Valve Dynamics

The actual geometric shape of MacClatchie valve and seat assembly used commonly in Dowell frac pumps is quite complicated. A simplified the model is presented in Figure 1 to make the valve / seat geometry more amenable to parametric variation for dynamic analysis. The valve is represented as a circular disc with a beveled edge of the same angle α. The valve seat is treated as a hollow cylinder with a beveled surface to accommodate the valve.

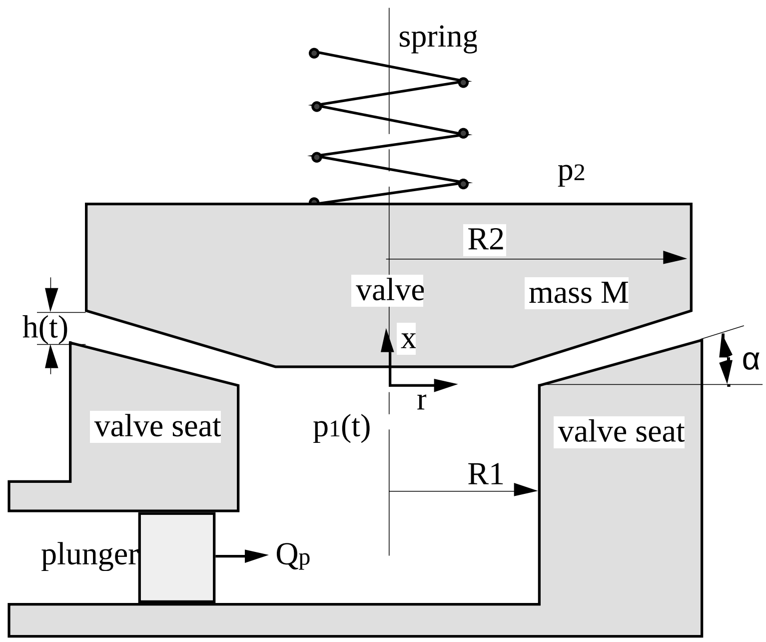

Figure 1. Geometrical Parameters of the Simplified Valve / Seat System.

The key to the model is to find the flow force acting on the valve. The flow force will be determined based on the dynamics of the flow in the gap between the valve strike face and its seat. There are two contributions to this flow force. One is from the viscous fluid motion through the gap where metal strike face and the elastomeric insert act as a step bearing. The other is from the squeeze (or suction) motion of the valve as the valve is closing onto (or pulling away from) its seat. These two contributions to the flow force will be always crucial for the valve dynamics, since the valve spends an important part of the pumping cycle when the gap between the valve strike face and its seat is small, even though the final valve lift may be high.

The calculation of this flow force is based on the lubrication theory using the geometry shown in figure 2. We think that the beveled surfaces on the valve and its seat only create a centering force for the valve and do not contribute to the flow force in the direction of the valve motion. Using the cylindrical coordinate system (r, x) fixed on the valve seat, as shown in figure 2, we can write the equations for the fluid motion as (based on the lubrication theory),

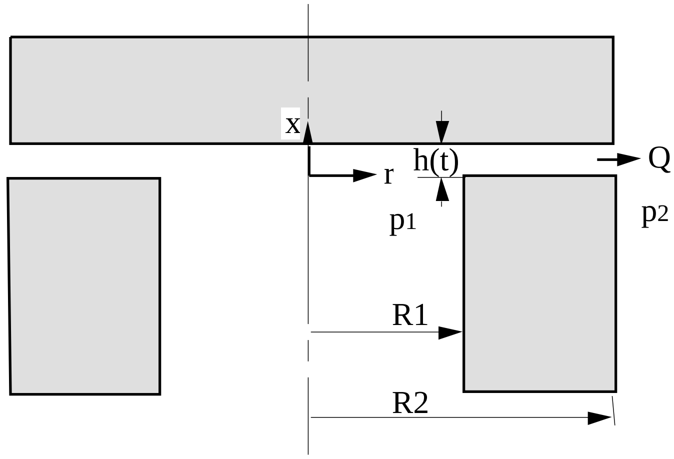

Figure 2. Geometry for calculating the flow force.

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial}{\partial x}(u_x)=0 \tag{1}$$

$$0=-\frac{\partial p}{\partial r}+\mu\frac{\partial u_r}{\partial x^2} \tag{2}$$

and

$$0=-\frac{\partial p}{\partial x} \tag{3}$$

where p is the fluid pressure, (u_r,u_x) is the components of the fluid velocity, and μ is the fluid viscosity.

Equation (3) indicates that the pressure does not change with x (across the gap between the valve and its seat). Thus we can integrate the equation (2) over x to get the velocity u_r,

$$u_r=\frac{1}{2\mu}\frac{dp}{dr}x(x-h) \tag{4}$$

where $h=h(t)$ is a function of time, and the no-slip boundary conditions, $u_r(r,0)=u_r(r,h)=0$, are used.

