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On the Stability of the Boussinesq Equations

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In this paper we generalize the method of energy to discuss the stability of thermally-driven convective flows governed by the Boussinesq equations. The energy method as applied to non-convective flows has the striking advantage that it may be applied to difference (in contrast to perturbation) motions and can, therefore, accommodate effects of inertially non-linear disturbances. SERRIN [*1*] has exploited this special feature of the method to obtain sufficient conditions for the stability of non-convective viscous flows in bounded and unbounded regions. The sufficiency conditions can be expressed as a Reynolds number estimate and also lead to a uniqueness theorem for steady bounded flows and to a variational algorithm for improving the Reynolds number estimate. The results are important because they apply for arbitrary disturbances in regions of unspecified geometry. The energy method has a long history in classical hydrodynamics, and relevant references can be found in SERRIN's paper. Recently CONRAD & CRIMINALE [*2*] have used the method to discuss the stability of time-dependent laminar motions.

The simplified Boussinesq system, treated in this paper, involves a coupling of internal (thermal) energy to kinetic energy by the action of buoyancy. In these flows density differences established by thermally imposed temperature gradients can induce fluid motion by driving less dense (typically hotter) fluid elements against the direction of the gravity vector and, in the process, liberating internal energy. In the simplified (Boussinesq) system density differences are ignored except as they induce buoyant forces. Also neglected is the effect of the variation of thermal properties and the effects of viscous dissipation on the temperature distribution (see CHANDRASEKHAR [*3*], pp. 16–18, for full discussion).

The generalization developed in this paper proceeds from the observation that suitable energy equations for the integrated motion may be formed from a difference motion. The quadratic and bilinear integrals which appear in these equations can be easily estimated and lend themselves naturally to the formulation of a variational problem.

Our results largely extend those of SERRIN [*1*, *4*] and include:

(1) A universal stability estimate (expressed as a relation between the Reynolds and Rayleigh numbers) for arbitrary disturbances in bounded regions of unspecified geometry and for periodic disturbances in an unbounded fluid layer or cylinder (Section 2). The estimate establishes the existence of an open region in the Rayleigh-Reynolds number plane in which the flow is universally stable.

(2) A uniqueness theorem for steady convective flow in a bounded region (Section 2).

(3) A variational algorithm for computing the minimum value of the relevant dimensionless parameters (Section 3). The algorithm is used to obtain improved limits for universal stability; *i.e.*, to extend the region of Rayleigh-Reynolds number plane in which the flow is certainly stable. In the absence of thermo-convective effects (Ra = 0) our variational formulation reduces to the principle of SERRIN.

(4) An application of the variational formulation to problems of thermal convection (Section 4). The variational problem governing the stability of the *difference flow* in a quiescent fluid layer (Re = 0) can be made to coincide with the variational problem governing the stability of this same flow to time-independent *infinitesimal perturbations*. This latter problem, which has been posed by SANI & SCRIVEN [5], makes appeal to the theory of linear operators in Hilbert space and has the rigorous linear theory for a foundation. UKHOVSKII & IUDOVICH [6] and, recently, SANI [7] have demonstrated, by comparing the linear and non-linear problems of steady convection, that non-linear instabilities do not exist at values of the Rayleigh number below the critical value given by linear theory (time-independent subcritical instabilities do not exist). This result also follows easily from our formulation of the problem as a difference motion (Section 4). The Rayleigh number given as the least eigenvalue of our variational equations is precisely the greatest value for which arbitrary periodic disturbances (including time-dependent ones) necessarily decay. Thus, the result of references [6] and [7] relative to sub-critical time-independent instabilities applies generally to *arbitrary* spatially periodic disturbances. This conclusion strengthens the supposition that the linear time-independent results are an appropriate starting point for non-linear theories [8, 9, 10, 11].

1. Energy Identities

We consider a basic fluid motion occupying a region $\mathscr{V}=\mathscr{V}(t)$ of space subject to a prescribed velocity distribution and temperature condition of the third kind on the boundary $\mathscr{S}$ of $\mathscr{V}$. At an instant $t=0$ the basic fluid motion (which may be no motion) and/or temperature distribution is altered. We wish to determine if as $t\to\infty$ the subsequent flow, which must satisfy the same conditions at the boundary as the basic flow, differs radically from the basic flow or approaches it asymptotically. In treating this problem we introduce the kinetic energy of the difference motion

$$K=\int_{\mathscr{V}}\left(\frac{u^2}{2}\right) \tag{1}$$

and a quantity

$$\Theta=\int_{\mathscr{V}}\left(\frac{\vartheta^2}{2}\right) \tag{2}$$

where ϑ is the temperature of the difference motion and Θ, the temperature modulus, scales the magnitude of changes in internal (thermal) energy. If both K and Θ tend to zero as $t\to\infty$, then we say that the basic motion is (asymptotically) stable in the mean.

First consider the bounded region $\mathscr{V}$ in which motions may be induced by the motion of material walls or by convection. Let V and T and V^* and T^*

be the velocity and temperature of the basic and altered flows, respectively. The vector $\boldsymbol{u}=\boldsymbol{V}^*-\boldsymbol{V}$ and temperature difference $\vartheta=T^*-T$ of the difference motion satisfy the conditions

$$\boldsymbol{u}=\frac{\partial\vartheta}{\partial n}+\sigma\vartheta=0 \quad \text{on} \quad \mathscr{S}. \tag{3}$$

Here $\boldsymbol{n}$ is the outward normal to $\mathscr{S}$ and $\sigma(\boldsymbol{r},t)\geqq 0$ is a piecewise continuous function of position. The coefficient σ is determined by a combined conduction-convection (or radiation) condition expressing an energy balance at the boundary surface. The limit $\sigma\to\infty$ corresponds to a prescribed temperature condition and $\sigma=0$ to a prescribed heat flux.

