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L. N. TAO
Professor,
Department of Mechanics,
Illinois Institute of Technology,
Chicago, Ill.
Assoc. Mem. ASME

D. D. JOSEPH
Health, Education and Welfare Fellow,
Illinois Institute of Technology,
Chicago, Ill.

Fluid Flow Between Porous Rollers

The problem of fluid flow between two porous rollers with a small gap is investigated. Solutions for both large and small values of the porosity of the minimum gap distance are derived. It is found that increasing porosity will decrease the maximum suction and shift its position away from the origin.

In recent years many phenomena have been studied which are directly concerned with fluid flow through porous channels. Examples of these are boundary-layer control, gaseous diffusion, transpiration cooling, and so on. These problems are essentially concerned with the sucking away of boundary layers or the injection of fluid into them through a porous surface. They are commonly studied based on the assumption that the rate of flow through the pores is known. Mathematically, this means that problems of this sort are equivalent to those of impermeable surfaces with the exception that the velocity boundary condition at the walls is not restricted to vanishing normal velocity components.

However, in some problems of fluid flow involving porous media, the mathematical description of physical phenomena will not be identical to the corresponding impermeable case. When a viscous fluid is entrained between two rotating cylinders or a moving surface and a cylinder which are separated by a small gap, a pressure gradient is developed in the gap. When the cylinders are impervious, the solution is known [1].[1] The pressure distribution is antisymmetric. There exists a region of negative pressure or suction.

From the physical point of view, a suction or negative pressure of large magnitude cannot exist, and this mathematical solution thus indicates possibilities of the existence of some other physical phenomena in opposition to the basic assumptions. Banks and Mill [2] have demonstrated that, depending upon the physical properties of the fluid, there is a maximum suction which can be maintained. Beyond this value the fluid, in general, cavitates.

Recently, G. I. Taylor and J. C. P. Miller [3] suggest another possibility. They suggest that there may be rollers which are not perfectly impermeable. If the rollers are considered to be slightly porous, and if the geometrical configuration is such that suction would be produced by impermeable cylinders, fluid would be sucked through the surfaces, and this might prevent the suction from rising to such a level that cavitation would occur. As they have indicated, a study of such a problem would be of particular interest to find out how porous a cylinder must be if cavitation is to be avoided. These results would be very useful to such industrial processes as those of paper-making machines and of wall paint rollers. Based on such reasoning, Taylor and Miller [3] have investigated a problem of a simpler nature. They assume that the two cylinders are in direct contact and obtain a solution in terms of Bessel and associated functions.

[1] Numbers in brackets designate References at end of paper.

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The present paper intends to investigate the problem of fluid flow between two rotating porous cylinders with a finite gap. In a similar approach to that of Taylor and Miller, the problem is formulated in the next section. We then find the solution is expressible in terms of spheroidal wave functions; it is, however, cumbersome in numerical applications. An entirely different series solution is then constructed which is convenient for numerical computation, particularly for small values of the porosity. Moreover, we intend to compare the solution to that of zero gap given by Taylor and Miller. A solution for very small gap which approaches the zero-gap solution as the limit is derived. Finally, some numerical illustrations and discussion are given in the last section.

Mathematical Formulation

We consider two porous rollers of equal radii R located with a small gap $2h_0$ and rotating at constant speed $\omega = U/R$ as shown in Fig. 1. It is known that the pressure at locations away from the minimum gap is nearly atmospheric. Our main concern is, therefore, the neighborhood of the minimum gap. Since the gap is small compared to the radii of the rollers, the Reynolds number of the flow is small, and the motion may indeed be described using the approach of the theory of hydrodynamic lubrication. Then the governing equation is

$$\frac{d}{dx}\left\{\frac{h^3}{3\mu}\frac{dp}{dx}\right\} - U\frac{dh}{dx} + W = 0 \tag{1}$$

where p is the pressure, and μ is the viscosity of the fluid, $2h$ is the local gap, and W the through-flow at the surface of the rollers. Following Taylor and Miller [3] we may relate W and the pressure in the gap by

