\beta_m\omega - \beta_m^2 - \gamma_m^2) + \gamma_m^2(\omega - 2\beta_m) \end{array} \right] + \right. \\ &\left. \left. + (e^{-\beta_m\tau} \cosh \gamma_m\tau - e^{-\omega\tau}) \left[\begin{array}{l} (\omega - 2\beta_m) R_a \\ \alpha_m^2(\omega - 2\beta_m) + \beta_m^2 - \gamma_m^2 \end{array} \right] \right\rangle \right). \end{aligned}$$

Temperature and velocity profiles for various values of the Prandtl and Grashof numbers are to be found in Figs. 2 to 7.

When $Pr > 1$, the decay of temperature and velocity is stimulated more by viscosity than by thermal diffusivity, and the temperature is driven to decay faster than the fluid, as a thermal medium, can allow.

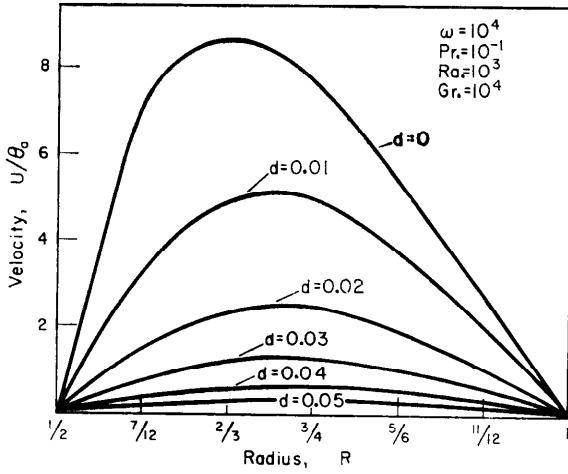


FIG. 2. Transient velocity profiles.

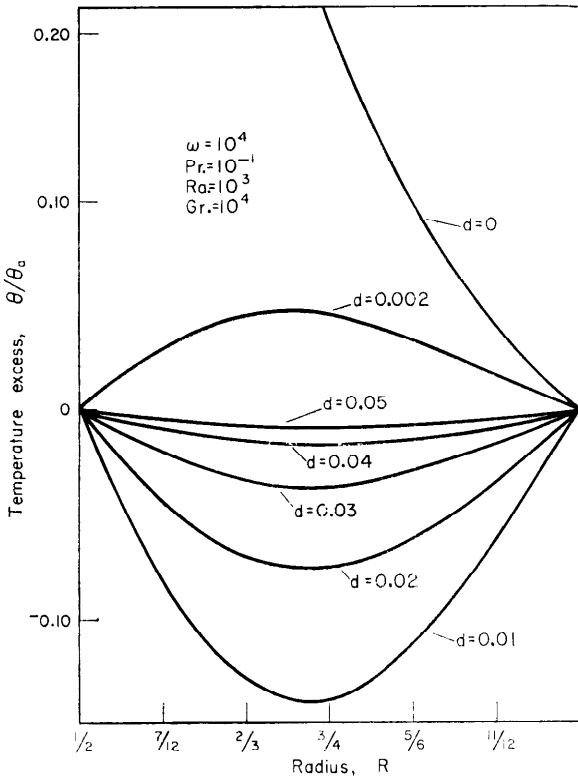


FIG. 3. Transient temperature profiles.

The velocity, on the other hand, is constrained to change more slowly through combined effects than is required by locally induced velocity changes which are being rapidly spread by shear throughout the whole fluid.

Hence, for changes which are induced thermally, as in the step removal of temperature excess from the inner wall of an annular cylinder, one expects that at positions removed from the thermal dis-

