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The Effect of Permeability on the Slow Motion of a Porous Sphere in a Viscous Liquid

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Es wird ein Verfahren vorgeschlagen, mit dem der Einfluß von durchlässigem Material auf eine mit niedriger Reynoldszahl strömende viskose Flüssigkeit berechnet werden kann. Speziell wird gezeigt, daß man das Gesetz von Darcy und die asymptotischen Gleichungen ($Re \to 0$) von Stokes benutzen kann, um Randwertprobleme zu formulieren, die Lösungen liefern, die für poröse und nichtporöse Gebiete gelten.

Es wird das Problem der Strömung einer viskosen Flüssigkeit hinter einer durchlässigen Kugel betrachtet. Dafür werden geschlossene Lösungen abgeleitet, die in einfacher Weise von der Durchlässigkeit der Kugel abhängen. Es wird gezeigt, daß der Widerstand einer durchlässigen Kugel derselbe ist wie der einer undurchlässigen mit kleinerem Radius.

A technique is suggested by which the effects of permeable materials on the low Reynolds number flow of viscous liquids may be evaluated. In particular, we show that Darcy's law and the asymptotic equations ($Re \to 0$) of Stokes may be used to formulate boundary value problems generating solutions valid for both porous and non-porous regions and matched at common boundaries.

The coupled problem of the streaming of a viscous liquid past a permeable sphere is considered and closed solutions which depend simply on the permeability on the sphere are derived. The drag on a permeable sphere is shown to be the same as the drag on an impermeable sphere of reduced radius.

В работе предлагается метод, с помощью которого может быть определено влияние проницаемого материала на течение вязкой жидкости с малым числом Рейнольдса. В частности показано, что используя закон Дарси и асимптотические уравнения ($Re \to 0$) Стокса, можно формулировать краевые задачи, решения которых имеют место для пористой и непористой областей.

Рассматривается проблема течения вязкой жидкости, прошедшей через проницаемую сферу. Выводятся конечные решения, которые в простой форме зависят от проницаемости сферы. Показано, что сопротивление проницаемой сферы такое же, как сопротивление непроницаемой сферы с меньшим радиусом.

1. Introduction

In this note we suggest a technique by which the effects of permeable materials on the low REYNOLDS number flow of viscous liquids may be evaluated. In particular, we show that DARCY's law and the asymptotic equations ($\text{Re} \to 0$) of STOKES may be used to formulate boundary value problems generating solutions valid for both porous and non-porous regions and matched at common boundaries. Generally, these matching problems are difficult and lead to complicated recursion formulae for the coefficients of the solution expansions [4].

However, in the problem of streaming motion past a permeable sphere no complicated matching is involved and the solution which shows clearly the effect of the permeability k is given in a simple closed form which is not more complicated than the solution given by STOKES to which it may be reduced by setting $k = 0$. This circumstance makes the problem of the permeable sphere particularly useful in exposing the methods by which the effect of naturally permeable materials on the flow of a viscous liquid may be evaluated.

2. Mathematical Formulation

In order to discuss the motion of a fluid in the presence of a porous material, one should ideally have at one's disposal sufficient information to determine the flow at the surface of and within the porous bulk [2], [3], [5], [6]. Mathematically, such information may be obtained from a generalization of an empirical law given by DARCY. BIOT [1] has shown how one may recover this law analytically from a phenomenological formulation of irreversible thermodynamics.

For statistically homogeneous and isotropic porous materials one may write

$$\vec{Q} = -k\,(\nabla P - \vec{F})/\mu\,, \tag{1}$$

where k, μ, P and F are the permeability, dynamic viscosity, pressure and vector body force, respectively. The vector $\vec{Q}$ is a volume flow rate per unit cross-section area. As such it represents the filter velocity rather than the true velocity of the fluid in the pores.