The rates of change of K and Θ are governed by the formulae

$$\frac{dK}{dt}=-\int_{\mathscr{V}}(\boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u}+\alpha\,\boldsymbol{g}\cdot\boldsymbol{u}\,\vartheta+\nu\,\nabla\boldsymbol{u}:\nabla\boldsymbol{u}) \tag{4}$$

and

$$\frac{d\Theta}{dt}=-\int_{\mathscr{V}}(\vartheta\,\boldsymbol{u}\cdot\nabla T+\kappa\,\nabla\vartheta\cdot\nabla\vartheta)-\kappa\int_{\mathscr{S}}\sigma\,\vartheta^2 \tag{5}$$

where $\nabla\boldsymbol{u}$, $\boldsymbol{D}$ and ∇T are the dyadic gradient of the difference motion, the strain rate tensor of the basic flow and the gradient of the basic flow temperature field, respectively. Here $\kappa(t)$, $\alpha(t)$, $\nu(t)$ and $\boldsymbol{g}(\boldsymbol{r},t)$ are the thermometric coefficient, the coefficient of thermal expansion, the kinematic viscosity and the field force (typically gravity) vector, respectively.

To prove equations (4) and (5) we note that the starred and unstarred flows both satisfy the Boussinesq equations

$$\nabla\cdot\boldsymbol{V}=0, \tag{6}$$

$$\frac{d\boldsymbol{V}}{dt}=-\nabla\frac{p}{\rho_0}+(1-\alpha(T-T_0))\,\boldsymbol{g}+\nu\,\nabla^2\boldsymbol{V}, \tag{7}$$

$$\frac{dT}{dt}=\kappa\,\nabla^2 T, \tag{8}$$

where $T_0(t)$ is a reference temperature at which material properties are evaluated. By subtraction one finds that

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{V}^*\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{V}=-\nabla\frac{(p^*-p)}{\rho_0}-\alpha\,\boldsymbol{g}\,\vartheta+\nu\,\nabla^2\boldsymbol{u} \tag{9}$$

and

$$\frac{\partial\vartheta}{\partial t}+\boldsymbol{V}^*\cdot\nabla\vartheta+\boldsymbol{u}\cdot\nabla T=\kappa\,\nabla^2\vartheta. \tag{10}$$

Form the scalar product of (9) with $\boldsymbol{u}$ and (10) with ϑ, and use $\operatorname{div}\boldsymbol{V}=\operatorname{div}\boldsymbol{V}^*=0$ to produce

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)+\boldsymbol{V}\cdot\nabla\left(\frac{u^2}{2}\right)=-\boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u}-\alpha\boldsymbol{g}\cdot\boldsymbol{u}\,\vartheta-\nu\,\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\operatorname{div}\boldsymbol{A} \tag{11}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\vartheta^2}{2}\right)+\boldsymbol{V}\cdot\nabla\left(\frac{\vartheta^2}{2}\right)=-\vartheta\,\boldsymbol{u}\cdot\nabla T-\kappa\,\nabla\vartheta\cdot\nabla\vartheta+\operatorname{div}\boldsymbol{B} \tag{12}$$

where

$$A = \nu \nabla \left(\frac{u^2}{2}\right) - \boldsymbol{u}\left(\frac{u^2}{2} + \frac{p^* - p}{\rho_0}\right) \tag{13}$$

and

$$\boldsymbol{B} = \kappa \vartheta \nabla \vartheta - \boldsymbol{u}\vartheta^2/2. \tag{14}$$

Equations (4) and (5) follow from (3), (11), (12) and the divergence theorem.

Under suitably restrictive assumptions on the asymptotic behavior of the functions, (4) and (5) continue to hold for unbounded regions. Another important case for which (4) and (5) are valid occurs when the flow geometry is such that disturbances can be considered spatially periodic at each instant. This is the case for cellular convection induced by heating below. For periodic disturbances non-vanishing contributions on planes normal to symmetry axes cancel one another. Flows with planar free surfaces, $\mathscr{S}_f$, are also described by (4) and (5). On such surfaces the requirement that the surface displacement and shear stress vanish imply that

$$0 = \boldsymbol{n} \cdot \nabla \left(\frac{u^2}{2}\right) = \boldsymbol{n} \cdot \boldsymbol{u} \quad \text{on} \quad \mathscr{S}_f.$$

This eliminates gravity inertial waves as a possible form for the disturbance, but accommodates all of the boundary conditions usually associated with cellular convection problems.

2. Criteria for Universal Stability

In this section we generalize the stability criteria developed by SERRIN to convective motions with thermal coupling. We supplement the condition $K \to 0$ as $t \to \infty$ for arbitrary non-vanishing divergence-free vectors $\boldsymbol{u}$ with the additional condition $\Theta \to 0$ as $t \to \infty$ for non-vanishing arbitrary scalars ϑ.

It is clear from (4) and (5) that heat conduction and viscosity tend to stabilize the flow by dissipating the kinetic and thermal energy of the disturbance. On the other hand the Reynolds stress (first term of the right-hand side of (4)) is known to be destabilizing. The coupling terms can also induce instability by inducing disturbance kinetic energy through the action of buoyancy (second term of the right-hand side of (4)) and by increasing the thermal energy of the disturbance by liberating energy carried up from below (first term of the right-hand side of (5)). The surface term in (5) is more stabilizing as σ is larger. The prescribed temperature condition is the most stabilizing, and the prescribed heat flux least stabilizing [*12*].

