Reprinted from *Printed in U.S.A.*
THE PHYSICS OF FLUIDS VOLUME 7, NUMBER 11 NOVEMBER 1964

Variable Viscosity Effects on the Flow and Stability of Flow in Channels and Pipes

DANIEL D. JOSEPH
University of Minnesota, Minneapolis, Minnesota
(Received 20 January 1964; final manuscript received 13 August 1964)

Variable viscosity and frictionally heated channel and pipe flows are investigated. The solutions are bounded and improved estimates of the critical stress (beyond which there are no steady solutions) developed. The stress first increases, then decreases, with increasing maximum temperature. At this stress maximum there is a neutral solution and neighboring unstable solutions to an associated stability problem. Points of inflection in the velocity profile can develop in Poiseuille flows and must develop in Couette flows. The Poiseuille profiles which develop are inviscidly unstable in channels but stable in pipes.

I. INTRODUCTION

IT is well known that the viscosity of most Newtonian fluids depends markedly on the temperature. This dependence, which is already quite strong in relatively inviscid liquids like water, becomes even more marked in the relatively viscous liquids; e.g., oils. In fact the effect of frictionally induced thermal gradients on the viscosity may so dominate motions of the Couette and Poiseuille type as to make steady flow impossible. As a result of the work of Regirer[1] and Kaganov,[2] we now know that for most liquids as the shear stress (or pressure gradient) is increased beyond a certain critical value, the heat generated by viscous friction cannot be conducted to the exterior environment rapidly enough to establish a steady state temperature field. If, for large temperature differences, the fluidity is proportional to the first or a greater power of the temperature, there exists a finite critical value of the shear stress beyond which steady solutions do not exist. This feature is clearly apparent in each of the substantial number of exact solutions[3-7] which have been developed for special fluidity functions. For the nonlinear dependence there is the additional complication that the solutions which do exist are nonunique in that at least two (Sec. VI) solutions may be found which satisfy the governing energy equation and boundary conditions.

In this paper we examine some implications of these heat effects on the laminar motion and stability of the motion in straight pipes and channels. Four aspects of these effects are discussed.

(1) The temperature and velocity distribution as well as the shear stress in Couette flow (Sec. V) and the pressure gradient in Poiseuille flow (Sec. VII) are bounded as a function of the maximum channel temperature. The bounds are used to recover the results of Regirer[1] and Kaganov[2] relative to the

[1] S. A. Regirer, Prikl. Math. Mekh. **22**, 414 (1958).
[2] S. A. Kaganov, Intern. Chem. Eng. **3**, 33 (1963).
[3] H. Hausenblas, Ing-Arch. **18**, 151 (1950).
[4] C. R. Illingworth, Proc. Cambridge Phil. Soc. **46**, 469 (1950).
[5] R. Nahme, Ing-Arch. **11**, 191 (1940).
[6] T. Gorazdovskii and S. Regirer, Zh. Tekh. Fiz. **26**, 1532 (1956) [English transl.: Soviet Phys.-Tech. Phys. **1**, 1493 (1956)].
[7] E. A. Kearsley, Trans. Soc. Rheol. **6**, 253 (1962).

existence and uniqueness of solutions, to improve the estimates of the critical values of the stress (pressure) parameter (beyond which there are no steady solutions) and to generally describe the analytical character of solutions when exact nonlinear solutions cannot be obtained; e.g., Poiseuille flow. The critical stress parameter for the nonlinear problem is bounded from above by the first characteristic value generated by a linearized comparison equation (Sec. IX).

(2) When the curvature of the fluidity function is positive the first stationary value of the stress parameter is maximum (Sec. V). The stress parameter increases with temperature on the first branch of the double-valued solution and decreases on the second branch.

(3) This second branch is unstable to infinitesimal temperature disturbances of a restricted class (Sec. X). More precisely there exists a neutral solution to the characteristic value problem when the stress parameter is at its critical (maximum) value. There are also neighboring unstable solutions which are associated with that branch of the steady-state solution for which the maximum temperature increases as the shear stress parameter is decreased.

(4) Inflection points in the velocity profiles of the variable viscosity flow are analyzed in detail. We show that the velocity profile of liquids and gases in Couette flow always possesses a point of inflection at mid-channel (Sec. IV). For liquids with a linearized viscosity temperature relation, the velocity profile is of a type shown stable by Lin despite the presence of the inflection point (Sec. VI).

Somewhat greater interest is attached to the existence of inflection points in the symmetric velocity profiles which develop in the generalized Poiseuille flow, as it is known that inviscid profiles of this kind are unstable.[8] It is an interesting consequence of this generalization of Poiseuille flow (Sec. VII) that for certain critical values of the pressure gradient, points of inflection do develop. The same technique which we use to discuss the existence and uniqueness of solutions for Poiseuille flow are employed to generate close estimates of the pressure gradients necessary to produce inflection points. The Reynolds number which corresponds to this condition and which contains the channel separation as a free parameter may formally be made to approach an "inviscid-limit" in which the plane Poiseuille is unstable by theorem. Inflection points can also develop in pipe flows. However, inviscid profiles of the type generated are always stable despite the presence of inflection points (Sec. VIII).

[8] W. Tollmien, Nachr. Ges. Wiss. Göttingen (Neue Folge) 1, 79 (1935).

II. DEPENDENCE OF VISCOSITY ON TEMPERATURE

We first observe that for most Newtonian liquids the thermal conductivity k is essentially constant over ranges of temperature in which the viscosity undergoes marked variation. Hence it is appropriate to take thermal conductivity as constant but to account for the change of viscosity in treating the flow of liquids.

We describe the viscosity variation with the fluidity function

$$\varphi(\theta) = \mu_R/\mu(\theta), \qquad \theta = T - T_R, \tag{1}$$

where $\mu_R = \mu(0)$ is the dynamic viscosity evaluated at a reference temperature T_R. For most liquids φ is a monotonically increasing and continuous function of the temperature $d\varphi/d\theta \geq 0$ and, of course $\varphi \geq 1$ as $T \geq T_R$. The curvature of φ in liquids is typically nonincreasing so that $d^2\varphi/d\theta^2 \geq 0$.

The empirical relation of Poiseuille

$$\varphi_2(\theta) = 1 + \alpha_R\theta + \beta_R\theta^2 \tag{2}$$

is widely used and the positive constants α_R and β_R are tabulated in literature.[9] For water referred to a temperature of 0°C Lamb[10] gives

$$\alpha_0 = 0.03368\ {}^\circ\mathrm{C}^{-1}, \quad \beta_0 = 0.000221\ {}^\circ\mathrm{C}^{-2},$$

$$\mu_0 = 0.01779\ \mathrm{g/cm\ sec}$$

Equation (2) is accurate to three places for water at atmospheric pressure and for temperatures ranging from freezing to boiling. Most empirical relations have not the combination of mathematical simplicity and wide range of validity as the Poiseuille law applied to water.