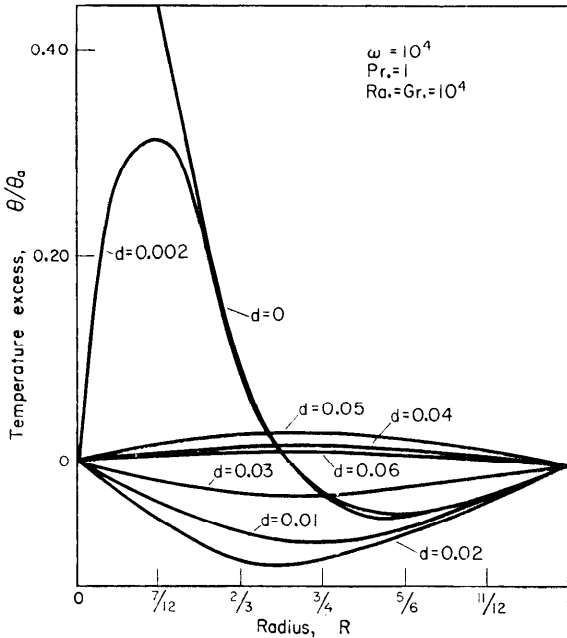


FIG. 4. Transient temperature profiles.

turbance, there will be a secondary increase of the temperature difference to compensate for the relatively poor conductivity. On the other hand, the velocity changes, which will be initiated by the rapid decline of the body force near the inside wall, propagate rapidly into the interior, depressing the velocity, and thereby aiding the decay with the effect of the shear. This secondary velocity effect (Fig. 7), and the secondary temperature effect (Fig. 6), is very dramatic for time $T = 0.01$. It is apparent from these graphs that the velocity, as predicted, does tend to overtake and assume the same sign as the temperature.

For $Pr = 0.1$ (Figs. 3 and 4), the fluid is relatively more conducting than viscous, and the temperature drop caused by the removal of θ_a

is propagated into the interior of the fluid relatively quickly. The velocity, which is reduced both by the action of shear and by the reduction of the buoyant force, is reduced relatively less because of the relative smallness of the viscosity. The fluid thus continues to transport cold fluid from below into the cross-section, further depressing the temperature. In this way rather large negative values of the temperature

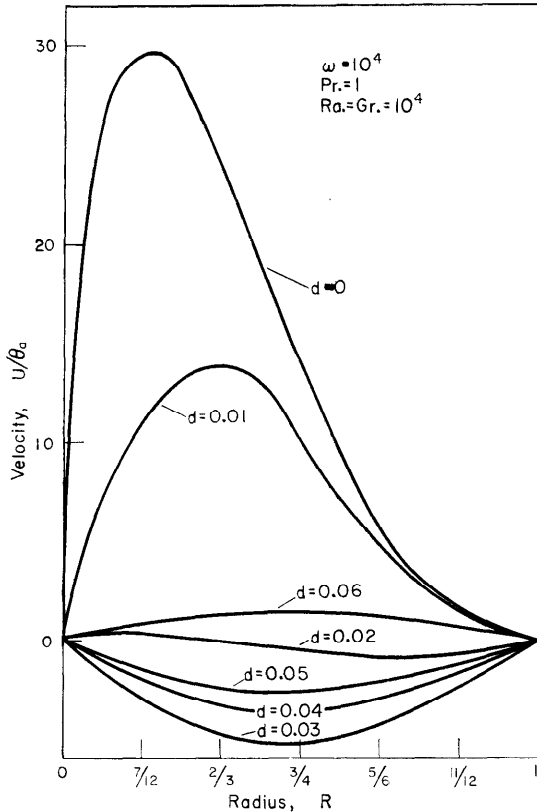


FIG. 5. Transient velocity profiles.

develop. The Grashof number is not large enough to allow to develop a negative buoyant force of magnitude sufficient to overcome the inertia of the fluid, and no negative velocities develop. The transient response is thus of the overdamped variety.

For $Pr = 1$ (Figs. 4 and 5), the fluid is as responsive to the effects of shear as to the conduction of heat. The temperature, as before, becomes negative, but the velocity, being more responsive to the combined action of the shear and the reversed buoyant force, can cool less

by convection than formerly, and the negative temperatures which do develop are of a smaller magnitude than for $Pr = 0.1$. Also, because the increased effect of the shear reinforces the negative buoyant force to decelerate the fluid strongly, the direction of the velocity is reversed. This, of course, initiates heating by convection, and the buoyant force increases, tending again to accelerate the fluid and to reverse the direction of the velocity. The approach to the static terminal state