It is commonly believed that the energy method has not the potential of the linear perturbation theory for fine discriminations of the limits of stability. The virtue of the method is that it gives insight into the physical situation, applies to arbitrary disturbances and is both mathematically simple and rigorous. It is of interest that in certain convective motions this method gives a quite respectable estimate of the limits of stability. In fact, the variational algorithm which is developed from the method leads precisely in special cases of physical interest to the linear perturbation result. In view of the fact that the method accommodates non-linear and even hydrodynamically or thermally inadmissible disturbances, these results are all the more striking and testify to the value and power of the energy method.

We turn now to the principal result of this section: an estimate of the kinetic energy of the temperature modulus of the mean motion at a time t.

Theorem 1*. *Let $\mathscr{V}=\mathscr{V}(t)$ be a bounded region of space which can be included in a sphere of diameter d. Let V and T be the velocity vector and temperature satisfying prescribed conditions at the boundary of $\mathscr{V}$. Then the kinetic energy K of any disturbance motion ($\boldsymbol{u}=V^*-V$) and the modulus Θ of any temperature disturbance ($\vartheta=T^*-T$) satisfy the following inequalities:*

$$\frac{\mathrm{Re}}{m}\sqrt{K}+\frac{\gamma\sqrt{\mathrm{Ra}}}{\beta}\sqrt{\Theta}$$

$$\leqq\left(\frac{\mathrm{Re}}{m}\sqrt{K_0}+\frac{\gamma\sqrt{\mathrm{Ra}}}{\beta}\sqrt{\Theta_0}\right)\begin{cases} e^{-\frac{\nu\gamma}{d^2}(3\pi^2\gamma-\sqrt{\mathrm{Ra}})t}, & \mathrm{Pr}\,\gamma^2\leqq 1,\\ e^{-\frac{\kappa}{d^2\gamma}(3\pi^2\gamma-\sqrt{\mathrm{Ra}})t}, & \mathrm{Pr}\,\gamma^2\geqq 1,\end{cases} \tag{15}$$

provided that

$$0\leqq\mathrm{Ra}\leqq 3\pi^2(\delta-\mathrm{Re})=9\pi^4\gamma^2. \tag{16}$$

Here K_0 and Θ_0 are the initial disturbance values, $-m$ is the least characteristic value of $\mathbf{D}$ and $\beta=\mathrm{Max}\,|\nabla T|$; moreover, g, ν, κ and α are maximum values in the time interval $[0, t]$. The dimensionless numbers are defined by $\mathrm{Pr}=\nu/\kappa$, $\mathrm{Ra}=\alpha g\beta d^4/\nu\kappa$, $\mathrm{Re}=d^2 m/\nu$, and $\delta\cong 80$ is the least positive root of $\mathrm{Tan}\sqrt{\delta}/2=\sqrt{\delta}/2$. If $0\leqq\mathrm{Ra}<3\pi^2(\delta-\mathrm{Re})$ for all t, then $K\to 0$ and $\Theta\to 0$ as $t\to\infty$, and the flow is stable.

Proof. We start with some known inequalities.

(1) By the Schwarz inequality

$$\left|\int_{\mathscr{V}} g\cdot\boldsymbol{u}\,\vartheta\right|\leqq\int_{\mathscr{V}} g\,|\boldsymbol{u}|\,|\vartheta|\leqq 2g\sqrt{K\Theta}, \tag{17}$$

and

$$\left|\int_{\mathscr{V}}\vartheta\,\boldsymbol{u}\cdot\nabla T\right|\leqq\int_{\mathscr{V}}\beta\,|\boldsymbol{u}|\,|\vartheta|\leqq 2\beta\sqrt{K\Theta}. \tag{18}$$

(2) With $\vartheta=0$ on $\mathscr{S}$ by Poincaré's inequality for a suitably chosen positive number ε** we have

$$\int_{\mathscr{V}}\nabla\vartheta\cdot\nabla\vartheta\geqq\frac{\varepsilon\pi^2}{d^2}\int_{\mathscr{V}}\vartheta^2=\frac{2\varepsilon\pi^2}{d^2}\Theta. \tag{19}$$

* The condition $\gamma>0$ has been shown by SERRIN to be sufficient for the stability of flow without thermal convection. A Reynolds number definition involving the maximum velocity of the basic flow has also been given by SERRIN and may be used in Theorem 1 (see references [*1*] and [*4*]).

** The value $\varepsilon=3$ applies for spherical regions, and it is this value which is used in Theorem 1. This theorem is valid for other regions provided that the appropriate value for ε is used. Several other geometries are briefly treated in the sequel. Theorems 1 and 2 also hold for the more general thermal boundary condition (3). The bound (19) must, however, be replaced by

$$\int_{\mathscr{V}}\nabla\vartheta\cdot\nabla\vartheta+\int_{\mathscr{S}}\sigma\vartheta^2\geqq\frac{\varepsilon\pi^2}{d^2}\int_{\mathscr{V}}\vartheta^2.$$

This estimate is valid for finite positive σ and leads to a smaller value for ε than (19) (see [*13*], p. 410). The value of ε depends on the distribution of σ and is not always easy to compute.