$$W = -kp/\mu \tag{2}$$

where k is the porosity of the material. Hence the governing equation becomes

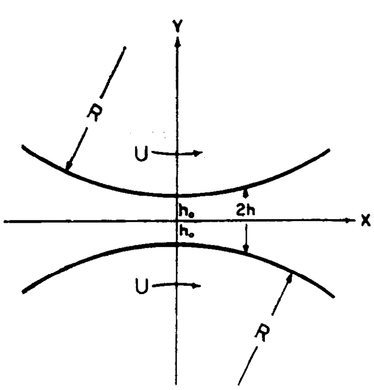

Fig. 1 Geometrical configuration

$$\frac{d}{dx}\left\{\frac{h^3}{3}\frac{dp}{dx}\right\} - \mu U \frac{dh}{dx} - kp = 0 \quad (3)$$

From geometry, Fig. 1, we have $(h - R - h_0)^2 = R^2 - x^2$. In our problem the main concern, as mentioned, is the region near the minimum gap. For large R and small h_0 the local gap may be expressed by

$$h = (x^2 + 2Rh_0)/2R \quad (4)$$

Then equation (3) becomes

$$\frac{d}{dx}\left\{(x^2 + 2Rh_0)^3 \frac{dp}{dx}\right\} - 24R^3kp - 24\mu UR^2x = 0 \quad (5)$$

Now, introducing

$$F(\zeta) = -\frac{(2Rh_0)^{3/2}}{24\mu UR^2} p(x)$$
$$\zeta = x/(2Rh_0)^{1/2} \qquad \gamma^2 = 6kR/h_0^2 \quad (6)$$

Equation (5) becomes

$$\frac{d}{d\zeta}\left[(1 + \zeta^2)^3 \frac{dF}{d\zeta}\right] - \gamma^2 F = -\zeta \quad (7)$$

Since the pressure must be atmospheric at infinity, we take that pressure as our reference and require that

$$F(\infty) = F(-\infty) = 0 \quad (8a)$$

Because of the symmetry of the geometry the effect of reversing the direction of the rotation of the rollers is to change the sign of the pressure. This implies that $F(\zeta) = -F(-\zeta)$. Hence another condition to be imposed on the solution is that

$$F(0) = 0 \quad (8b)$$

Since the pressure distribution is antisymmetric, we may confine our attention to the interval $(0 \leqslant \zeta < \infty)$.

Spheroidal Wave Equation

By the substitution of

$$\zeta = z/(1 - z^2)^{1/2}, \quad F(\zeta) = (1 - z^2)^{5/4}\phi(z) \quad (9)$$

equation (7) is transformed to

$$(1 - z^2)\phi'' - 2z\phi' + \left[\frac{15}{4} - \gamma^2(1 - z^2) - \frac{25/4}{1 - z^2}\right]\phi = -\frac{z}{(1 + z^2)^{3/4}} \quad (10)$$

The homogeneous part of this equation is a spheroidal wave equation [4, 5]:

$$(1 - z^2)\phi'' - 2z\phi' + \left[\lambda + \rho^2(1 - z^2) - \frac{\mu^2}{1 - z^2}\right]\phi = 0 \quad (11)$$

The solution of the homogeneous part of equation (10) may then be written in terms of spheroidal wave functions of first and second kinds

$$APs_\nu^\mu(z,\rho) + BQs_\nu^\mu(z,\rho) \quad (12)$$

where $\mu(=5/2)$ and $\rho(= i\gamma)$ are parameters of the functions. The parameter λ $(=15/4)$ and the "characteristic exponent" ν are related transcendentally by compatibility conditions to the other parameters. In our problem λ is a constant and ν must be determined as an implicit function of λ and the other parameters [5]. Furthermore, by means of the Wronskian of spheroidal wave functions

$$W[Ps_\nu^\mu(z, \rho), Qs_\nu^\mu(z, \rho)] = c/(1 - z^2) \quad \mathbf{(13)}$$

where c is a constant, the particular integral is

$$\phi_p = c\int^z \omega(1 - \omega^2)^{1/4}[Ps_\nu^\mu(z, \rho)Qs_\nu^\mu(\omega, \rho) - Ps_\nu^\mu(\omega, \rho)Qs_\nu^\mu(z, \rho)]d\omega \quad (14)$$