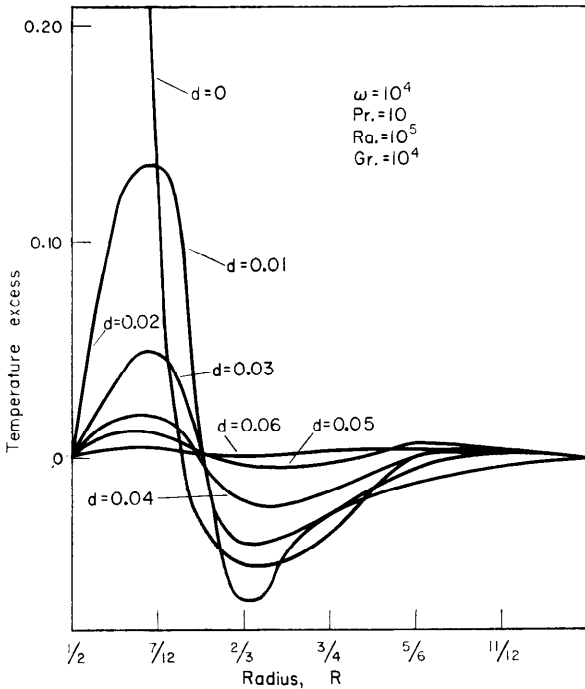


FIG. 6. Transient temperature profiles.

is thus (as for $Pr = 10$) of the underdamped variety. The velocity and temperature will thus oscillate at fixed frequencies and decaying amplitudes about the new configuration of equilibrium.

RESPONSE TO PERIODIC PRESSURE-RESONANCE

In order to discuss the resonance phenomena we shall consider the effect of a pulsating pressure gradient on the velocity and temperature fields. There is some question about the practicability of this solution and Siegel and Perlmutter⁽¹³⁾ have suggested the fully developed solu-

tions will not be applicable because the thermal entrance region under these conditions is of great length. However, a resonance phenomenon in a viscous and heat conducting fluid is a novel feature and it is of some interest that the governing differential system does accommodate such solutions.

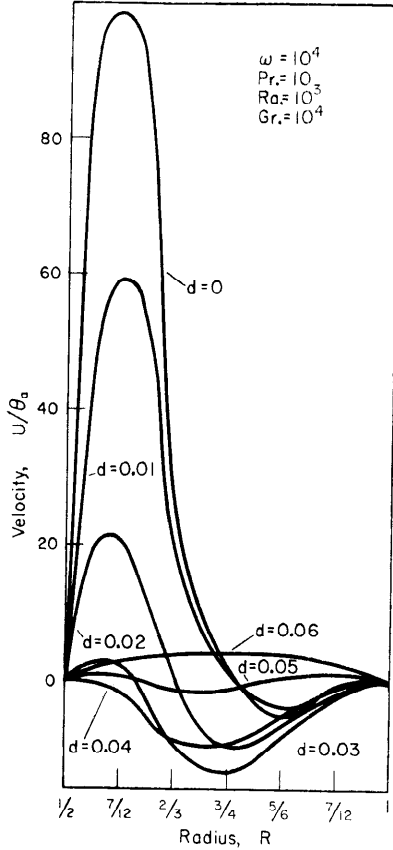


FIG. 7. Transient velocity profiles.

In particular we seek the response of the fluid to the pressure pulsation under the conditions that the temperature excesses at the inner outer walls of an annular channel are θ_a and zero, respectively, at initial and subsequent times. The non dimensional pressure gradient is given as a sinusoidal variation from the constant pressure gradient E_i of the initial state

$$E(\tau) = E_i(1 - \sin \omega \tau).$$

The solution of the problem given by Eq. (21) is evaluated explicitly for the given conditions as