The linearized version of (3),

$$\varphi_1(\theta) = 1 + \alpha_R\theta, \tag{3}$$

has been used by Hausenblas[3] and Regirer[6] to resolve problems of Couette and Poiseuille flow.

For many lubricating oils a fluidity function given by

$$\varphi_e(\theta) = \exp(\gamma_R\theta) \tag{4}$$

may be used to represent the variation of viscosity over a temperature range of the order of 40°C. Nahme[5] and Kearsley[7] have used this relation to

[9] E. Hatschek, *The Viscosity of Liquids* (D. Van Nostrand, Inc., New York, 1928).

[10] H. Lamb, *Hydrodynamics* (Cambridge University Press, London, 1940), 6th ed.

discuss the problems of plane Couette flow and of pipe flow, respectively.

The viscosity of some liquids, normally insensitive to variations in pressure, may exhibit a marked pressure dependence at very high pressures. We do not consider pressure effects.

III. DISCUSSION OF THE GOVERNING EQUATIONS

There are a number of exact solutions to one dimensional flow problems of variable viscosity liquids. These problems have in common the feature that the variation of the shear stress $\tau(\eta)$ may be obtained from the momentum equation by a single integration. If P is the axial pressure gradient, η the transverse coordinate and $h(\eta)$ the variable metric coefficient, then

$$\begin{aligned}\tau(\eta) &= C && \text{(plane Couette flow)}\\ &= P\eta && \text{(plane Poiseuille flow)}\\ &= C + P\eta && \text{(combined plane Couette and Poiseuille flow)}\\ &= P\eta/2 && \text{(cylindrical tube Poiseuille flow)}\\ &= C/\eta^2 && \text{(Couette flow between cylinders)}\end{aligned}$$

where C is a constant

The variation of the shear stress does not depend on the viscosity, and one can make a natural comparison of the distribution of strain rate for constant and variable viscosity conditions under the condition of equal shear stress.

The strain rate is given by

$$2\dot{\epsilon}(\eta) = \varphi(\theta)\tau(\eta)/\mu_R$$

and the temperature is governed by

$$\frac{1}{h(\eta)}\frac{d}{d\eta}\left(h\frac{d\theta}{d\eta}\right) + \tau^2(\eta)\varphi(\theta)/k\mu_R = 0.$$

Since μ_R is referred to the coldest place in the liquid, it is more viscous here than elsewhere. Thus at interior points $\varphi(\theta) > 1$. To maintain a constant shear stress the variable viscosity liquid must generate $\varphi(\theta)$ times the strain rate of the constant viscosity fluid. Hence the shear stresses do more work in the variable viscosity case and this increase is reflected in an increase [$\varphi(\theta)$ times constant viscosity dissipation] in the dissipation of work to heat.

In gases the effects are opposite and the shear stress does less work and produces less in the way of frictional heating than in the constant viscosity case.

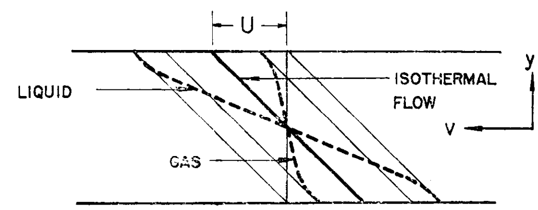

Fig. 1. Comparison of Couette flow of liquids, gases, and constant viscosity fluid for equal shear stress.

IV. PLANE COUETTE FLOW—POINTS OF INFLECTION

Consider the plane Couette flow of a Newtonian fluid with no restriction as to the compressibility. The origin of coordinates is located at the channel center and the plates are separated by a distance $2L$. The upper and lower plates are held at a single temperature ($\theta = 0$) and move to the left and right respectively with a velocity U. We introduce dimensionless variables and parameters

$$x = y/L, \quad u = U/(\tau L/\mu_R),$$
$$\psi = \theta/\theta_1, \quad \lambda = \tau^2L^2/k\mu_R\theta_1,$$

where $\theta_1^{-1} = d\varphi(0)/d\theta$ is a scaling temperature.

The dimensionless velocity and temperature fields are obtained as solutions to the differential system

$$du/dx = \varphi(\psi), \tag{5}$$

$$d^2\psi/dx^2 + \lambda\varphi(\psi) = 0, \tag{6}$$

$$u(1) = u(-1) = \mu_R U/\tau L, \tag{7}$$

$$\psi(1) = \psi(-1) = 0. \tag{8}$$

The fluidity function $\varphi(0) = 1$ at both walls, and it follows from (5) that

$$\frac{du}{dx}(1) = \frac{du}{dx}(-1). \tag{9}$$

From (6) we learn that the temperature must vary across the channel. Since φ must vary but has the same value at the channel walls, it must possess at least one maxima or minima, i.e., an interior point at which[11] $d\varphi/dx = 0$. From (5)

$$\frac{d^2u}{dx^2} = \frac{d\varphi}{d\psi}\frac{d\psi}{dx} = -\lambda\varphi'\int_0^x \varphi(\eta)\,d\eta. \tag{10}$$

It follows from (10) that if φ' is a monotone function that d^2u/dx^2 vanishes at $x = 0$ but not elsewhere.

In Fig. 1 we have sketched the velocity distribution for fluids of constant viscosity. We compare

[11] Similar inferences have been made by C. C. Lin in connection with inflection points in boundary layer profiles. See *Theory of Hydrodynamic Stability* (Cambridge University Press, London, 1955), p. 92.

this distribution with the velocity distribution for liquids (gases) in which the viscosity is known to be a decreasing (increasing) function of temperature. We have assumed equal shear stress (τ) in all three distributions. It follows from (1) and (5) that the slope of the profile at each wall is the same for the three distributions. As compared with fluids of constant viscosity the decrease in the viscosity of liquids caused by frictional heating will require that the plates move with a velocity greater than U to maintain the same shear. In gases the fluid will act more viscous by virtue of the dissipation, and the velocity will be smaller than U.