Therefore the complete solution of equation (10) is

$$\phi = APs_\nu^\mu(z,\rho) + BQs_\nu^\mu(z,\rho) + \phi_p \quad (15)$$

The spheroidal wave equation has three singular points; namely, two regular points at 0 and 1, and one irregular point at ∞. Its fundamental solutions, spheroidal wave functions, are directly obtainable in power-series form [6]. However, owing to the complicated behavior of the recurrence formulas, the spheroidal wave functions are usually expressed in terms of some other special functions or polynomials, particularly Bessel or associated Legendre functions. These series then require the determination of the characteristic values ν from the infinite continued fraction formulated from the recurrence relationship. Although for real integer μ and ν these functions have been well investigated, the case of fractional μ and ν has only been scarcely explored. This greatly handicaps the application of these functions to our present problem. We intend in the following text to establish a series solution which is valid for a wide range of the parameters and more convenient for numerical computation.

Series Solution

Let us express the solution of equation (7) in the form of

$$F(\zeta) = \sum_0 F_n(\zeta)\gamma^{2n} \quad (16)$$

Substituting this series into equation (7) and equating the coefficients of equal powers of γ, we obtain

$$\frac{d}{d\zeta}\left[(1 + \zeta^2)^3 \frac{dF_0}{d\zeta}\right] + \zeta = 0 \quad (17)$$

and

$$\frac{d}{d\zeta}\left[(1 + \zeta^2)^3 \frac{dF_n}{d\zeta}\right] - F_{n-1}(\zeta) = 0 \quad (18)$$

The leading term $F_0(\zeta)$ satisfies the same equation as that of impermeable rollers $(\gamma = 0)$. Its solution is readily determined by direct integrations

$$F_0(\zeta) = -\int_{C_2}^{\zeta} (1 + \omega^2)^{-3}d\omega \int_{C_1}^{\omega} \eta d\eta \quad (19)$$

Using boundary conditions $F_0(0) = F_0(\infty) = 0$, we then find

$$C_1 = 1/3, \qquad C_2 = 0$$

or

$$F_0(\zeta) = -\int_0^{\zeta} (1 + \omega^2)^{-3}\, d\omega \int_{1/3}^{\omega} \eta d\eta = \frac{1}{6}\zeta(1 + \zeta^2)^{-2} \quad (20)$$

The other terms $F_n(\zeta)$ are obtained by repeated integrations of equation (18)

$$F_n(\zeta) = \int_{B_2}^{\zeta} \frac{d\omega}{(1 + \omega^2)^3} \int_{B_1}^{\omega} F_{n-1}(\eta)\, d\eta \quad (21)$$

Since the integration limits B_1 and B_2 are at our disposal, they may be chosen to satisfy the homogeneous boundary conditions at $\zeta = 0$ and $\zeta = \infty$. The solution constructed in this way clearly satisfies the given boundary conditions. Adopting this point of view, we then find that $F_n(\zeta)$ are of the form

$$F_n(\zeta) = (1 + \zeta^2)^{-n-2} \sum_{r=1}^{n} \alpha_r^n \zeta^{2r-1} \tag{22}$$

with

$$\begin{aligned} \alpha_1^{n-1} &= 6[\alpha_2^n - (1 + n)\alpha_1^n] \\ \alpha_r^{n-1} &= 4(r - n - \tfrac{7}{2})(r - n - 1)\alpha_{r-1}^n \\ &\quad + [8r(r - n) - 2(8r - n - 1)]\alpha_r^n + 2r(2r + 1)\alpha_{r+1}^n \\ \alpha_{n-1}^n &= (n - \tfrac{1}{7})\alpha_n^n \end{aligned} \tag{23}$$