(3) With $\boldsymbol{u}=0$ on $\mathscr{S}$ and $\operatorname{div}\boldsymbol{u}=0$ in $\mathscr{V}$

$$\int_{\mathscr{V}} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \geqq \frac{\delta}{d^2} \int_{\mathscr{V}} u^2 = \frac{2\delta}{d^2} K \tag{20}$$

where $\delta \geqq \varepsilon \pi^2$ because of the divergence constraint.

(4) With $-m$ a lower bound for the characteristic values of the strain rate tensor $\boldsymbol{D}$ in the time interval $[0, t]$

$$\int_{\mathscr{V}} \boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} \geqq -m \int_{\mathscr{V}} u^2 = -2mK \tag{21}$$

and $m \geqq 0$ because Trace $\boldsymbol{D}=0$.

A combination of (4), (5) and (17)–(21) can be made to produce the estimates

$$\frac{dK}{dt} \leqq \left(2m - \frac{2\delta\nu}{d^2}\right) + 2\alpha g \sqrt{\Theta K}, \tag{22}$$

$$\frac{d\Theta}{dt} \leqq -\frac{2\varepsilon\pi^2\kappa}{d^2}\Theta + 2\beta\sqrt{\Theta K}. \tag{23}$$

A change of variables

$$K = \hat{K}^2, \tag{24}$$

$$\Theta = \hat{\Theta}^2, \tag{25}$$

where $\hat{K}$ and $\hat{\Theta}$ are positive definite, leads to the equations

$$\frac{d\hat{K}}{dt} - \left(m - \frac{\delta\nu}{d^2}\right)\hat{K} - \alpha g \hat{\Theta} \leqq 0, \tag{26}$$

$$\frac{d\hat{\Theta}}{dt} + \frac{\varepsilon\pi^2\kappa}{d^2}\hat{\Theta} - \beta\hat{K} \leqq 0, \tag{27}$$

which may be rewritten as

$$\frac{d\hat{K}}{dt} + \frac{\varepsilon\pi^2\nu\gamma^2}{d^2}\hat{K} - \frac{\nu}{d^2}\sqrt{\mathrm{Ra}}\left(\hat{\Theta}\sqrt{\frac{\alpha g\kappa}{\beta\nu}}\right) \leqq 0, \tag{28}$$

$$\gamma\,\mathrm{Pr}\frac{d}{dt}\left(\hat{\Theta}\sqrt{\frac{\alpha g\kappa}{\beta\nu}}\right) + \frac{\varepsilon\pi^2\nu\gamma}{d^2}\left(\hat{\Theta}\sqrt{\frac{\alpha g\kappa}{\beta\nu}}\right) - \frac{\nu}{d^2}\gamma\sqrt{\mathrm{Ra}}\,\hat{K} \leqq 0, \tag{29}$$

where $\gamma^2 = (\delta - \mathrm{Re})/\varepsilon\pi^2$, β, g, m, Pr and Ra are defined in Theorem 1. By addition of (28) and (29) it follows that

$$\frac{d}{dt}\left(\hat{K} + \gamma\,\mathrm{Pr}\,\hat{\Theta}\sqrt{\frac{\alpha g\kappa}{\beta\nu}}\right) + \frac{\gamma\nu}{d^2}\left(\varepsilon\pi^2\gamma - \sqrt{\mathrm{Ra}}\right)\left(\hat{K} + \hat{\Theta}\sqrt{\frac{\alpha g\kappa}{\gamma^2\beta\nu}}\right) \leqq 0 \tag{30}$$

when $\gamma^2 = (\delta - \mathrm{Re})/\varepsilon\pi^2 > 0$. Then with $\varepsilon^2\pi^4\gamma^2 > \mathrm{Ra}$ and $1 \geqq \mathrm{Pr}\,\gamma^2$

$$\frac{d}{dt}\left\{\left(\hat{K} + \gamma\,\mathrm{Pr}\,\hat{\Theta}\sqrt{\frac{\alpha g\kappa}{\beta\nu}}\right) e^{\frac{\nu\gamma}{d^2}(\varepsilon\pi^2\gamma - \sqrt{\mathrm{Ra}})t}\right\} \leqq 0, \tag{31}$$

and with $1 \leqq \mathrm{Pr}\,\gamma^2$

$$\frac{d}{dt}\left\{\left(\hat{K} + \gamma\,\mathrm{Pr}\,\hat{\vartheta}\sqrt{\frac{\alpha g\kappa}{\beta\nu}}\right) e^{\frac{\kappa}{d^2\gamma}(\varepsilon\pi^2\gamma - \sqrt{\mathrm{Ra}})t}\right\} \leqq 0. \tag{32}$$

Theorem 1 follows directly from integration of equations (31) and (32) over the time interval $[0, t]$, the substitutions (24) and (25) and

$$\hat{K}+\gamma \Pr \hat{\vartheta} \sqrt{\frac{\alpha g \kappa}{\beta \nu}}=\frac{\nu}{d^2}\left(\frac{\mathrm{Re}}{m} \hat{K}+\frac{\gamma}{\beta} \sqrt{\mathrm{Ra}}\, \hat{\Theta}\right).$$

For spherical regions $\varepsilon=3$ and $\delta \cong 80$ (see PAYNE & WEINBERGER [*14*]). The value of δ for regions between parallel plates, in circular cylinders and in two- and three-dimensional bounded spaces has been given by SERRIN [*1*] and improved by VELTE [*15*]. The values $\varepsilon=1$ and $\delta=3.74\pi^2$ apply to the horizontal fluid layer between rigid walls of height d and are of special interest.

