Transverse Velocity Components in Fully Developed Unsteady Flows

D. D. JOSEPH[1] **and L. N. TAO**[2]

It is known that if an incompressible fluid is confined to straight pipe or channel, and if the axial velocity is steady and fully developed, then, under certain very general conditions, no transverse velocity components can exist.[3] This conclusion is not valid for unsteady flows, and it is the purpose of this note to develop the appropriate restrictions for the unsteady case.

By *fully developed* we mean that the velocity components are two-dimensional functions of the transverse coordinates. It is commonly assumed that when a fully developed flow is confined to a straight channel, then the transverse components of the velocity are zero, i.e., $u_i = (0, 0, u_3(x_1, x_2))$, where x_3 is along the axial direction of the channel. Indeed, if one stipulates in addition that the transverse components of the body force should have a potential, and that there should be no relative motion of the boundaries so that the no-slip condition will require that all velocity components vanish at the walls, then it can be rigorously demonstrated that in steady flows the transverse velocity components must vanish.[3] However, it does not follow that the transverse velocity components must vanish for motions which are fully developed and transient.

We note that the governing equations for a viscous incompressible fluid having constant material properties and being fully developed are

$$u_i = u_i(x_1, x_2, t)$$

$$u_{\alpha,\alpha} = 0 \qquad (\alpha = 1, 2) \tag{1}$$

$$\frac{\partial u_3}{\partial t} + u_\alpha u_{3,\alpha} = -\frac{1}{\rho}\frac{\partial p}{\partial x_3} + F_3 + \nu\nabla^2 u_3 \tag{2}$$

$$\frac{\partial u_\alpha}{\partial t} + u_\beta u_{\alpha,\beta} = P_{,\alpha} + \nu u_{\alpha,\beta\beta} \tag{3}$$

where p, ρ, F_3, P, and ν are the pressure, density, axial body force, combined pressure and transverse force potential, and kinematic viscosity, respectively.

We shall first demonstrate, by example, that equations (1) and (3) have nontrivial solutions which give transverse velocity components and satisfy that no-slip condition at the stationary walls bounding the cross section of the pipe. Let the pipe have a circular cross section and the pipe wall be located at $r = a$. Initially, let the pipe and fluid contained rotate about the pipe axis with angular velocity ω. The initial flow is fully developed. The radial velocity component is initially not present and it may be assumed from symmetry that it will not develop. Let the pipe be brought impulsively to a state of rest. Equations (1) and (3) become

$$\frac{\partial u_\theta}{\partial \theta} = 0$$

$$\frac{\partial u_\theta}{\partial t} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2}\right]$$

subject to the initial and boundary conditions

$$t = 0 \qquad u_\theta(r, 0) = \omega r$$

$$t > 0 \qquad u_\theta(a, t) = 0$$

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[1] Assistant Professor of Mechanical Engineering, Illinois Institute of Technology, Chicago, Illinois.

[2] Professor of Mechanics, Illinois Institute of Technology, Chicago, Illinois.

[3] S. H. Maslen, "Transverse Velocities in Fully Developed Flows," Quarterly of Applied Mathematics, vol. 16, 1958, pp. 173–175.

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The solution of this initial boundary-value problem is

$$u_\theta = 2\omega\Sigma e^{-k^2\nu t}J_1(kr)/k^3J_2(ka)$$

where the parameters k are the zeros of $J_1(ka)$.

Hence, unlike the steady case, the assumption of a fully developed velocity profile and the no-slip condition at stationary walls does not necessarily imply that $u_i(0, 0, u_3)$.

However, the foregoing assumptions do limit dramatically the possible types of transverse velocity fields. In particular, it is true that for fully developed transient flows in stationary pipes or channels, (*a*) the transverse velocity components must be monotonically decreasing function of the time; (*b*) if transverse velocity components are not present initially, they will not develop thereafter.

To prove these propositions, we introduce the stream function

$$u_\alpha = e_{\alpha\beta}\psi_{,\beta} \tag{4}$$

where $e_{\alpha\beta}$ are the components of the two-dimensional alternating tensor. The stream function, so defined, satisfies equation (1) identically. If one substitutes equation (4) into (3) and operates on the resulting equation with $e_{\alpha\delta}\partial/\partial x_\delta$, one finds that the stream function satisfies

$$\frac{\partial\psi_{,\alpha\alpha}}{\partial t} + e_{\alpha\beta}(\psi_{,\beta}\psi_{,\alpha\delta})_{,\delta} = \nu\psi_{,\alpha\alpha\beta\beta} \tag{5}$$

which is the two-dimensional equation governing the diffusion of the axial component of vorticity.

Integration of equation (5) over a channel cross-sectional area A bounded by a solid stationary curve C reveals that

$$\frac{\partial}{\partial t}\int_C \psi_{,\alpha}n_\alpha dC + \int_C e_{\alpha\beta}n_\delta\psi_{,\alpha\delta}\psi_{,\beta}dC = \nu\int_A \psi_{,\alpha\alpha\beta\beta}dA$$

where Green's theorem has been used to effect the integration. Since the no slip condition $\psi_{,\alpha} = 0$ on C, one concludes that

$$\int_A \psi_{,\alpha\alpha\beta\beta}dA = 0 \tag{6}$$

The integration of equation (5) multiplied by ψ over the cross section A shows, after some manipulations, that

$$\frac{\partial}{\partial t}\left[\psi(C)\int_C \psi_{,\alpha}n_\alpha dC - \int_A[(u_1^2 + u_2^2)/2]dA\right] + \frac{1}{2}e_{\alpha\beta}\int_C[(\psi^2)_{,\beta}\psi_{,\alpha\delta}n_\delta + \psi_{,\beta}(\psi_{,\delta})^2n_\alpha]dC$$
$$= \nu\left[\psi(C)\int_A \psi_{,\alpha\alpha\beta\beta}dA - \int_C \psi_{,\beta}\psi_{,\alpha\alpha}n_\beta dC + \int_A(\psi_{,\alpha\alpha})^2dA\right]$$

which in view of equation (6) may be written as

$$-\frac{\partial}{\partial t}\int_A (u_1^2 + u_2^2)dA = 2\nu\int_A(\psi_{,\alpha\alpha})^2dA \tag{7}$$

Since the integral on the right is positive definite,

$$\frac{\partial}{\partial t}\int_A (q^2/2)dA \leqslant 0 \tag{8}$$

where $q^2 = u_1^2 + u_2^2$.

Furthermore, the equality sign holds only when $q \equiv 0$. Let us suppose consistent with equation (8) that it is possible for the kinetic energy of the transverse motion to be constant in time

$$\int_A (q^2/2)dA = \text{const} > 0$$

It follows from equation (7) that $\psi_{,\alpha\alpha} = 0$ everywhere. Since, on the boundary, $\partial\psi/\partial n = \partial\psi/\partial s = 0$ on C, ψ can at most be a constant at every point of A. This, however, necessarily implies that $q^2 = 0$ at every point of A. Consequently our original supposition was mistaken and the total kinetic energy of the transverse motion either (*a*) decreases in time or (*b*) vanishes.

Since this energy can neither increase nor assume a constant value other than zero, we may conclude that $\int_A (q^2/2)dA$ decreases monotonously to the null limit. This can be true if and only if q^2 itself, that is, u_1^2 and u_2^2, tend monotonously to zero.

Hence transverse velocity components can never develop in a fully developed unsteady flow in which they are initially absent. If these components are initially present they must decay in time.