The convergence of this series solution is shown in the Appendix. For small values of γ this series converges rapidly. Only a few $F_n(\zeta)$ are needed to give the necessary accuracy. Here we list the first few $F_n(\zeta)$

$$F_0(\zeta) = \frac{1}{6}\zeta(1 + \zeta^2)^{-2} \tag{24}$$

$$F_1(\zeta) = -\frac{1}{72}\zeta(1 + \zeta^2)^{-3} \tag{25}$$

$$F_2(\zeta) = \frac{\zeta}{72(192)} \frac{7\zeta^2 + 13}{(1 + \zeta^2)^4} \tag{26}$$

$$F_3(\zeta) = -\frac{53}{48(72)(192)} \frac{(7/20)\zeta^4 + \zeta^2 + (157/212)}{(1 + \zeta^2)^5} \tag{27}$$

$$F_4(\zeta) = \frac{53\zeta}{48(72)(192)(10240)} \times \left[\frac{5010}{29}\zeta^6 + 690\zeta^4 + 1013\zeta^2 + 460.8\right] \Big/ (1 + \zeta^2)^6 \tag{28}$$

Solution for Very Small Gap

For large values of γ, the validity of the solution given in the preceding section may be in doubt. We intend to give here a solution which is valid for large values of γ and approaches in the limit the solution given by Taylor and Miller.

Before we proceed to the mathematical solution, we will discuss the physical meaning of the parameter $\gamma^2 = 6kR/h_0^2$. Term γ^2 is controlled by the ratio of porosity k to the square of minimum gap-distance h_0. As stated previously

$$|h/h_0 - 1| \ll 2R/h_0 \tag{29}$$

and so h_0 cannot be allowed to tend to infinity independent of R. As a consequence, vanishing γ can only be the case of zero porosity; i.e., with impermeable rollers, and small γ corresponds to those rollers which are not very porous. On the other hand, h_0 may approach zero value independent of other parameters, which represents physically the case of two rollers in contact. Though the porosity may be large (this is not typical), the continuity equation rules out the possibility that the medium is completely porous. This implies a trivial solution $p = 0$ everywhere. Consequently, infinite values of γ mean a direct contact of the rollers and large values of γ imply very small gap distances between the rollers.

To find the solution for large values of γ, we first write

$$F(\zeta) = F_0(\zeta) + G(\zeta) \tag{30}$$

where $F_0(\zeta)$ is the impermeable solution. This means that $G(\zeta)$ represents the porous effect and is governed by

$$\frac{d}{d\zeta}\left\{(1 + \zeta^2)^3 \frac{dG}{d\zeta}\right\} - \gamma^2 G = \frac{\gamma^2\zeta}{6(1 + \zeta^2)^3} \tag{31}$$

To facilitate the necessary simplifications, we now introduce

$$G(\zeta) = \left\{\frac{\xi}{\gamma}\left(2 - \frac{\xi}{\gamma}\right)\right\}^{3/4} H(\xi) \tag{32}$$

$$\zeta = \left(1 - \frac{\xi}{\gamma}\right)\left\{\frac{\xi}{\gamma}\left(2 - \frac{\xi}{\gamma}\right)\right\}^{-1/2} \quad (0 \leq \xi \leq \gamma) \tag{33}$$

Equation (31) is then transf rmed to

$$\frac{d^2H}{d\xi^2} - \left\{1 + \frac{21/16}{\xi^2} - \frac{1}{16\gamma^2} \times \left[\frac{9}{\xi/\gamma} + \frac{9}{2 - \xi/\gamma} + \frac{21}{(2 - \xi/\gamma)^2}\right]\right\} H = \tfrac{1}{6}(\xi/\gamma)^{3/4}(1 - \xi/\gamma)(2 - \xi/\gamma)^{3/4} \tag{34}$$

It is seen that for large γ the terms in the square bracket times $(16\gamma^2)^{-1}$ are small compared to other terms of the coefficient of H. It may therefore be approximated by

$$\frac{d^2H}{d\xi^2} - \left(1 + \frac{21/16}{\xi^2}\right) H = \tfrac{1}{6}(\xi/\gamma)^{3/4}(1 - \xi/\gamma)(2 - \xi/\gamma)^{3/4} \tag{35}$$