V. PLANE COUETTE FLOW-EXISTENCE AND UNIQUENESS OF SOLUTIONS

The integral formulation employed here leads to improved bounds on the solutions corresponding to arbitrary viscosity variations. The results of Regirer[1] relative to the existence and uniqueness of solutions to equation (6) and boundary conditions (8) are recovered and extended. The stress parameter $\lambda(\psi_M)$ possesses a first maximum and there are only two branches corresponding to two values of ψ_M for each λ near the first maximum of $\lambda(\psi_M)$.

Since $\varphi(\psi) \geq 1$, the curvature of ψ is negative. There is but one stationary point in $(-1, 1)$, and it is a maximum. By symmetry the maximum occurs at the channel center, and we may replace conditions (8) with

$$\frac{d\psi}{dx}(0) = \psi(1) = 0. \qquad (11)$$

Equation (8) and conditions (11) are replaced by the equivalent integral equation,

$$\psi = \lambda \int_x^1 d\eta \int_0^\eta \varphi[\psi(\gamma)]\, d\gamma. \qquad (12)$$

Since ψ is convex

$$\psi_M(1 - x) \leq \psi \leq \psi_M,$$

where $\psi_M = \psi(0)$. This last inequality is combined with (12) to produce

$$\lambda \int_x^1 d\eta \int_0^\eta \varphi[\psi_M(1 - \gamma)]\, d\gamma \leq \psi \leq \tfrac{1}{2}\lambda\varphi(\psi_M)(1 - x^2), \qquad (13)$$

with λ restricted by the condition that (13) holds when $x = 0$

$$LB(\lambda) = \frac{2\psi_M}{\varphi(\psi_M)} \leq \lambda \leq \psi_M\left\{\int_0^1 d\eta \int_0^\eta \varphi[\psi_M(1 - \gamma)]\, d\gamma\right\}^{-1} = UB(\lambda). \qquad (14)$$

It follows from (14) that no solutions exist when the shear stress parameter exceeds Max $[UB(\lambda)]$. The behavior of the solutions depends largely on the order with which $\varphi(\psi_M)$ increases with ψ_M as ψ_M tends to infinity. We let φ_{k_ν} represent the asymptotic development of φ so that

$$\lim_{\psi_M \to \infty} \varphi[\psi_M] \to A_\nu \psi_M^{k_\nu} \qquad (A_\nu > 0)$$

and from (14)

$$\lim_{\psi_M \to \infty} \left\{\frac{2\psi_M}{A_\nu \psi_M^{k_\nu}} \leq \lambda \leq \frac{(k_\nu + 2)\psi_M}{A_\nu \psi_M^{k_\nu}}\right\}.$$

Three cases may be distinguished.

(a) $0 \leq k_\nu \leq 1$, $\lambda \to \infty$. The shear stress is a unique and increasing function of the maximum temperature. Solutions exist for all λ.

(b) $k_\nu = 1$, $2/A_\nu \leq \lambda \leq 3/A_\nu$. The shear stress is a unique and increasing function of the maximum temperature but possesses a finite asymptote $(3/A_\nu)$ beyond which solutions do not exist.

(c) $k_\nu > 1$, $\lambda \to 0$. The shear stress is not a unique function of the maximum temperature. $UB(\lambda)$ has two zeros and must possess at least one maximum. Beyond Max $UB(\lambda)$ there are no solutions. Below Max $LB(\lambda)$ there are at least two solutions.

The common liquids are characterized by (c) and this case will be considered in the sequel. That a solution exists for all ψ_M may be verified by direct quadrature of (6).[1]

We now establish that the first stationary point of $\lambda(\psi_M)$ is maximum when $\varphi'' = d^2\varphi/d\psi^2 > 0$. From (6) and (11) ($\dot{\psi} = \partial\psi/\partial\psi_M$)

$$d^2\dot{\psi}/dx^2 = -\dot{\lambda}\varphi - \lambda\varphi'\dot{\psi}, \qquad (15)$$

$$\frac{d\ddot{\psi}}{dx^2} = -\ddot{\lambda}\varphi - 2\dot{\lambda}\varphi'\dot{\psi} - \lambda\varphi''\dot{\psi}^2 - \lambda\varphi'\ddot{\psi}, \qquad (16)$$

$$\dot{\psi}(1) = d\dot{\psi}(0)/dx = \ddot{\psi}(1) = d\ddot{\psi}(0)/dx = 0.$$

Multiply (15) by $\ddot{\psi}$ and (16) by $\dot{\psi}$ and integrate over (0, 1) to obtain

$$\ddot{\lambda}|_{\dot{\lambda}=0} = -\lambda\frac{\int_0^1 \varphi''\dot{\psi}^3\, d\gamma}{\int_0^1 \varphi\dot{\psi}\, d\gamma}.$$

We observe that with $\dot{\lambda} \to 0$ and $\dot{\psi}(0) = 1$ that (15) must generate $\dot{\psi} \geq 0$, $d\dot{\psi}/dx \leq 0$. Moreover (15) cannot generate $\dot{\psi}(x)$ with an interior zero unless $d\dot{\psi}/dx$ also possesses an interior zero. But

$$\frac{d\dot{\psi}}{dx} = -\dot{\lambda}\int_0^x \varphi(\psi)\, dx - \lambda\int_0^x \varphi'\dot{\psi}\, dx$$

cannot vanish before $\dot{\lambda}$ changes sign. This implies that $\dot{\psi} \geq 0$ when $\dot{\lambda}$ first vanishes. That this point is maximum follows from $\ddot{\lambda} < 0$. The possibility that $\ddot{\lambda} > 0$ at other stationary points would also seem very restricted and only this one maximum appears in the exact solutions.

The stability of the second branch is discussed in Sec. IX.

The behavior of solutions and of the function $\lambda(\psi_M)$ can be quite closely approximated by the bounding technique which leads to (14). The bounds can be replaced with a tighter relation. From (12) we form the ratio

$$\frac{\psi_M}{\psi} = \frac{\int_0^1 d\eta \int_0^\eta \varphi(\psi)\, d\gamma}{\int_x^1 d\eta \int_0^\eta \varphi(\psi)\, d\gamma} = 1 + \frac{\int_0^x \eta\, d\eta\, \bar{\varphi}(\eta)}{\int_x^1 \eta\, d\eta\, \bar{\varphi}(\eta)}$$

$$\geq 1 + \frac{\int_0^x \eta\, d\eta \operatorname{Min} \bar{\varphi}}{\int_x^1 \eta\, d\eta \operatorname{Max} \bar{\varphi}} = \frac{1}{1 - x^2}.$$

Hence

$$\psi_M(1 - x) \leq \psi(x) \leq \psi_M(1 - x^2) \tag{17}$$

and the solution is found in a region bounded by parabolas of the first and second degree. It then follows from (12) that λ must lie in a region defined by

$$\frac{\psi_M}{\int_0^1 d\eta \int_0^\eta \varphi[\psi_M(1 - \gamma^2)]\, d\gamma} \leq \lambda \leq \frac{\psi_M}{\int_0^1 d\eta \int_0^\eta \varphi[\psi_M(1 - \gamma)]\, d\gamma}. \tag{18}$$

The technique of bounding the solutions as a function of the maximum temperature may also be applied to the velocity. Thus we may replace equation (5) with

$$u = \int_0^x \varphi[\psi(\eta)]\, d\eta.$$

From this and the inequality (17) we obtain

$$\int_0^x \varphi[\psi_M(1 - \eta)]\, d\eta \leq u(x) \leq \int_0^x \varphi[\psi_M(1 - \eta^2)]\, d\eta.$$

The unknown ψ_M may be related to given data by setting $x = 1$ in the above.