The permeability constant is an experimentally determined quantity which depends upon the porous material and scales the ease with which a fluid may pass through the given porous material. Ordinarily, the magnitude of this constant is very small and an order of magnitude of 10^{-5} in. [2] is to be considered very large.

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To find the velocity and pressure fields in a permeable medium it is necessary to solve a POISSON equation for the pressure

$$\nabla^2 P - \operatorname{div} \vec{F} = 0 \tag{2}$$

which is obtained from the DARCY law (1) by application of the principle of conservation of mass ($\operatorname{div} \vec{Q} = 0$). The appropriate boundary conditions specify the value of the pressure or the normal derivative of the pressure, i. e., the normal velocity, or a linear combination of these two on the boundaries. In problems which involve flow between materials of different permeability boundary pressures and velocities may be unknown. In this case the missing boundary conditions are supplied by the requirement that both pressure and the normal component of velocity are continuous across surfaces separating regions of different permeability. As in the hydrodynamics of ideal fluids, no requirements are imposed on the tangential components of velocity. This condition is consistent with the nature of the DARCY law which postulates that changes in the empirically equivalent velocity $\vec{Q}$ do not depend on the mechanisms of shear. Obviously, the governing equation (2) is not of sufficiently high order to accommodate conditions on the tangential component of velocity and a discontinuity of this component of velocity across the surface separating regions of different permeability is to be expected.

If the porous boundary separates a region of permeable material from a region where such material is absent it is necessary, if the governing differential system is not to be underdeterminate, to specify the behavior of the viscous liquid immediately outside the porous region.

In the case of a viscous flow the order of the equations governing the external motion is raised by two though the equations governing the internal motion are unchanged. It is therefore necessary to impose some requirement on the tangential component of velocity at the interface. When much of what would nominally be an impermeable surface is really hole entrance the appropriate boundary condition is not easy to anticipate. However, when the permeability is small relative to some typical gross area of the porous body, e. g., the square of the sphere radius, it would seem valid to retain the no slip condition relative to the wall. Since the permeability is slight in most natural materials this condition is not severely restrictive.

However, it has not, to the authors' knowledge, been established either empirically or analytically that the no slip condition applies at the boundary of the region of permeable material. It may even be argued that there is in fact some net tangential drag on the external fluid due to the tangential motion of the liquid within the permeable material. This question can only be resolved by experiment.

In the steady slow motion of a viscous fluid with constant material properties, the convective terms of the NAVIER-STOKES equations may be dropped, and the governing equations describing the conservation of mass and momentum become

$$\partial q_j / \partial x_j = 0\,, \tag{3}$$

$$\partial p / \partial x_j = \mu\, \partial^2 q_j / \partial x_i\, \partial x_i\,. \tag{4}$$

Differentiation of equation (4) with respect to x_j reveals that the pressure is a harmonic function:

$$\nabla^2 p = 0\,. \tag{5}$$

Collecting the conditions which we have established as consistent with the formalism, we require that:

1. The external field satisfies the momentum equation (4) and the continuity condition (3).
2. The internal field satisfies POISSON's equation (2) for the pressure.
3. The pressure and normal velocity component are continuous at the boundary.
4. The tangential velocity vanishes on the exterior of a porous wall.

The passage from one medium to another will be marked by discontinuities in the tangential component of velocity and in the derivatives of pressure and normal component of velocity. Zero permeability restores the ordinary boundary conditions corresponding to impermeable media, i. e., the vanishing of the velocity components.

Such is the structure of fluid problems consistent with the use of DARCY's law. The advantage of such a formalism is twofold. First, an attempt is made to satisfy physical laws internal as well as external to the porous media. Secondly, this formulation leads to a mathematically tractable system. We next consider the application of this formalism to a particular problem.

3. Permeable Sphere in a Uniform Stream

We consider a stationary sphere of radius as immersed in a uniform streaming viscous fluid. The sphere is porous and the within flow the sphere is governed by the generalization of DARCY's law. The body force per unit volume $\vec{F} = 0$ and the pressure within the sphere satisfies LAPLACE's equation.