One further substitution

$$H(\xi) = \left\{\frac{1}{6}\left(\frac{2}{\gamma}\right)^{3/4}\right\} \xi^{1/2} V(\xi) \tag{36}$$

transforms equation (35) into

$$\xi^2 V'' + \xi V' - \left(\frac{25}{16} + \xi^2\right) V = \xi^{9/4}\left(1 - \frac{\xi}{\gamma}\right)\left(1 - \frac{\xi}{2\gamma}\right)^{3/4} \tag{37}$$

The homogeneous part of this equation is a modified Bessel equation. Therefore the complementary solution is

$$AI_{5/4}(\xi) + BI_{-5/4}(\xi) \tag{38}$$

To find the particular integral of equation (37), we expand the right-hand side of equation (37) by the binomial theorem. Equation (37) is

$$\xi^2 V'' + \xi V' - \left(\frac{25}{16} + \xi^2\right) V = \gamma^{5/4}(\gamma - \xi) \times \sum_{m=0} \frac{\Gamma(7/4)}{m!2^m\,\Gamma(7/4 - m)} \left(\frac{\xi}{\gamma}\right)^{9/4+m} \tag{39}$$

Although the particular integral of this equation may be expressed as an infinite series of "modified Lommel functions," the numerical computation is again not simple since these functions have not been tabulated. Now we take the particular solution of the form

$$V_p = cL_{5/4}(\xi) + \sum_{m=0} a_m \left(\frac{\xi}{\gamma}\right)^{9/4+m} \tag{40}$$

where $L_\nu(\xi)$ is the modified Struve function [7] which is defined by

$$L_\nu(\xi) = \sum_0 \frac{(\xi/2)^{\nu+2m+1}}{\Gamma(m + 3/2)\Gamma(m + \nu + 3/2)} \tag{41}$$

Substitution of equation (40) into equation (39) reveals that

$$c = 2^{1/4}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{7}{4}) \tag{42}$$

and

$$a_1 = -\frac{7\gamma^{1/4}}{2[117/4 + 13/\gamma - 25/4\gamma^2]} \quad (43)$$

$$a_2 = -\frac{59\gamma^{1/4}}{16[221/4 + 17/\gamma - 25/4\gamma^2]} \quad (44)$$

$$a_m\,[\gamma^2(9/4 + m)(5/4 + m) + \gamma(5/4 + m) - 25/16] = \gamma^2\left\{\frac{\Gamma(7/4)\gamma^{1/4}}{2^{m-1}(m-1)!\Gamma(3/4 - m)} \times \left(\frac{1}{2m(3/4 - m)} - 1\right) - a_{m-2}\right\} \quad (45)$$

From the recurrence formula we may observe that

$$\frac{a_m}{a_{m-2}} = \left[\frac{1}{2^m} - 1\right] O\left(\frac{1}{m^2}\right) \quad (46)$$

This shows that the series

$$\sum_0 a_m(\xi/\gamma)^{9/4+m} \quad (47)$$

is convergent for all values of ξ in the interval $(0 \leq \xi \leq \gamma)$. Therefore the complete solution of equation (39) is

$$V(\xi) = AI_{5/4}(\xi) + BI_{-5/4}(\xi) + V_p \quad (48)$$

Accordingly

$$G(\zeta) = \frac{2^{3/2}}{6\gamma^{1/4}}\left\{\left(\frac{\xi}{\gamma}\right)^{5/3} - \frac{1}{2}\left(\frac{\xi}{\gamma}\right)^{8/3}\right\}^{3/4} V(\xi) \quad (49)$$

Since pressure vanishes at $x = \infty$ or $V(0) = 0$, this requires $B = 0$. Using the other boundary condition $p(0) = 0$ or $V(\gamma) = 0$, we find

$$A = -\left\{2^{1/4}\Gamma(\tfrac{1}{2})\Gamma(7/4)L_{5/4}(\xi) + \sum_{m=1} a_m\right\}\Big/ I_{5/4}(\gamma) \quad (50)$$