In Fig. 2 we have used (18) to bound the exact solution for φ_2. The bounds (18) are evaluated as

$$\frac{2\psi_M}{1 + \frac{5}{6}\psi_M + \frac{11}{15}\delta\psi_M^2} \leq \lambda \leq \frac{2\psi_M}{1 + \frac{2}{3}\psi_M + \frac{1}{2}\delta\psi_M^2}. \tag{19}$$

We have also plotted the exact solution corresponding to φ_0 and φ_1 giving a graphical representation of the conclusions of this section. The maximum value of the shear stress is smaller for the quadratic than for the linear case. The exact solution for φ_e which is not shown lies below φ_2.

VI. PLANE COUETTE FLOW—EXACT SOLUTIONS

Illingsworth[4] and Morgan[12] have shown how the problem of plane Couette flow with temperature dependent properties may be reduced to quadratures. The functions φ_e, φ_0, φ_2, and φ_3 lead to explicit solutions in terms of tabulated functions. In the case of the quadratic and cubic viscosity-temperature relation solutions may be expressed in terms of elliptic functions. We shall examine the linear, quadratic, and exponential cases. The quadratic case, which has not been treated previously, is discussed in some detail, for φ_2 is exact for water over the whole of the temperature range in which water is liquid.

A. Linear Solution

For small temperature differences it is appropriate to use the linearized relation (4), with $\theta_1 = \alpha_R^{-1}$ and $\varphi_1 = 1 + \psi$, to describe the variation of viscosity with temperature. The solutions to equation (5) and (6) subject to conditions (7) and (11) are

$$\psi = -1 + \cos \lambda^{\frac{1}{2}}\eta / \cos \lambda^{\frac{1}{2}}, \tag{20}$$

$$u = \sin \lambda^{\frac{1}{2}}\eta / \lambda^{\frac{1}{2}} \cos \lambda^{\frac{1}{2}}. \tag{21}$$

The shear stress parameter λ is found from the condition that

$$u(1) = \tan \lambda^{\frac{1}{2}} / \lambda^{\frac{1}{2}}.$$

These representations, which were first obtained in a somewhat different form by Hausenblas, possess certain interesting features.

The temperature in the channel interior increases without bound as $\lambda^{\frac{1}{2}} \to \frac{1}{2}\pi$. This behavior is expected from the consideration of the previous section, for as $\psi_M \to \infty$, the inequalities (19) yield ($\delta = 0$)

$$\tfrac{12}{5} \leq \lambda \leq 3,$$

which are satisfied with $\lambda = \frac{1}{4}\pi^2$. The liquid will have vaporized for values of λ below the critical.

Curiously enough Lin[13] used a velocity profile of the form (21) to demonstrate that certain antisymmetric velocity profiles would not tolerate either neutral or self-excited oscillations. Hence based on Lin's work we may assert that the inviscid profile (21) is stable to velocity disturbances despite the presence of an inflection point in the velocity profile.

[12] A. J. Morgan, J. Aeron. Sci. **24**, 315 (1957).
[13] C. C. Lin, Quart. J. Appl. Math., II **3**, 218 (1945).

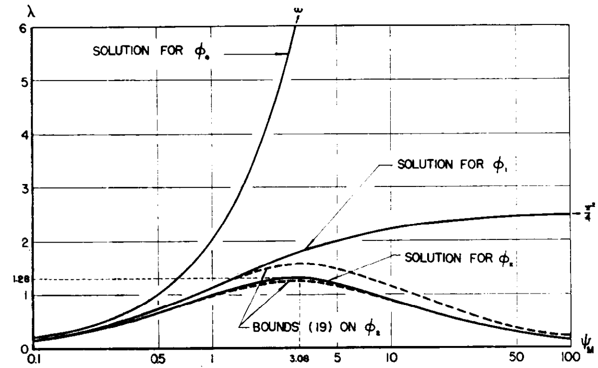

Fig. 2. Shear stress parameter as function of maximum temperature for Couette flow.

B. Quadratic Solution

The solution of Eq. (6) with $\varphi = \varphi_2 = 1 + \psi + \delta\psi^2$ subject to the conditions (11) is given implicitly by

$$x = \left(\frac{3\cos\gamma}{2\delta\lambda^{\frac{1}{2}}\sigma}\right)^{\frac{1}{2}} \int_0^A \frac{d\Phi}{(1 - K^2\sin^2\Phi)^{\frac{1}{2}}}, \quad (22)$$

where $A = 2\tan^{-1}[(\psi_M - \psi)\cos\gamma/\sigma]$, or explicitly by

$$\psi_M - \psi = \sigma(\tan^2\Phi/2)/\cos\gamma, \qquad (23)$$
$$\Phi = \sin^{-1}\left\{\mathrm{sn}\, x\left(\frac{2\lambda\sigma\delta}{3\cos\gamma}\right)^{\frac{1}{2}}\right\},$$

where

$$\sigma^2 = \tfrac{3}{4}(\psi_M^2 + \psi_M/\delta + 4/\delta - 3/4\delta^2),$$
$$\tan\gamma = 3(\psi_M + 1/2\delta)/2\sigma,$$
$$K^2 = \tfrac{1}{2}(1 + \sin\gamma),$$
$$\delta = \beta_R/\alpha_R^2.$$

The shear stress parameter λ and unknown maximum temperature ψ_M are related by

$$1 = \left(\frac{3\cos\gamma}{2\delta\lambda^{\frac{1}{2}}\sigma}\right)^{\frac{1}{2}} \int_0^B \frac{d\Phi}{(1 - K^2\sin^2\Phi)^{\frac{1}{2}}}, \quad (24)$$

where $B = 2\tan^{-1}[\psi_M\cos\gamma/\sigma]^{\frac{1}{2}}$

The graph of this equation is compared (for $\delta = 0.195$) with the inequalities (19) in Fig. 2. We note that solutions do not exist when the shear stress parameter exceeds 1.28. For each value of λ below the critical value there are two solutions corresponding to a higher and lower maximum temperature.