The condition $\gamma^2=(\delta-\mathrm{Re})/\varepsilon\pi^2>0$ is essentially SERRIN's result as improved by PAYNE & WEINBERGER and by VELTE. For $\mathrm{Ra}=0$ we have universal stability when

$$\mathrm{Re}=\frac{m d^2}{\nu}<80, \quad \text{or} \quad \frac{V d}{\nu}<\sqrt{80}=8.98$$

where $V=\mathrm{Max}\,|V|$ (see [*1*], [*14*]).

For the basic state of rest ($\mathrm{Re}=0$) in a bounded region which can be contained in a sphere of diameter d, we have universal stability when $\mathrm{Ra}<2368$. The estimates given here also apply for arbitrary periodic disturbances between parallel rigid surfaces on which the temperature is prescribed. For these configurations $\varepsilon=1$ and $\delta=3.74\pi^2$, and we have universal stability when $\mathrm{Ra}<368$. This compares with the result $\mathrm{Ra}<1707$ of the linear perturbation theory (CHANDRASEKHAR [*3*], p. 43).

Theorem 1 may also be used to obtain the following uniqueness theorem concerning steady flow in a fixed bounded region.

Theorem 2. *Let V^* and T^* and V and T be two steady flows in $\mathscr{V}$ subject to a prescribed time-independent velocity and temperature distribution on the boundary of $\mathscr{V}$. Suppose further that*

$$0 \leqq \mathrm{Ra}<\varepsilon \pi^2(\delta-\mathrm{Re}). \tag{33}$$

Then the two flows must be identical.

Proof of Theorem 2. The kinetic energy K and the temperature modulus Θ are constant if the flow is steady. Since (33) is satisfied, so is (15) or (16). The two conditions are compatible if and only if $K_0=\Theta_0=K=\Theta=0$. This implies that $V^*=V$ and $T^*=T$.

3. Variational Techniques

The limits of stability given by Theorem 1 do not depend on specific details of the geometrical configuration or the distribution of the basic field variables. This generality is obtained at the expense of rather more conservative estimates than can be obtained by using all of the available information. It is of course desirable to obtain the best possible estimate of the limits of stability. An alternate procedure for finding these limits in non-convective motions which has been widely used by other authors employs a variational algorithm based on the energy method ($\mathrm{Ra}=0$) for finding the minimum Reynolds number as the maximum eigenvalue of a bounded set. This concept can be extended in a natural manner to thermally-coupled flows. That there exists a finite positive Reynolds or Rayleigh number (the maximum eigenvalue of a bounded set) follows from Theorem 1. We shall now develop a variational formalism for finding this maximum eigenvalue.

First we accentuate the role of the Rayleigh number by introducing the variables:

$$\tau=\frac{\nu}{d^2}t, \quad \psi=T/\beta, \quad \boldsymbol{g}=g\boldsymbol{f},$$

and

$$\boldsymbol{v}=\boldsymbol{u}\sqrt{\frac{\nu\beta}{\alpha g\kappa}}, \quad \boldsymbol{\varepsilon}=\boldsymbol{D}/m,$$

where the dimensionless numbers Pr, Ra and Re are defined as before, and $g=\operatorname{Max}|\boldsymbol{g}|$. In these variables equations (4) and (5) may be written as

$$\frac{d}{d\tau}\int_{\mathscr{V}}(v^2/2)=-\int_{\mathscr{V}}(\mathrm{Re}\,\boldsymbol{v}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{v}+\sqrt{\mathrm{Ra}}\,\boldsymbol{f}\cdot\boldsymbol{v}\vartheta+\nabla\boldsymbol{v}:\nabla\boldsymbol{v}) \tag{34}$$

and

$$\mathrm{Pr}\frac{d}{d\tau}\int_{\mathscr{V}}(\vartheta^2/2)=-\int_{\mathscr{V}}((\sqrt{\mathrm{Ra}}\,\nabla\psi\,.\,\boldsymbol{v}\vartheta+\nabla\vartheta\cdot\nabla\vartheta-\int_{\mathscr{S}}\sigma\vartheta^2, \tag{35}$$

Our problem is to determine the greatest values of the Reynolds and Rayleigh numbers and the "best" value of the positive parameter λ for which the inequality

$$\frac{d}{d\tau}\int_{\mathscr{V}}(v^2+\lambda\,\mathrm{Pr}\,\vartheta^2)/2=\frac{d}{d\tau}(K+\lambda\,\mathrm{Pr}\,\Theta)\leqq 0 \tag{36}$$

holds for all $\tau>0$. This criterion, like that of Theorem 1, allows for the possibility that either K or Θ may momentarily increase while the sum $K+\lambda\mathrm{Pr}\,\Theta$ decreases monotonously in time. It is obvious that $K+\lambda\mathrm{Pr}\,\Theta$ plays the role of an energy for the system and (36) is an energy criteria. For (36) to be true, it is necessary that for a fixed $\mu=\mathrm{Re}/\sqrt{\mathrm{Ra}}$ * and all $\tau>0$ that

$$\sqrt{\mathrm{Ra}}\{I_1(\boldsymbol{v},\vartheta)+\lambda I_2(\boldsymbol{v},\vartheta)\}+D(\boldsymbol{v},\boldsymbol{v})+\lambda\mathscr{D}(\vartheta,\vartheta)\geqq 0 \tag{37}$$