In summarizing, the complete solution valid for all values of ξ in the interval $(0 \leq \xi \leq \gamma)$ is

$$V(\xi) = AI_{-5/4}(\xi) + 2^{1/4}\Gamma(\tfrac{1}{2})\Gamma(7/4)L_{5/4}(\xi) + \sum_{m=1} a_m(\xi/\gamma)^{9/4+m} \quad (51)$$

We now consider the behavior of this solution as γ approaches infinity. Combining equations (6), (20), (30), and (49), we find

$$p = -p_0 + C\gamma^{5/4}\{(\xi/\gamma)^{5/3} - (\xi/\gamma)^{8/3}/2\}^{3/4}\, V(\xi) \quad (52)$$

where

$$p_0 = \frac{4\mu UR^2 x}{(x^2 + 2Rh_0)^2} \quad (53)$$

$$C = \frac{-4\mu UR^2}{(6kR^3)^{3/4}} \quad (54)$$

Furthermore, a combination of substitutions on the independent variables reveals that

$$\xi = \frac{2(6kR^3)^{1/2}}{x(x^2 + 2Rh_0)^{1/2} + (x^2 + 2Rh_0)} \quad (55)$$

In the limit as $\gamma \to \infty$ ($h_0 \to 0$) we have from equations (52)–(55) that

$$p = -p_0 + C\xi^{5/4}V(\xi) \quad (56)$$

and

$$V(\xi) = 2^{1/4}\Gamma(\tfrac{1}{2})\Gamma(7/4)\left[L_{5/4}(\xi) - \frac{L_{5/4}(\gamma)}{I_{5/4}(\gamma)} I_{5/4}(\xi)\right] \quad (57)$$

In this solution there is one point which needs special attention. As pointed out by Taylor and Miller, both $I_{5/4}(\xi)$ and $L_{5/4}(\xi)$ tend to infinity as $\xi \to \infty$ but the difference between them is of a lower order of magnitude than either separately. From their integral representations or series expansions we may observe that

$$\lim_{z \to \infty} \frac{L_{5/4}(z)}{I_{5/4}(z)} \to 1 \quad (58)$$

The pressure distribution for $h_0 \to 0$ or $\gamma \to \infty$ is then

$$p = C\xi^{5/4}[2^{1/4}\Gamma(1/2)\Gamma(7/4)\{L_{5/4}(\xi) - I_{5/4}(\xi)\} + \xi^{1/4}] \quad (59)$$

This is the same result given by Taylor and Miller as we expected.

Discussion

In previous sections we have obtained solutions for both small and large values of the parameter γ. The resulting pressure distributions in dimensionless forms for various values of γ are plotted in Figs. 2 and 3.

The effect of the variation of porosity at a fixed gap distance is depicted in Fig. 2 while Fig. 3 illustrates the effect of various gap distances at a fixed value of porosity. For direct comparison the solution valid for impermeable rollers ($\gamma = 0$) is to be found in Fig. 2 while that valid for zero gap is to be found in Fig. 3.

From these graphs we observe that an increase of porosity will tend to reduce the maximum suction and to shift the station of maximum suction away from the origin. A similar result may be produced by increasing the gap size. In a sense increasing