The velocity and temperature may be directly related by quadratures so that

$$\tfrac{1}{2}\lambda u^2 = (\psi_M - \psi)\{1 + \tfrac{1}{2}(\psi_M - \psi) + \tfrac{1}{3}\delta(\psi_M - \psi)^2\}.$$

Finally the shear stress parameter may be found in terms of the boundary velocity by requiring that

$$\lambda u(1) = 2\psi_M(1 + \tfrac{1}{2}\psi_M + \tfrac{1}{3}\delta\psi_M^2).$$

C. Exponential Solution

This has been given in a somewhat different form by Nahme.[5] The temperature distribution is represented by

$$\psi = 2\log\left\{\frac{\cosh(\lambda^{\frac{1}{2}}e^{\psi_M/2}/\sqrt{2})}{\cosh(x\lambda^{\frac{1}{2}}e^{\psi_M/2}/\sqrt{2})}\right\}. \quad (25)$$

The relation between the maximum temperature and shear stress parameter is obtained by setting $x = 0$ in equation (25)

$$\psi_M = 2\log\{\cosh(\lambda^{\frac{1}{2}}e^{\psi_M/2}/\sqrt{2})\}. \quad (26)$$

The velocity is represented by

$$\lambda u = 2e^{\psi_M} - \frac{2\cosh^2(\lambda^{\frac{1}{2}}e^{\psi_M/2}/\sqrt{2})}{\cosh^2(x\lambda^{\frac{1}{2}}e^{\psi_M/2}/\sqrt{2})}$$

and the maximum temperature is obtained from the boundary velocity by putting $x = 1$ in the above equation, i.e.,

$$\psi_M = \log(1 + \tfrac{1}{2}\lambda u(1)).$$

This solution has the same gross features as that corresponding to φ_2. Equation (26) possesses a maximum $\lambda = 0.893$ when $\psi_M = 1.18$, above which there are no solutions and below which there are two solutions for each λ.

This solution derives its importance from the fact that it describes with good accuracy the effects of variable viscosity and dissipation over a wide range of temperatures in the very viscous oils where these effects are greatest.

D. Some Numerical Results

From the above solutions we now compute the values of important parameters when the stress parameter has its maximum value.

As has been mentioned, the quadratic variation φ_2 is exact over the whole of the range of temperatures for which water at atmospheric pressure is liquid. Hence the solutions which have been developed for φ_2 apply universally to the Couette flow of water.

The reference temperature for water is taken at 0°C and for oil at 15°C. The material constants are evaluated at the reference temperature. The fluidity function constants are given for water in Sec. II. The physical parameters of the oil are given by Nahme.[5] The fluidity function constant for oil

TABLE I. Values of parameters for $\lambda = \lambda_{max}$ (Couette flow).

	Water	Oil
k (g cm/sec² °C)	58 100	12 100
μ_R (g/cm sec)	0.01779	4.26
ν_R (cm²/sec)	0.01779	4.0
θ_{max} (°C)	91.5	18.7
$L\tau_{max}$ (dynes/cm)	196	853
U (cm/sec)	38 800	4535
Re $= \tau_{max}L^2/\nu_R\mu_R$	556 000 L	49 L

is given as $\gamma_{15°} = 0.0631$°C. This fluidity function is accurate in a range of over 40°C.

The Reynolds number is based on isothermal flow at the wall temperature. The results are summarized in Table I.

There are no steady laminar solutions when $\lambda > \lambda_{max}$. This value of the shear stress parameter is reached before the water boils. For oils the temperature rise corresponding to this condition is quite moderate and well within the range of the analytical approximation (φ_e).

It will be observed the Reynolds number contains the channel separation as a free parameter. It follows that the critical condition can occur at any Reynolds number.

VII. PLANE POISEUILLE FLOW

All of the qualitative results of Sec. V and the arguments which lead to them may be duplicated for Poiseuille flow. Somewhat greater importance is attached to the existence of points of inflection in the velocity profile. Such profiles are known to be unstable. In this section we show that inflection points do develop in the Poiseuille flow of common liquids when the problem is generalized to include the effects of frictionally induced thermal gradients on the viscosity. We also show that the points of inflection will ordinarily appear when the temperature excess is rather modest and the pressure gradient is considerably below that for which steady solutions are nonexistent.

The integral equations

$$u = \int_1^x \eta\varphi[\psi(\eta)]\,d\eta, \tag{27}$$

$$\psi = \lambda_P \int_x^1 d\eta \int_0^\eta \gamma^2\varphi[\psi(\gamma)]\,d\gamma, \tag{28}$$

automatically satisfy the conditions that

$$\frac{du}{dx}(0) = u(1) = \frac{d\psi}{dx}(0) = \psi(1) = 0,$$

where the geometry and symbols are as for Couette flow with PL replacing τ and λ_P (pressure gradient parameter) replacing λ.

As in Couette flow ψ is convex, and an inequality parallel to that leading to (17) may be used to bound with a parabola, in this case a quartic. Thus the solution is to be found in the region between parabolas of first and fourth degree

$$\psi_M(1-x) \le \psi \le \psi_M(1-x^4). \tag{29}$$

The integral equation (28) is combined with (29) to produce

$$\lambda_P \int_x^1 d\eta \int_0^\eta \gamma^2\varphi[\psi_M(1-\gamma)]\,d\gamma \le \psi \le \lambda_P \int_x^1 d\eta \int_0^\eta \gamma^2\varphi[\psi_M(1-\gamma^4)]\,d\gamma, \tag{30}$$

where λ_P is restricted, so that (30) will be true when $x = 1$. Thus the relation

$$\frac{\psi_M}{\int_0^1 d\eta \int_0^\eta \gamma^2\varphi[\psi_M(1-\gamma^4)]\,d\gamma} \le \lambda_P \le \frac{\psi_M}{\int_0^1 d\eta \int_0^\eta \gamma^2\varphi[\psi_M(1-\gamma)]\,d\gamma} \tag{31}$$

establishes bounds on the maximum temperature when the pressure gradient is prescribed. Bounds on the velocity distribution are obtained from (27) and (29), and one may easily obtain from this result bounds on the maximum velocity by setting $x = 0$.