where

$$I_1(\boldsymbol{v},\vartheta)=\int_{\mathscr{V}}(\mu\boldsymbol{v}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{v}+\boldsymbol{f}\cdot\vartheta), \tag{38}$$

$$I_2(\boldsymbol{v},\vartheta)=\int_{\mathscr{V}}\nabla\psi\cdot\boldsymbol{v}\vartheta, \tag{39}$$

$$D(\boldsymbol{v},\boldsymbol{v})=\int_{\mathscr{V}}\nabla\boldsymbol{v}:\nabla\boldsymbol{v}, \tag{40}$$

$$\mathscr{D}(\vartheta,\vartheta)=\int_{\mathscr{V}}\nabla\vartheta\cdot\nabla\vartheta+\int_{\mathscr{S}}\sigma\vartheta^2, \tag{41}$$

$$v=\frac{\partial\vartheta}{\partial n}+\sigma\vartheta=0 \quad \text{on} \quad \mathscr{S}, \tag{42}$$

* Theorem 1 insures that there is an open region in the first quadrant of the $\sqrt{\mathrm{Ra}}$, Re-plane in which the flow is stable. The relation $\mathrm{Re}=\mu\sqrt{\mathrm{Ra}}$ describes a ray through the origin and $0\leqq\mu\leqq\infty$ in the first quadrant. The limits of stability can then be regarded as the curve, say $F(\sqrt{\mathrm{Ra}},\mathrm{Re})=0$. Then for each fixed μ there is one Ra and one Re below which the flow is stable. The family of such values, obtained as μ ranges over its possible values, is the curve $F(\sqrt{\mathrm{Ra}},\mathrm{Re})=0$.

and

$$\operatorname{div} \boldsymbol{v} = 0. \tag{43}$$

The problem set here is conveniently divided into two parts. In the first of these we regard $\lambda > 0$ as given and seek the minimum value of $\sqrt{\mathrm{Ra}}$ for which (37), (42) and (43) hold. We call this $\tilde{R}_\lambda = \tilde{R}_\lambda(\mu)$. We then seek the value of λ for which $\tilde{R}_\lambda$ is greatest. We call this value $R = R(\mu) = \operatorname{Max} \tilde{R}_\lambda$. If $\sqrt{\mathrm{Ra}} < R$ for a fixed μ, then the energy will decrease monotonically, and the flow is stable.

The first part of this problem is easily formulated in the framework of variational calculus. We require that

$$-\{I_1(\boldsymbol{v}, \vartheta) + \lambda I_2(\boldsymbol{v}, \vartheta)\} = \mathrm{Max} \tag{44}$$

hold for a class of suitably differentiable functions* $\boldsymbol{v}$ and ϑ satisfying (42), (43) and the normalizing condition

$$D(\boldsymbol{v}, \boldsymbol{v}) + \lambda \mathscr{D}(\vartheta, \vartheta) = 1. \tag{45}$$

Lagrange multipliers R_λ and $P(x, y, z, t)$ are then introduced and the system (42), (43), (44) and (45) reformulated as a system of partial differential equations by requiring

$$\delta \left\{ I_1(\boldsymbol{v}, \vartheta) + \lambda I_2(\boldsymbol{v}, \vartheta) - 2 \frac{P}{R_\lambda} \nabla \cdot \boldsymbol{v} + \frac{1}{R_\lambda} \left(D(\boldsymbol{v}, \boldsymbol{v}) + \lambda \mathscr{D}(\vartheta, \vartheta) \right) \right\} = 0. \tag{46}$$

The Euler-Lagrange equations corresponding to (46) are

$$\frac{R_\lambda}{2\lambda} (\lambda \nabla \psi + \boldsymbol{f}) \cdot \boldsymbol{v} = \nabla^2 \vartheta \tag{47}$$

and

$$\mu R_\lambda \boldsymbol{v} \cdot \boldsymbol{\varepsilon} + \frac{R_\lambda}{2} (\lambda \nabla \psi + \boldsymbol{f}) \vartheta = -\nabla P + \nabla^2 \boldsymbol{v} \tag{48}$$

which are to be solved subject to (42) and (43).

Now for any solution of the variational equations (42), (43), (47), and (48) we have by suitable multiplications and the divergence theorem that

$$\frac{R_\lambda}{2} \int_{\mathscr{V}} (\lambda \nabla \psi + \boldsymbol{f}) \cdot \boldsymbol{v} \vartheta + \lambda \mathscr{D}(\vartheta, \vartheta) = 0$$

and

$$\frac{R_\lambda}{2} \int_{\mathscr{V}} (\lambda \nabla \psi + \boldsymbol{f}) \cdot \boldsymbol{v} \vartheta + \mu R_\lambda \int_{\mathscr{V}} \boldsymbol{v} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{v} + D(\boldsymbol{v}, \boldsymbol{v}) = 0.$$

By addition and equation (45)

$$-I_1(\boldsymbol{v}, \vartheta) - \lambda I_2(\boldsymbol{v}, \vartheta) = \frac{1}{R_\lambda}. \tag{49}$$

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* It is not in the spirit of this paper to develop a detailed description of this class; suffice it to say that this leads to a class of strongly differentiable functions. The formal development of the theory given here can be unambiguously applied in any case, and the interested reader may formulate the detailed restrictions for himself (see Sobolev [*16*]).