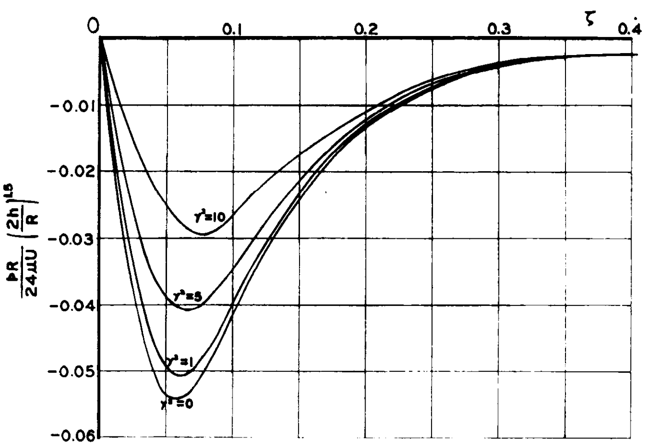

Fig. 2 Pressure distribution with variable porosity and fixed gap

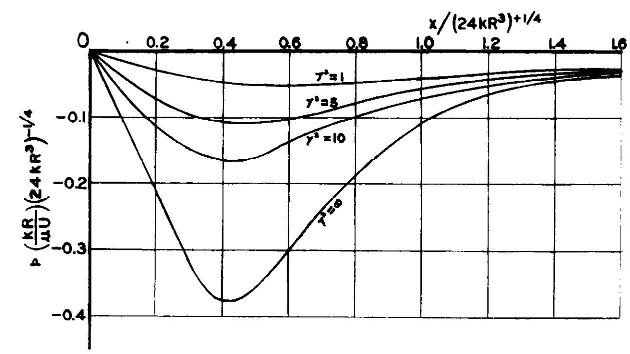

Fig. 3 Pressure distribution with variable minimum gap and fixed porosity

the porosity has the same physical effect in reducing the maximum suction as increasing the gap distance.

The derived results are also applicable to the case of a porous roller and an impermeable flat plate. As shown in [3], owing to the property of symmetry a direct substitution of some new variables readily gives the required solution.

Acknowledgment

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APPENDIX

Convergency of Series Solution

To establish the interval of convergence of the series, equation (16), we substitute equation (22) into equation (18) and find

$$F_{n-1} = 2\left[(1+\zeta^2)(2n^2+5n) + \frac{(1+\zeta^2)^2}{\zeta^2}\right]F_n + \frac{2(2n^2+5n)\zeta^{2n+1}\alpha_n{}^n}{(\zeta^2+1)^{n+1}} + \sum_{r=1}^{n}\frac{2r\xi^{2r-3}\alpha_r{}^n}{(1+\zeta^2)^{n+1}} \times [(2r-3)+4(r-n-2)\zeta^2+(2r-4n-5)\zeta^4] \quad (60)$$

Considering ζ as a parameter (excluding $\zeta = 0$) we then have symbolically

$$F_{n-1} = f_nF_n + g_n$$
and
$$F_n = f_{n+1}F_{n+1} + g_{n+1} \quad (61)$$

The elimination of the inhomogeneous parts leads to a three-term homogeneous recurrence relationship

$$L_nF_{n-1} = M_nF_n - N_nF_{n+1} \quad (62)$$

where

$$L_n = g_{n+1}$$
$$M_n = f_ng_{n+1} + g_n \quad (63)$$
$$N_n = f_n + 1g_n$$

Introducing $R_n = F_n/F_{n-1}$, equation (62) becomes

$$R_n = \frac{L_n}{M_n - N_nR_{n+1}} \quad (64)$$

By repeated substitutions equation (64) transforms to an infinite continued fraction

$$R_n = \frac{L_n}{|M_n} - \frac{L_{n+1}N_n|}{|M_{n+1}} - \frac{L_{n+2}N_{n+1}|}{|M_{n+2}} - \ldots\ldots. \quad (65)$$

With L, M, and N as the respective limits of L_n, M_n, N_n, the continued fraction is convergent [8] if the roots (ρ_1, ρ_2) of

$$N\rho^2 - M\rho + L = 0 \quad (66)$$

are either $|\rho_1| \neq |\rho_2|$ or $\rho_1 = \rho_2$, divergent if $\rho_1 \neq \rho_2$ but $|\rho_1| = |\rho_2|$. Moreover, the convergent case has

$$\lim_{n\to\infty} R_n \to \rho_1 \text{ or } \rho_2$$

depending on the parameters of the fraction. In our case, with f and g as the respective limit of f_n and g_n, then

$$L = g$$
$$M = g(f+1) \quad (67)$$
$$N = fg$$

The roots of equation (66) are $1/f$ and 1; viz., 0 and 1. This indicates series (16) is either an entire function of γ or convergent for $|\gamma| < 1$.

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