The critical value of $\lambda_P = \lambda_{P\max}$ is bounded between the maximum of the left- and right-hand of (31), and the remarks which follow Eq. (14) and apply the Couette flow retain their validity for Poiseuille flow. In particular, solutions exist for all ψ_M when $\lambda_P = \lambda_{P\max}$.[2] When the curvature of φ is positive the first stationary point of λ_P is a maximum and ψ is nonnegative.

For φ_2 and φ_e we evaluate (31) as

$$\frac{12\psi_M}{1+\frac{11}{4}\psi_M+\frac{51}{77}\delta\psi_M^2} \le \lambda_P \le \frac{12\psi_M}{1+\frac{2}{5}\psi_M+\frac{3}{5}\psi_M^2} \tag{32}$$

and

$$\frac{\psi_M}{e^{\psi_M}\int_0^1 d\eta \int_0^\eta \gamma^2 e^{-\gamma^4\psi_M}\,d\gamma} \le \lambda_P \le \frac{\psi_M}{e^{\psi_M}\int_0^1 d\eta \int_0^\eta \gamma^2 e^{-\psi_M\gamma}\,d\gamma}, \tag{33}$$

respectively.

We now establish bounds on the pressure gradient parameter λ_{P_i}, above which the velocity profile

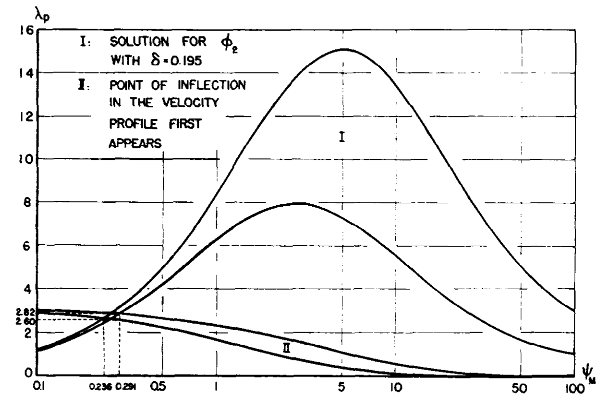

FIG. 3. Pressure gradient parameter as a function of maximum temperature for plane Poiseuille flow (water).

possesses inflection points. Differentiate (27) twice and (28) once to obtain

$$\frac{d^2u}{dx^2} = x\frac{d\varphi}{d\psi}\frac{d\psi}{dx} + \varphi$$
$$= \varphi\left\{1 - x\frac{d\log\varphi}{d\psi}\int_0^x \lambda_P\eta^2\varphi(\psi)\,d\eta\right\}.$$

If $d\varphi/d\psi \leq 0$, as is true for gases and isothermal flow, the curvature of u cannot vanish. For the common liquids

$$d\log\varphi/d\psi \leq 1$$

and this condition is sufficient to insure that an inflection point will first appear at a cold wall. The pressure gradient parameter corresponding to this condition is

$$\lambda_{P_i} = \left\{\int_0^1 \eta^2\varphi[\psi]\,d\eta\right\}^{-1}, \tag{34}$$

where $x = \varphi = d\varphi/d\psi = 1$. It follows from (29) that

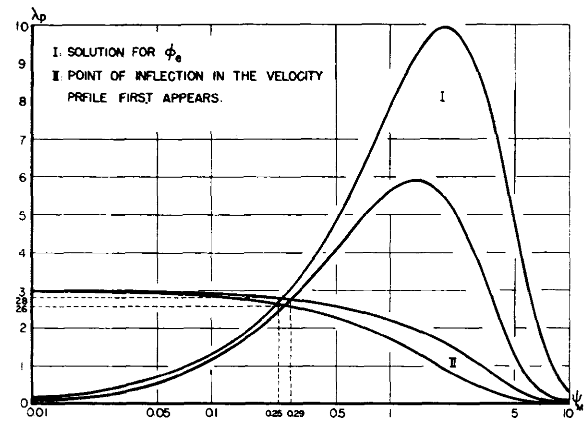

FIG. 4. Pressure gradient parameter as a function of maximum temperature for plane Poiseuille flow (oil).

$$\left\{\int_0^1 \eta^2\varphi[\psi_M(1-\eta^4)]\,d\eta\right\}^{-1} \leq \lambda_{P_i}$$
$$\leq \left\{\int_0^1 \eta^2\varphi[\psi_M(1-\eta)]\,d\eta\right\}^{-1}. \tag{35}$$

Of course λ_{P_i} also satisfies (31) and is found on the intercept of these two regions. The condition that

$$\lambda_{P_i} \leq \frac{1}{\int_0^1 \eta^2\varphi[\psi_{M_i}(1-\eta)]\,d\eta}$$
$$\leq \operatorname{Max}\frac{\psi_M}{\int_0^1 d\eta\int_0^\eta \gamma^2\varphi[\psi_M(1-\gamma^4)]\,d\gamma} \leq \lambda_{P_{Max}},$$

where ψ_{M_i} is defined by the least positive root of

$$\int_0^1 \eta^2\varphi[\psi_M(1-\eta)]\,d\eta$$
$$= \frac{1}{\psi_M}\int_0^1 d\eta\int_0^\eta \gamma^2\varphi[\psi_M(1-\gamma)]\,d\gamma$$

is sufficient to insure that a point of inflection in the velocity profile will appear for values of the pressure gradient parameter below those for which no steady solutions exist.

For φ_2 and φ_e, respectively, we obtain from (35)

$$\frac{3}{1+\frac{4}{7}\psi_M+\frac{32}{77}\delta\psi_M^2} \leq \lambda_{P_i}$$
$$\leq \frac{3}{1+\psi_{M/4}+\delta\psi_M^2/10} \tag{36}$$

$$\frac{1}{e^{\psi_M}\int_0^1 \eta^2 e^{-\psi_M\eta^4}\,d\eta} \leq \lambda_{P_i} \leq \frac{1}{e^{\psi_M}\int_0^1 \eta^2 e^{-\psi_M\eta}\,d\eta}. \tag{37}$$

In Fig. 3 we have plotted (32) and (36) for $\delta = 0.195$ (typically water). In Fig. 4 we have plotted (33) and (37) (typically oil). In both cases the region in the (λ_P, ψ_M) plane for which an inflection point first appears in the physical plane is defined by the intersection of regions I and II.

In Table II we have computed the values of the important parameters when points of inflection have appeared. The physical constants are those used in Table I.

Again the Reynolds number contains L as a free

TABLE II. Values of parameters for inflection point condition (Poiseuille flow).

