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THE PHYSICS OF FLUIDS VOLUME 7, NUMBER 5 MAY 1964

Incompatibility of Beltrami Flow with Viscous Adherence

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(Received 4 October 1963; revised manuscript received 16 December 1963)

Boundary conditions on the vorticity are deduced to infer conditions under which Beltrami flows of a viscous fluid are possible. The inconsistency of steady and unsteady Beltrami flow with adherence to rigid surfaces is established for a broad class of rigid motions of the bounding surfaces. The implications of isochoric motions are explored, and the possibility of an isochoric Beltrami motion of a Newtonian fluid is eliminated for rigid translatory motions of the boundary surface. The inconsistency of Beltrami flows of any fluid with rotation about an axis of geometric symmetry is also established. It is shown that the adherence condition implies either a vanishing or infinite vorticity at boundary surfaces for certain kinematically possible types of Beltrami motions.

I. INTRODUCTION

A FLOW in which the velocity $\mathbf{q}$ and the vorticity $\boldsymbol{\omega}$ = curl $\mathbf{q}$ ($\neq 0$) have the same direction is called a Beltrami flow. Examples of such flows are reviewed by Bjørgum.[1] There is an extensive literature on Beltrami fields which has been reviewed by Truesdell.[2] It would appear that no examples of Beltrami motions consistent with the adherence condition usually applied to viscous fluids have been constructed. As Professor Truesdell remarks, "It would be valuable to investigate in general the question of the consistency of a Beltrami motion with the adherence condition at a rigid boundary." Since such flows are obviously of a very special nature (they cannot occur in either plane motion or in axially symmetric motions without swirl), there is reason to believe that the conditions under which they are possible must be severely restricted.

[1] O. Bjørgum, Arbok Univ. Bergen, Mat. Nat. Ser. Nr. 1 (1951).
[2] C. Truesdell, *The Kinematics of Vorticity* (Indiana University Publication Sci. Ser. No. 19, 1954).

The possibility of steady Beltrami flows in viscous fluids has been discussed by Rajeswari and Rathna.[3] These authors considered only special solutions of their equations and their surmise that "... viscous steady Beltrami flow under conservative forces is not possible" is not conclusive.

In this paper we establish, under quite general conditions, the inconsistency of steady and unsteady Beltrami flows with adherence to rigid surfaces which (1) translate, (2) rotate about an axis of geometric symmetry, (3) both translate and rotate about an axis of symmetry.

Though we have, in some cases, employed dynamical arguments, a number of the results follow from kinematics alone.

II. GOVERNING EQUATIONS

We assume that the flow is of the Beltrami type so that

[3] G. K. Rajeswari and S. L. Rathna, Appl. Sci. Res. **A10**, No. 5, 311 (1961).

$$\boldsymbol{\omega} = \lambda \mathbf{q}, \tag{1}$$

$$\lambda \not\equiv 0, \tag{2}$$

where the scalar λ is called the "abnormality" of the Beltrami field.

When mass conservation arguments are used, we assume that the motion is isochoric (div $\mathbf{q} = 0$). It follows from the fact that $\boldsymbol{\omega}$ is solenoidal that

$$0 = \mathbf{q} \cdot \nabla \lambda. \tag{3}$$

In an isochoric, Beltrami motion the velocity vector is tangential to surfaces of constant abnormality ($\mathbf{q} \not\equiv 0$, $\nabla \lambda \not\equiv 0$).[4]

When dynamical arguments are used, we assume that the fluid is Newtonian, that extraneous body forces are conservative and that the motion is isochoric. In this case, the flow may be dynamically characterized by

$$0 = \boldsymbol{\omega} \times \mathbf{q} = \partial \mathbf{q}/\partial t - \nabla \varphi - \nu \nabla^2 \mathbf{q}, \tag{4}$$

and vorticity by

$$\partial \boldsymbol{\omega}/\partial t = \nu \nabla^2 \boldsymbol{\omega}. \tag{5}$$

We assume that the velocity, vorticity, and abnormality are continuous functions and that all derivatives which appear exist and where necessary are continuous.