It is a known result of variational calculus (*cf.* [*17*], Chap. 7) that there exist maximizing functions $\tilde{\boldsymbol{v}}$, $\tilde{\vartheta}$ which solve (42), (43), (44) and (45). These functions are also eigenfunctions of the variational equations (*cf.* [*13*], pp. 400–401) with the eigenvalue

$$\frac{1}{\tilde{R}_\lambda} = -I_1(\tilde{\boldsymbol{v}}, \tilde{\vartheta}) - \lambda I_2(\tilde{\boldsymbol{v}}, \tilde{\vartheta}) = \mathrm{Max}\{-I_1(\boldsymbol{v}, \vartheta) - \lambda I_2(\boldsymbol{v}, \vartheta)\}. \tag{50}$$

From (49) and (50) it follows that

$$\tilde{R}_\lambda \leqq R_\lambda \tag{51}$$

for any eigenvalue R_λ. Moreover, it is also known that the variational problem generates a complete set of eigenfunctions $(\boldsymbol{v}_i, \vartheta_i)$ and a corresponding set of eigenvalues $(R_\lambda^{(i)})$ (*cf.* [*17*], Chap. 7). This complete set may be assumed to satisfy the orthogonality conditions*

$$\int_{\mathscr{V}} \{2\mu\, \boldsymbol{v}_i \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{v}_j + (\lambda \nabla \psi + \boldsymbol{f}) \cdot (\boldsymbol{v}_i \vartheta_j + \boldsymbol{v}_j \vartheta_i)\} = D(\boldsymbol{v}_i, \boldsymbol{v}_j) + \lambda \mathscr{D}(\vartheta_i, \vartheta_j) = 0, \quad i \neq j.$$

From this it follows that for *any* admissible $\boldsymbol{v}$ and ϑ

$$\begin{aligned} -I_1(\boldsymbol{v}, \vartheta) - \lambda I_2(\boldsymbol{v}, \vartheta) &= \sum a_i a_j \int_{\mathscr{V}} (\mu\, \boldsymbol{v}_i \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{v}_j + (\lambda \nabla \psi + \boldsymbol{f}) \cdot \boldsymbol{v}_i \vartheta_j) \\ &= -\sum_i a_i^2 \{I_1(\boldsymbol{v}_i, \vartheta_i) + \lambda I_2(\boldsymbol{v}_i, \vartheta_i)\} \leqq \text{l.u.b.} \left(\frac{1}{R_\lambda^{(i)}}\right). \end{aligned} \tag{52}$$

Combining (51) and (52), we obtain

$$-I_1(\boldsymbol{v}, \vartheta) - \lambda I_2(\boldsymbol{v}, \vartheta) \leqq \text{l.u.b.} \left(\frac{1}{R_\lambda^{(i)}}\right) \leqq \frac{1}{\tilde{R}_\lambda}. \tag{53}$$

Since the left-hand side can be made arbitrarily close to its maximum value by a suitable choice of $\boldsymbol{v}$ and ϑ, it follows that

$$\frac{1}{\tilde{R}_\lambda} = \text{l.u.b.} \left(\frac{1}{R_\lambda^{(i)}}\right).$$

Hence we have

Theorem 3. *Let $\boldsymbol{v}$ and ϑ be solutions of the variational problem* (42), (43), (44) *and* (45) *for a fixed $\mu = \mathrm{Re}/\sqrt{\mathrm{Ra}}$ and any positive λ, and let*

$$\frac{1}{\tilde{R}_\lambda} = \mathrm{Max}\{-I_1(\boldsymbol{v}; \vartheta) - \lambda I_2(\boldsymbol{v}, \vartheta)\}.$$

Then the eigenvalue problem (42), (43), (47) *and* (48) *has a least eigenvalue $\tilde{R}_\lambda$, and the basic flow will be stable provided $\sqrt{\mathrm{Ra}} < \tilde{R}_\lambda$. Moreover, given a complete set of eigenfunctions $(\boldsymbol{v}_i, \vartheta_i)$ and the corresponding eigenvalues $R_\lambda^{(i)}$, we have $\tilde{R}_\lambda =$ g.l.b. $(R_\lambda^{(i)})$.*

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* This, in fact, follows necessarily from the variational equations (47) and (48) whenever the eigenvalues are not equal.

Proof of Theorem 3. For any suitably normalized solution of (34) and (35)

$$\frac{d}{d\tau}(K+\lambda \Pr \Theta) = -\sqrt{\mathrm{Ra}}\,\{I_1(\boldsymbol{v},\vartheta)+\lambda I_2(\boldsymbol{v},\vartheta\}-1. \tag{54}$$

For *any* admissible $\boldsymbol{v}$ and ϑ we have by (53)

$$\frac{d}{d\tau}(K+\lambda \Pr \Theta) \leqq \frac{\sqrt{\mathrm{Ra}}}{\tilde{R}_\lambda}-1. \tag{55}$$

Theorem 3 can also be stated as a criterion for the critical Reynolds number. With $\sqrt{\mathrm{Ra}}=\mathrm{Re}/\mu$ and $\tilde{R}_\lambda=\tilde{\mathrm{Re}}_\lambda/\mu$ the requisite restatement is obvious, and in the limit $\mu\to\infty$ one easily obtains from (50)

$$-\int \boldsymbol{v}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{v}=\frac{1}{\mathrm{Re}} \tag{56}$$

and from (47) and (48)

$$\mathrm{Re}\,\boldsymbol{v}\cdot\boldsymbol{\varepsilon} = -\nabla P+\nabla^2\boldsymbol{v}. \tag{57}$$

These are SERRIN's results [*1*]. There are a few special flows for which (57) and boundary conditions have been solved and values for the limiting Reynolds numbers obtained [*1*, *18*].