	Water	Oil
Max λ_{P_i}	2.82	2.8
Max ψ_{M_i}	0.291	0.29
Max PL^2 (dynes/cm²)	291.00	1490.
Max θ_{M_i} (°C)	8.75	3.6
Re $= PL^3/2\mu_R\nu_R$	46 000 L	44 L

parameter and an inflection point may appear in the velocity profile at any Reynolds number.

VIII. HAGEN-POISEUILLE FLOW

The integral equations

$$u(r) = \int_r^1 \eta\varphi(\psi)\, d\eta,$$

$$\psi(r) = \lambda_H \int_r^1 \frac{d\eta}{\eta} \int_0^\eta \gamma^3\varphi(\psi)\, d\gamma,$$

replace the equations

$$du/dr = r\varphi, \tag{38}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) + \lambda_H r^2\varphi(\psi) = 0, \tag{39}$$

and the boundary condition

$$\frac{d\psi}{dr}(0) = \frac{du}{dr}(0) = u(1) = \psi(1) = 0.$$

The nondimensional quantities are as for plane Poiseuille flow with r_0, $\frac{1}{2}P$, and λ_H replacing L, P, and λ.

The analysis is similar to that of the previous section, except that it is not possible to apply a convexity argument to the temperature profile. One may establish that solutions are found in the region defined by

$$\frac{\psi_M}{\int_0^1 \frac{d\gamma}{\gamma}\int_0^\gamma \eta^3\varphi[\psi_M(1-\eta^4)]\, d\eta} \leq \lambda_H \leq \frac{\psi_M}{\int_0^1 \frac{d\gamma}{\gamma}\int_0^\gamma \eta^3\varphi\left[\frac{\lambda_H}{16}(1-\eta^4)\right] d\eta}. \tag{40}$$

For this flow as well as Poiseuille flow in channels the remarks which follow Eq. (14) and apply to Couette flow retain their validity. In particular $\dot\lambda$ is not negative at the first stationary point of $\lambda_H(\psi_M)$. By differentiating (39) twice with respect to ψ_M we may obtain

$$\ddot\lambda_H|_{\dot\lambda_H=0} = -\lambda_H \frac{\int_0^1 r^3\varphi''\dot\psi^3\, dr}{\int_0^1 r^3\varphi\dot\psi\, dr},$$

which shows that this stationary point is a relative maximum. Though the possibility of relative minima cannot be definitely excluded by the above consideration it seems unlikely that these can be generated even if $\dot\psi$ should change sign in (0, 1).

Points of inflection do appear in the velocity profile. When $d \log \varphi/d\psi \leq 1$ they first appear at the wall, and the pressure gradient parameter for this condition is to be found in the region defined by

$$\frac{1}{\int_0^1 \gamma^3\varphi[\psi_M(1-\gamma^4)]\, d\gamma} \leq \lambda_{H_i} \leq \frac{1}{\int_0^1 \gamma^3\varphi\left[\frac{\lambda_H}{16}(1-\gamma^4)\right] d\gamma}$$

and (40).

All the above features, i.e., the existence of a maximum for the stress parameter, the double valued solutions and the existence of inflection points in the velocity profile, are clearly apparent in Kearsley's[7] solution for φ_e. This author's suggestion that the appearance of the inflection points in the velocity profile of the pipe flow suggests instability must be regarded with caution. As is known[14] an inviscid pipe flow cannot be unstable unless the quantity

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{du}{dr}\right)$$

changes sign in (0, 1). This criteria is analogue of Rayleigh's inflection point criteria. Since

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{du}{dr}\right) = r\frac{d\psi}{dr}\varphi' = -\lambda_H\varphi'\int_0^r r^3\varphi\, dr < 0$$

the dissipation induced effects cannot generate a profile which satisfies the necessary condition for inviscid instability.

IX. AN UPPER BOUND ON THE CRITICAL STRESS PARAMETER

In this section we show that the lowest characteristic value of a linear comparison equation gives an upper bound on the critical stress parameter.

For common liquids [Case (c), $\varphi''(\psi) \geq 0$] the fluidity function may be represented by

$$\varphi(\psi) = \psi + G(\psi), \qquad G(\psi) > 1, \qquad G(0) = 1.$$

The equation governing the temperature

$$\frac{d}{dx}\left(p(x)\frac{d\psi}{dx}\right) + \lambda f(x)(\psi + G(\psi)) = 0, \tag{41}$$

$$p(x) > 0, \qquad f(x) > 0, \qquad 0 < x < 1,$$

and the boundary conditions

$$\frac{d\psi}{dx}(0) = \psi(1) = 0 \tag{42}$$

[14] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, London, 1961), p. 361.

resemble the Sturm–Liouville system[15] to which they reduce with $G = 0$. Let $\hat{\psi}$ be the solution to the reduced system. From the Sturm–Liouville theory it follows that the linear system will generate a sequence of positive characteristic values. To the first such value λ_0 will correspond a characteristic function $\hat{\psi}_0$ which satisfies the boundary conditions and has no zero in (0, 1).

Assume that a solution of (41) and (42) exists when $\lambda = \lambda_0$. From (41) and the comparison equation we obtain

$$0 = \int_0^1 \left\{ \hat{\psi}_0 \frac{d}{dx}\left(p \frac{d\psi}{dx}\right) - \psi \frac{d}{dx}\left(p \frac{d\hat{\psi}_0}{dx}\right) + \lambda_0 f(x)\hat{\psi}_0 G(\psi) \right\} dx$$

or upon integration

$$0 = \lambda_0 \int_0^1 f(x)\hat{\psi}_0(x)G(\psi)\, dx > 0.$$

It follows that our assumption was erroneous and that there is no solution of (41) and (42) when $\lambda = \lambda_0$ (the first characteristic value of the reduced linear system).

Now let $\lambda \neq \lambda_0$ be any value of λ for which the nonlinear system has a solution. From (41) and the comparison equation

$$\frac{\lambda}{\lambda_0} = \left\{1 + \frac{\int_0^1 f\hat{\psi}_0 G\, dx}{\int_0^1 f\psi\hat{\psi}_0\, dx}\right\}^{-1} \le \left\{1 + \frac{1}{\psi_M}\right\}^{-1} = \frac{\psi_M}{\psi_M + 1}$$

it follows that λ_0 is a universal upper bound on $\lambda(\psi_M)$ and that

$$\lambda(\psi_M) \le \lambda_0\psi_M/(\psi_M + 1).$$

The values of λ_0 are as follows:

(a) Couette flow, $\lambda_0 = \frac{1}{4}\pi^2$
(b) Plane Poiseuille flow, $\lambda_0 = 16.3$
(c) Hagen Poiseuille flow, $\lambda_0 = 23.1$.