III. STEADY TRANSLATION OF BOUNDING SURFACES

Consider the translation of a rigid surface either in an unbounded fluid at rest at infinity or bounded externally by a translating rigid boundary. The velocity of the internal surface is $\mathbf{V}_{\Sigma_i}$. If this surface is bounded externally, the external boundary translates with a velocity $\mathbf{V}_{\Sigma_0}$. If there is no external boundary, the fluid is at rest at infinity.

Boundary conditions on the vorticity may also be deduced. Let C be a circuit on the surface of a solid boundary. Then

$$\int_S \boldsymbol{\omega} \cdot \mathbf{N} \, d\Sigma = \oint_C \mathbf{q} \cdot d\mathbf{r} = \mathbf{V}_\Sigma \cdot \oint_C d\mathbf{r}$$

where S is an arbitrary subelement of either Σ_i of Σ_0 and $\mathbf{N}$ points into the fluid. Since S is arbitrary, a continuity argument may be applied and the integrand set to zero. Hence

$$\boldsymbol{\omega}_\Sigma \cdot \mathbf{N} = 0. \tag{6}$$

We combine this familiar result with (1) and (6) to find that

$$\lambda_\Sigma (\mathbf{V}_\Sigma \cdot \mathbf{N}) = 0 \tag{7}$$

so that either λ_Σ or $\mathbf{V}_\Sigma \cdot \mathbf{N}$ vanishes at each point of the rigid surfaces.

Through the surface Σ we pass planes P which contain the normal $\mathbf{N}$ and the velocity vector $\mathbf{V}_\Sigma$. The intercept of these planes and Σ are surface curves. If $\mathbf{V}_\Sigma \cdot \mathbf{N} \not\equiv 0$ on arcs of these curves, then it follows from the continuity of λ and (1) that $\lambda_\Sigma = 0$ at points where $\mathbf{V}_\Sigma \cdot \mathbf{N} = 0$. This implies that $\boldsymbol{\omega}_\Sigma = 0$ at every point of Σ.

When the motion is isochoric $\mathbf{V}_\Sigma \cdot \nabla \lambda = 0$ so that λ cannot vary on these curves. If $\lambda_\Sigma = 0$ anywhere on one of these curves, it has this value everywhere on this curve. Again we conclude[5] that $\boldsymbol{\omega}_\Sigma = 0$ at every point of Σ.

Thus the Beltrami motion of an adhering fluid in the presence of translating rigid surfaces implies a vanishing vorticity at these surfaces if (a) the surface possesses no arc of zero curvature parallel to the direction of motion, or (b) the motion is isochoric. This result applies to steady and unsteady motions.

A less general but more powerful result may be obtained for the steady, isochoric flow of a Newtonian fluid with conservative extraneous body forces. From Eq. (5) we learn that in such flows the vorticity is harmonic. From the previous consideration we have for the bounded rigid surfaces that $\boldsymbol{\omega}_{\Sigma_i} = \boldsymbol{\omega}_{\Sigma_0} = 0$. If the fluid is unbounded and at rest at infinity $\boldsymbol{\omega}_\infty = 0$. Hence as a consequence of potential theory: *Isochoric, steady Beltrami motion of a Newtonian fluid in the presence of extraneous body forces is not consistent with the rectilinear translation of solid surfaces.*

IV. UNSTEADY TRANSLATION OF BOUNDING SURFACES

We may extend the previous result to unsteady flows by employing Greens second identity. First consider the simply connected bounded region τ enclosed by Σ_i and Σ_0. Just as in III we find that

$$\boldsymbol{\omega}_{\Sigma_i} = \boldsymbol{\omega}_{\Sigma_0} = 0.$$

For each Cartesian component of the vorticity (no summation)

$$\int_\tau \omega_i \nabla^2 \omega_i \, d\tau = -\int_{\Sigma_i + \Sigma_0} \omega_i \nabla \omega_i \cdot \mathbf{N} \, d\Sigma - \int_\tau (\nabla \omega_i)^2 \, d\tau = -\int_\tau (\nabla \omega_i)^2 \, d\tau. \tag{8}$$

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[4] R. Ballabh, Proc. Benares Math. Soc. 2, 69 (1940).