The equations governing the external flow are identical to those governing the flow around an impermeable sphere. The boundary conditions on the sphere, however, no longer require that the normal component of velocity vanish.

The governing equations for the external flow are (3) and (4) expressed in spherical coordinates (r, θ, Φ) for the axially symmetric case.

The boundary conditions are:

1) The pressure[1]) is continuous at the boundary of the sphere

$$p(a_+, \theta) = P(a_-, \theta)\,,$$

2) The normal component of velocity is continuous at the boundary of the sphere

$$q_r(a_+, \theta) = Q_r(a_-, \theta)\,,$$

3) The fluid does not slip relative to the sphere

$$q_\theta(a_+, \theta) = 0\,,$$

4) Within the sphere, the velocity components are related to the pressure by DARCY's law

$$Q_\theta(a_-, \theta) = -\frac{k}{\mu\, a}\left(\frac{\partial P}{\partial \theta}\right)_a,$$

$$Q_r(a_-, \theta) = -\frac{k}{\mu}\left(\frac{\partial P}{\partial r}\right)_a,$$

5) The velocity at infinity is that of the free stream.

Velocity and Pressure Fields

It may be readily verified that the following functions which are obtained by classical techniques satisfy the requisite differential equations and boundary conditions. The solution of the problem is thus given by

$$p = -3\,\mu\, a\, U \cos\theta / r^2\,(2 + k/a^2)\,,$$

$$q_r = \left\{\frac{-3\,a\,U}{(2 + k/a^2)\,r}\left(1 - \frac{a^2}{3\,r^2}\left(1 + \frac{2\,k}{a^2}\right)\right) + U\right\}\cos\theta\,,$$

$$q_\theta = \left\{\frac{3\,a\,U}{(2 + k/a^2)\,2\,r}\left(1 + \frac{a^2}{3\,r^2}\left(1 + \frac{2\,k}{a^2}\right)\right) - U\right\}\sin\theta\,,$$

$$P = -\,3\,\mu\, U\, r\cos\theta/(2\,a^2 + k)\,,$$

$$Q_\theta = -\,3\,U\,k\sin\theta/(2\,a^2 + k)\,,$$

$$Q_r = 3\,U\,k\cos\theta/(2\,a^2 + k)\,.$$

Drag on the Sphere

The problem of finding the drag on a permeable body when the external and internal velocity and pressure fields are known is complicated by the fact that the actual solid boundaries are defined by an intricate maze of channels which lace through the body. It is prohibitively difficult to describe these boundaries analytically.

Fortunately, we can quite easily evaluate the drag on the porous body from considerations of a D'ALEMBERT equilibrium of forces on the fluid which saturates the porous medium. Thus we find that the drag D on the solid must balance the forces on the spherical shell of fluid surrounding the porous body. The components of this drag are

$$D_i = \int_s (-p\, n_i + \tau_{ij}\, n_j - \varrho\, q_i q_j\, n_j)\, ds\,.$$

The effect of the last term must be small in an approximation in which the effect of inertial forces is counted as negligible. In the case of the sphere this integral vanishes and one obtains

$$D_2 = D_3 = 0\,,$$

$$D = D_1 = -6\,\pi\, a\, U\,\mu\Big/\left(1 + \frac{k}{2\,a^2}\right).$$

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[1]) Upper case letters denote quantities evaluated within the permeable material. Lower case letters refer to regions free of permeable material.

Hence the effect of the permeability of the sphere on the drag is equivalent to a reduction in the radius of the sphere by a factor $(1 + k/2\,a^2)^{-1}$. For $k \to 0$ one obtains $D = -6\,\pi\,a\,U\,\mu$ which recovers the well-known result of STOKES relative to the drag on a slowly moving sphere in a viscous liquid.

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