$$\begin{aligned}
 \left[\begin{matrix} U(R, \tau) \\ \Theta(R, \tau) \end{matrix} \right] = & - \sum \frac{I_{0m} C_0(\alpha_m R)}{N_{0m}(\alpha_m^4 + R_a)} \left(\frac{2\Theta_a}{\pi I_{0m}} \left[\begin{matrix} R_a \\ \alpha_m^2 \end{matrix} \right] + E_i(1 - sm\omega\tau) \left[\begin{matrix} \alpha_m^2 \\ -1 \end{matrix} \right] - \right. \\
 & - \frac{\omega E_i(R_a + \alpha_m^2 \delta_m - \alpha_m^2 \gamma_m)}{2\gamma_m[(\beta_m - \gamma_m)^2 + \omega^2]} \langle (\beta_m - \gamma_m) (\cos \omega\tau - e^{(\gamma_m - \beta_m)\tau}) + \\
 & + \omega \sin \omega\tau \rangle \left[\begin{matrix} 1 \\ \frac{\delta_m + \gamma_m}{R_a} \end{matrix} \right] + \frac{\omega E_i(R_a + \alpha_m^2 \delta_m + \alpha_m^2 \gamma_m)}{2\gamma_m[(\beta_m + \gamma_m)^2 + \omega^2]} \langle (\beta_m + \gamma_m) (\cos \omega\tau - \\
 & - e^{-(\gamma_m + \beta_m)\tau}) + \omega \sin \omega\tau \rangle \left[\begin{matrix} 1 \\ \frac{\delta_m - \gamma_m}{R_a} \end{matrix} \right] \Bigg). \tag{22}
 \end{aligned}$$

It will be observed that the solution is composed of a time-independent and time-dependent part.

$$\left[\begin{matrix} U(R, \tau) \\ \Theta(R, \tau) \end{matrix} \right] = \left[\begin{matrix} U_1(R) + U_2(R, \tau) \\ \Theta_1(R) + \Theta_2(R, \tau) \end{matrix} \right].$$

If circumstances are such that γ_m is imaginary, i.e. $G_r > \frac{\alpha_m^4}{4} \left(1 - \frac{1}{Pr}\right)^2$ then the transients of the time dependent part will be oscillatory.

In the underdamped case it is possible to distort the velocity and temperature profiles after the effects of transition by suitable choices of the exciting frequency ω .

To show this we consider the case for $Pr = 1$. For this case $\gamma_m = -R_a^{1/4}$ and $\beta_m = \alpha_m^2$. After the effects of the transition have decayed the time dependent part Eq. (22) may be rewritten

$$\begin{aligned}
 \left[\begin{matrix} U_2(R, \tau) \\ \Theta_2(R, \tau) \end{matrix} \right] = & \sum_{m=1} \frac{I_{0m} C_0(\alpha_m R)}{N_{0m}(\alpha_m^4 + R_a)} \left(E_i \sin \omega\tau \left[\begin{matrix} \alpha_m^2 \\ -1 \end{matrix} \right] + \right. \\
 & + \frac{\omega E_i(\alpha_m^4 + R_a) \cos \omega\tau}{(\alpha_m^4 + R_a)^2 + 2\omega^2(\alpha_m^4 - R_a) + \omega^4} \left[\begin{matrix} R_a - \alpha_m^4 - \omega^2 \\ 2\alpha_m^2 \end{matrix} \right] + \\
 & \left. + \frac{\omega^2 E_i \sin \omega\tau}{(\alpha_m^4 + R_a)^2 + 2\omega^2(\alpha_m^4 - R_a) + \omega^4} \left[\begin{matrix} \alpha_m^2(3R_a - \alpha_m^4 - \omega^2) \\ 3\alpha_m^4 - R_a + \omega^2 \end{matrix} \right] \right).
 \end{aligned}$$

A constant pressure gradient gives rise to a displacement in the oscillatory systems of magnitude $(Ra + \alpha_m^4)^{-1}$. The periodic gradient gives rise to distortion of magnitude

$$[(\alpha_m^4 + R_a)^2 + 2\omega^2(\alpha_m^4 - R_a) + \omega^4]^{-1/4}$$

This distortion is maximum for a given integer m if the exciting frequency

$$\omega = \omega_m = (G_r - \alpha_m^4)^{1/2}.$$

The distortion is then maximum and equal to $1/(2\alpha_m^2 R_d^{1/2})$. Hence we may conclude that any harmonic of the solution can be stimulated by a suitable choice of exciting frequency.

If $Pr \neq 1$ there will exist M such that for $M > M\gamma_m$ is purely real. Resonant frequencies for this general case are given for $m \leq M$ by

$$\omega_m = \left[G_r - \frac{\alpha_m^4}{2} \left(1 + \frac{1}{Pr} \right)^2 \right]^{1/2}.$$

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