[5] We exclude the possibility that there are surface curves which are straight and parallel to $\mathbf{V}_\Sigma$ and extend to infinity in both directions.

From Eqs. (5) and (8) we find that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\tau}\omega_i^2\,d\tau = -\nu\int_{\tau}(\nabla\omega_i)^2\,d\tau. \tag{9}$$

Equation (9) also applies when τ is a simply connected unbounded region provided that

$$\omega_i\nabla\omega_i = O(r^{-3}).$$

In this case

$$\lim_{r\to\infty}\int_{\Sigma_0}\omega_i\nabla\omega_i\cdot\mathbf{N}\,d\Sigma = 0$$

and (8) and (9) retain their validity.

Since the right-hand side of (9) is negative definite, it follows that the total vorticity of the fluid decays in time. If the vorticity is initially absent, it cannot develop. Moreover, as a result of III, there is no initial steady state from which this vorticity may decay. *Isochoric, Beltrami motions of a Newtonian fluid with conservative extraneous forces and bounded by translating surfaces cannot develop from initial states in which the vorticity is absent. Moreover, if the vorticity is initially present, it must decay in time.*

V. RIGID ROTATION OF BOUNDING SURFACES

Now we shall exclude the possibility of arbitrary rigid rotations. Let the rigid surface Σ translate at a velocity $\mathbf{V}$ and rotate with an angular velocity $\mathbf{\Omega}$. The direction of the angular velocity vector and an axis of geometric symmetry must coincide if the resulting motion is to be steady. This symmetry restriction will be assumed for unsteady motions. The velocity of the fluid on Σ is given by the adherence condition as

$$\mathbf{q}_\Sigma = \mathbf{V} + \mathbf{\Omega}\times\mathbf{r}, \tag{10}$$

where $\mathbf{r}$ has been referred to the axis of geometric symmetry. We again deduce a boundary condition on the vorticity by considering the circulation on an arbitrary subelement of Σ,

$$\int_S \boldsymbol{\omega}\cdot\mathbf{N}\,d\Sigma = \oint \mathbf{q}_\Sigma\cdot d\mathbf{r} = \int_S 2\mathbf{\Omega}\cdot\mathbf{N}\,d\Sigma. \tag{11}$$

Hence,

$$\int_S(\boldsymbol{\omega} - 2\mathbf{\Omega})\cdot\mathbf{N}\,dS = 0,$$

and since S is arbitrary

$$(\boldsymbol{\omega}_\Sigma - 2\mathbf{\Omega})\cdot\mathbf{N} = 0. \tag{12}$$

Equations (1) and (10) are applied to (12) to obtain

$$\lambda_\Sigma(\mathbf{V}\cdot\mathbf{N}) + \lambda_\Sigma(\mathbf{\Omega}\times\mathbf{r})\cdot\mathbf{N} = 2\mathbf{\Omega}\cdot\mathbf{N}. \tag{13}$$

We observe that because of the symmetry restrictions $(\mathbf{\Omega}\times\mathbf{r})\cdot\mathbf{N} = 0$ and $\lambda_\Sigma(\mathbf{V}\cdot\mathbf{N}) = 2\mathbf{\Omega}\cdot\mathbf{N}$. If Σ does not translate ($\mathbf{V} = 0$), then $\mathbf{\Omega}\cdot\mathbf{N} = 0$. Hence *Beltrami flow is inconsistent with adherence to any rigid surface (except an infinite circular cylinder) in steady rotation or in unsteady rotation about an axis of geometric symmetry.*

VI. GENERAL RIGID MOTION OF BOUNDING SURFACES

If the rigid surface both translates and rotates, we may use Eq. (13) to deduce the boundary values of the abnormality

$$\lambda_\Sigma = 2\mathbf{\Omega}\cdot\mathbf{N}/\mathbf{V}\cdot\mathbf{N}. \tag{14}$$