Theorem 3 is valid for any given $\lambda>0$. This leaves λ as a free parameter which may be chosen so as to give the best possible limits for stability. Clearly this is defined by the greatest value of the Rayleigh number below which the energy surely decreases. Analytically we seek that λ for which

$$R(\mu)=\underset{\lambda>0}{\mathrm{Max}}\,\tilde{R}_\lambda \tag{58}$$

for a fixed μ. The relation $\mathrm{Re}=\mu R$ then determines a point of the R, Re-plane. The set of such points obtained as μ ranges through positive values determines a region in the first quadrant in which the flow is necessarily stable.

In Section 4 we show that the best λ for the case of a motionless fluid heated from below ($\mu=0$) may be determined *a priori*, and that the time-dependent non-linear stability problem coincides with the time-independent linear stability problem. Thus, the method of energy not only accommodates finite disturbances giving a correct trend, but can in certain cases, give the same precise results as the small perturbation theory. It is to these cases that we now turn our attention.

4. Stability of a Motionless Fluid Heated from Below

There is an especially extensive literature on the problem of the stability of a state of rest in horizontal fluid layers [*3*, *19*]. Instability of these flows is generally manifested in the development of secondary cellular motions, and analytically it is usual to introduce the principle of exchange of stability into the small perturbation analysis. The onset of convection is presumed to occur when a critical Rayleigh number, determined from the linear analysis, is exceeded. The cell shape itself is indeterminate, as is the direction of the circulation within the cell, on the basis of the linear theory. Non-linear theories, which are presently in a stage of active development, typically start from the results of linear theory and examine the non-linear interactions after the onset of instability. In this section we show that this procedure is a valid one in that the state of rest is not more unstable to arbitrary disturbances than to infinitesimal disturbances at Rayleigh numbers below the critical values given by the linear (small perturbation) theory.

We first observe that a state of rest is compatible with the Boussinesq equations only if the external force per unit mass (assumed conservative) and temperature gradient fields are parallel. This may be established as a consequence of the curl of equation (7). In the unstable configuration the two fields have the same sense.* Let the direction of the fields be designated by the variable unit vector $\boldsymbol{i}(\boldsymbol{v}, t)$. Then

$$\nabla \psi = -\boldsymbol{i}, \tag{59}$$

$$\boldsymbol{f} = -\boldsymbol{i}. \tag{60}$$

The Euler-Lagrange equations (47) and (48) under these circumstances and with $\vartheta = \hat{\vartheta}/\sqrt{\lambda}$ are

$$-(\boldsymbol{i}\cdot\boldsymbol{v})(1+\lambda)\,R_\lambda/2\sqrt{\lambda} = \nabla^2\hat{\vartheta}, \tag{61}$$

$$-(\boldsymbol{i}\,\hat{\vartheta})(1+\lambda)\,R_\lambda/2\sqrt{\lambda} = -\nabla P + \nabla^2\boldsymbol{v}, \tag{62}$$

and these are to be solved subject to the conditions (42) and (43). The relation $|\nabla\psi| = |\boldsymbol{f}|$ is satisfied for motionless fluids in a constant gravitational field and heated below. For this configuration the temperature gradient is constant and necessarily parallel to the gravity vector. In fluid-filled cavities conditions (42) are also satisfied. This is also true of a horizontal fluid layer bounded by rigid plates. The boundary condition $\boldsymbol{v}=0$ is not valid at the free surfaces, but if the surface is fixed ($\boldsymbol{v}\cdot\boldsymbol{n}=0$) equations (47) and (48) (subject to div $\boldsymbol{v}=0$ and boundary conditions appropriate to free surfaces) govern.

It may be readily verified that the boundary-value problem (47) and (48), subject to boundary conditions appropriate to rigid or plane free surfaces, is identical to the boundary-value problem obtained from small perturbations from the Boussinesq system governing the state of rest. In this latter theory the principle of exchange of stability requires that partial time derivatives be set equal to zero for the (critical) condition of marginal stability. The resulting equations are (47) and (48) with $\sqrt{\mathrm{Ra}}$ replacing $(1+\lambda)R_\lambda/2\sqrt{\lambda}$.

Let us suppose that the critical Rayleigh number for the time-independent small perturbations is $\sqrt{\mathrm{Ra}_c}$. Then it follows that for any $\lambda>0$ the least eigenvalue for the variational equations

$$\tilde{R}_\lambda = \frac{2\sqrt{\lambda}}{1+\lambda}\sqrt{\mathrm{Ra}_c}. \tag{63}$$

Clearly the maximum value for $\tilde{R}_\lambda$ corresponds to $\lambda=1$ and

$$R = \sqrt{\mathrm{Ra}_c}. \tag{64}$$

It follows that the critical Rayleigh number given by the linear perturbation theory is identical to $R =$ of equation (58), and Theorem 3 applies. Hence we have

* The physical situation described by vectors of opposite sense is that corresponding to "heating from above". In this situation, for fluids which expand when heated, there is not the buoyant force mechanism necessary to induce a disturbance motion.

Theorem 4. *The Boussinesq system governing the stability fluid at rest, heated from below and bounded above and below by any combination of free (plane) and rigid surfaces, is not unstable to arbitrary periodic disturbances at Rayleigh numbers below those given by small perturbation theory. If the fluid is bounded, the disturbances need not be periodic.*

Theorem 4 establishes that the critical Rayleigh number given by the time-independent linear theory is sufficient for stability. *All* other periodic disturbances will decay in time if the Rayleigh number is below this critical value.

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