X. STABILITY OF THE TEMPERATURE DISTRIBUTION

For the common liquids [Case (c), $\varphi'' \ge 0$] the first stationary value of the stress parameter is a maximum. This maximum is the only stationary point for each of the known exact nonlinear solutions. In the neighborhood of this first maximum the solution is double valued. We shall show that this second branch, which could presumably be started by preheating and maintained by small stresses, is unstable in a sense described below.

The calculation will be developed for plane Poiseuille flow but, as will be seen, is valid for Couette flow and pipe flow. The one-dimensional energy equation governing the diffusion of heat created by the viscous dissipation of a steady plane Poiseuille flow is

$$\frac{\partial\psi^*}{\partial t} = \frac{\partial^2\psi^*}{\partial x^2} + \lambda_P x^2\varphi(\psi^*),$$

where t is a suitably reduced time and $\psi^*(x, t)$ is to vanish at the walls. We next seek solutions of the form

$$\psi^* = \psi(x) + \psi_c(x)e^{-ct}$$

where $\psi(x)$ is the steady solution (28) and $\psi_c(x)$ is an infinitesimal disturbance. The equation

$$\frac{d^2\psi_c}{dx^2} = -[c + \lambda_P x^2\varphi'(\psi)]\psi_c \tag{43}$$

and boundary conditions

$$\psi_c(-1) = \psi_c(1) = 0 \tag{44}$$

constitute the characteristic value problem from which we may determine stable ($c > 0$) and unstable ($c < 0$) characteristic values.

We first observe[16] that *there is a neutral solution of* (43) *and* (44) *where* $d\lambda_P/d\psi_M = 0$. This follows from the fact that $\psi = \psi(\psi_M, x)$ and that the solution of the problem

$$\frac{d^2\dot{\psi}}{dx^2} = -x^2\{\lambda_P\varphi'(\psi)\dot{\psi} + \dot{\lambda}_P\varphi(\psi)\}, \tag{45}$$

$$\dot{\psi}(1) = \dot{\psi}(-1) = 0, \quad \dot{\lambda}_P = d\lambda_P/d\psi_M$$

is given by differentiation of (28) with respect to ψ_M. When $\dot{\lambda}_P = c = 0$ we see that a solution of (43) and (44) exists and is given by

$$A\psi_c(x) = \dot{\psi}(x),$$

where A is an arbitrary constant. A neutral solution of the perturbation equation for a zero wave number disturbance is thus associated with Max $\lambda_P(\psi_M)$.

Now we examine the behavior of $C\,[\lambda_P(\psi_M)]$ in the neighborhood of this maximum value. From Eqs. (43) and (45) we obtain

$$0 = \int_{-1}^1 \left\{\dot{\psi}\frac{d^2\psi_c}{dx^2} - \psi_c\frac{d^2\dot{\psi}}{dx^2}\right\} dx = \int_{-1}^1 \{\dot{\lambda}_P x^2\varphi(\psi)\psi_c - c\psi_c\dot{\psi}\}\, dx.$$

[15] The singular cases of this theory which occur in the applications also have a first eigenfunction with no interior zero.

[16] I am greatly indebted to the referee of this paper for calling my attention to Eq. (45) and its implication.

This is rewritten as

$$\frac{c}{\lambda_P} = \frac{\int_{-1}^{1} x^2\varphi(\psi)\psi_c\, dx}{\int_{-1}^{1} \psi_c\dot{\psi}\, dx}$$

and in the passage to the limit

$$\left.\frac{dc}{d\lambda_P}\right|_{\lambda_P=0} = \frac{\int_{-1}^{1} x^2\varphi(\psi)\dot{\psi}\, dx}{\int_{-1}^{1} \dot{\psi}^2\, dx} > 0.$$

It follows that in *the neighborhood of* Max (λ_P) *the characteristic value C and the slope* $d\lambda_P/d\psi_M$ *have the same sign*. We conclude that the temperature distribution is stable on the first branch and unstable on the second.

For Couette flow we obtain

$$\left.\frac{dc}{d\lambda}\right|_{\lambda=0} = \frac{\int_{-1}^{1} \varphi(\psi)\dot{\psi}\, dx}{\int_{-1}^{1} \dot{\psi}^2\, dx} > 0$$

and for Hagen–Poiseuille flow

$$\left.\frac{dc}{d\lambda}\right|_{\lambda=0} = \frac{\int_{0}^{1} r^3\varphi(\psi)\dot{\psi}\, dr}{\int_{0}^{1} r\dot{\psi}^2\, dr} > 0,$$

so that our conclusion applies quite generally.

An investigation of the stability of the flow must treat the velocity–temperature coupling which has been ignored in the present analysis.

XI. COMMENTS ON THE LIMIT (Re $\rightarrow \infty$)

In both Poiseuille and Couette flow the critical condition (beyond which there are no steady solutions) and in Poiseuille flow the incipient inflection point condition (beyond which points of inflection appear in the velocity profile) are reached when the shear stress or pressure gradient parameter exceeds a certain value. The dimensionless parameters themselves depend on the channel separation as well as material parameters. On the other hand the maximum velocity and temperature corresponding to either the critical or incipient inflection point conditions do not depend explicitly on channel separation. Thus for either condition the shear stress (pressure gradient) and the corresponding Reynolds number (based on the constant viscosity flow with same distribution of shear) depend on the channel separation, the latter linearly. It follows that the critical condition can appear for any Reynolds number if the channel separation is suitably chosen. Very small channels or pipes will require quite large temperature (viscosity) gradients even though the maximum temperature difference is unaltered. For a given fluid and fixed geometry there can thus, in liquids at any rate, be no such thing as an inviscid limit (Re $\rightarrow \infty$), for no steady solutions can exist for this condition.

On the other hand, one may conceive of this limit either as a sequence of fluids of decreasing viscosity or as a sequence of flows in channels or pipes of increasing dimension. This latter viewpoint is of course preferable on physical grounds. If we suppose in Poiseuille flow that the value of the pressure gradient parameter lies above that for which points of inflection have appeared but below that for which steady solutions are anavailable, then a limiting process on the Reynolds number may be described by considering channels or pipes of increasing dimension. For the plane flow, since the profile does possess inflection points, this limit will yield flows unstable not only to the effects of viscosity but in the inviscid sense. For pipe flow inflection points also develop, but the profiles generated are stable.