The vorticity at each point on Σ is given by

$$\boldsymbol{\omega}_\Sigma = 2\mathbf{\Omega}\cdot\mathbf{N}(\mathbf{V} + \mathbf{\Omega}\times\mathbf{r})/\mathbf{V}\cdot\mathbf{N}. \tag{15}$$

If $\mathbf{\Omega}$ and $\mathbf{V}$ are not collinear, the vorticity tends to infinity where $\mathbf{V}\cdot\mathbf{N} = 0$. Hence a Beltrami motion with finite vorticity is not consistent with adherence to rigid surfaces which translate arbitrarily and rotate about an axis of geometric symmetry provided that: (a) The velocity and angular velocity are not collinear. (b) The surface possesses at least one tangent parallel to the velocity.

We note that this result encompasses all smooth rigid bodies satisfying the first of the above conditions.

We now infer restrictions which apply to the isochoric motion of a Newtonian fluid when the velocity and angular velocity of the surface are collinear.

First we establish the following result: *Isochoric Trkalian motions of a Newtonian fluid with conservative extraneous forces cannot develop from initial states in which the vorticity is absent. If the vorticity is initially present, it must decay monotonously. Moreover, steady motions of this type are impossible.*

A Trkalian motion is one for which the abnormality is independent of position. It is known that the curl of a Trkalian field is again a Trkalian field. Hence one may obtain a Helmholz equation for vorticity by forming the curl of the curl of Eq. (1):

$$\nabla^2\boldsymbol{\omega} + \lambda^2(t)\boldsymbol{\omega} = 0. \tag{16}$$

For the steady isochoric flow of a Newtonian fluid, $\boldsymbol{\omega}$ is harmonic so that $\lambda^2\boldsymbol{\omega} = 0$. Hence $\boldsymbol{\omega} = 0$, for if $\lambda = 0$ then by (1) $\boldsymbol{\omega} = 0$. This means no steady Trkalian flow of this type is possible.

If unsteady, from (5) and (16) we find that

$$\partial\boldsymbol{\omega}/\partial t + \lambda^2(t)\boldsymbol{\omega} = 0$$

which, when scalar multiplied by $\boldsymbol{\omega}$ and integrated, gives

$$\omega^2 = A \exp\left[-2\nu \int \lambda^2(t)\, dt\right], \qquad (17)$$

where A is a function of space coordinates only.

It follows from (17) that the amplitude of the vorticity decays monotonously. If not initially present it cannot develop.

We now show that if $\boldsymbol{\Omega}$ and $\mathbf{V}$ are collinear, then an isochoric Beltrami motion in the neighborhood of a rigid surface is necessarily Trkalian.

If $\boldsymbol{\Omega}$ and $\mathbf{V}$ are collinear, it follows from (14) that $\lambda_\Sigma = 2\Omega/V$, a function of time only. Let $\mathbf{V} = (V_x, 0, 0)$ and define a surface coordinate ξ to which the vectors $\boldsymbol{\Omega} \times \mathbf{r}$ are tangent and a third independent coordinate η embedded in Σ. Since $\lambda_\Sigma = \text{const}$,

$$(\partial\lambda/\partial\xi)_\Sigma = (\partial\lambda/\partial\eta)_\Sigma = 0.$$

If the motion is isochoric,

$$0 = \mathbf{q}_\Sigma \cdot \nabla\lambda = V(\partial\lambda/\partial x)_\Sigma + \Omega r \sin\theta(\partial\lambda/\partial\xi)_\Sigma$$
$$= V(\partial\lambda/\partial x)_\Sigma.$$

At points where these three partial derivatives are independent $\nabla\lambda = 0$. Hence, $\lambda = \lambda(t)$ in these regions, and the motion is Trkalian. The three derivatives are not independent where the surface possesses an arc of zero curvature parallel to x and our result does not hold in the neighborhood of such arcs.