Next we integrate equation (1) across the gap *x*,

$$\int_0^h\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right]2\pi r dx+\int_0^h\left[\frac{\partial}{\partial x}(u_x)\right]2\pi r dx=0 \tag{5}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\int_0^h u_r\,dx\right)+\left[u_x(r,h)-u_x(r,0)\right]=0 \tag{6}$$

Substituting (4) into (6) and noticing that $u_x(r,h) = \dot{h}$ and $u_x(r,0) = 0$, we have

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right) = 12\mu\frac{\dot{h}}{h^3} \tag{7}$$

Since *h* is not a function of *r*, integration of (7) gives,

$$p = 3\mu\frac{\dot{h}}{h^3}r^2 + c\ln r + d \tag{8}$$

where the integration constants *c* and *d* are determined by the boundary conditions for the pressure, $p(r = R_1) = p_1$ and $p(r = R_2) = p_2$, which leads to the final expression for the pressure,

$$p = p_2 + (p_1 - p_2)\frac{\ln(r/R_2)}{\ln(R_1/R_2)} - 3\mu\frac{\dot{h}}{h^3}\left(R_2^2 - r^2 - \frac{\ln(r/R_2)}{\ln(R_1/R_2)}\left(R_2^2 - R_1^2\right)\right) \tag{9}$$

The total flow force acting on both side of the valve surface valve resulted from the pressure distribution is calculated as,

$$F_{flow} = (p_1 - p_2)\pi R_1^2 + \int_{R_1}^{R_2}\left[p(r) - p_2\right]2\pi r dr \tag{10}$$

$$\int_{R_1}^{R_2}(p - p_2)rdr = -\frac{3}{4}\mu\frac{\dot{h}}{h^3}\left(R_2^2 - R_1^2\right)^2$$

where

$$+\left[\frac{p_1 - p_2}{\ln(R_1/R_2)} - 3\mu\frac{\dot{h}}{h^3}\frac{\left(R_2^2 - R_1^2\right)}{\ln(R_2/R_1)}\right]\frac{1}{2}\left(-\frac{\left(R_2^2 - R_1^2\right)}{2} + R_1^2\ln(R_2/R_1)\right)$$

$$= \frac{p_1 - p_2}{2}\left(\frac{\left(R_2^2 - R_1^2\right)}{2\ln(R_2/R_1)} - R_1^2\right) - \frac{3}{4}\mu\frac{\dot{h}}{h^3}\left[\left(R_2^4 - R_1^4\right) - \frac{\left(R_2^2 - R_1^2\right)^2}{\ln(R_2/R_1)}\right]$$

Thus,

$$F_{flow} = \frac{\pi}{2}\frac{\left(R_2^2 - R_1^2\right)}{\ln(R_2/R_1)}(p_1 - p_2) - \frac{3\pi}{2}\mu\frac{\dot{h}}{h^3}\left[\left(R_2^4 - R_1^4\right) - \frac{\left(R_2^2 - R_1^2\right)^2}{\ln(R_2/R_1)}\right] \tag{11}$$

Note that in the limit of $R_1 = 0$, the integration constant *c* in (8) should be zero, and the total flow force is

$$F_{flow}\big|_{R_1 = 0} = -\frac{3\pi}{2}\mu\frac{\dot{h}}{h^3}R_2^4 \tag{12}$$

in agreement with the solution for the force on a circular disk being pulled away from a rigid plan, as listed in Bachelor [19??, page??].

In addition, the flow rate at the exit of the gap between the valve and its seat, (at $r = R_2$), is calculated as

$$Q = 2\pi R_2 \int_0^h u_r\left(R_2, x\right) dx = \frac{\pi R_2}{6\mu} h^3 \left(-\frac{dp}{dr}\right)_{r=R_2}$$

$$= \frac{\pi R_2}{6\mu} h^3 \left(-6\mu \frac{\dot{h}}{h^3} R_2 - \frac{\left(p_1 - p_2\right) + 3\mu \frac{\dot{h}}{h^3}\left(R_2^2 - R_1^2\right)}{R_2 \ln\left(R_1 / R_2\right)} \right) \tag{13}$$

$$= \pi\left(-R_2^2 + \frac{R_2^2 - R_1^2}{2\ln\left(R_2/R_1\right)}\right)\dot{h} + \frac{\pi}{6\mu}\frac{p_1 - p_2}{\ln\left(R_2/R_1\right)} h^3$$

The first term in (13) represents the change of the fluid volume due to the valve movement; and the second term accounts for the flow rate driven by the pressure difference along the gap. Again, in the limit of $R_1 = 0$, $Q\big|_{R_1=0} = -\pi R_2^2 \dot{h}$, which is the additional space created by the movement of the disk away from the surface.

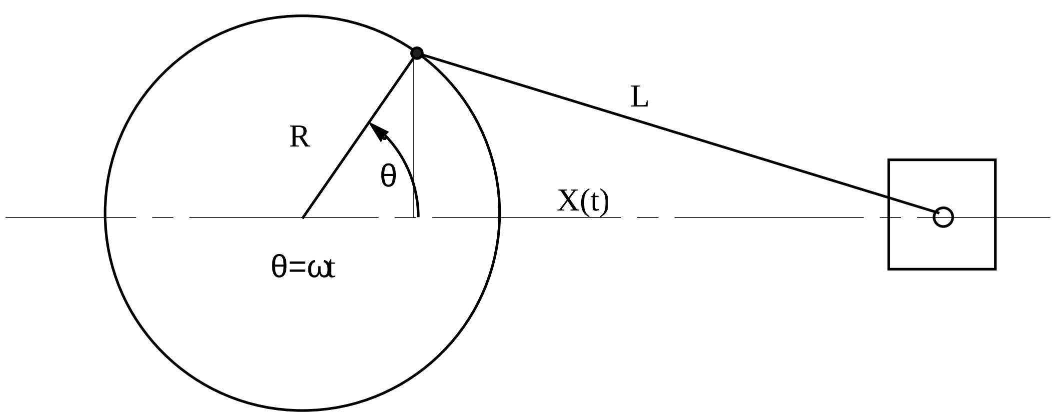

Figure 3 - Slider Crank Geometry.

In the frac pump, the dynamic behavior of the valve is driven by the action of the plunger due to its slider crank drive mechanism which is depicted in figure 3. For a given pump speed in rpm (given the angular speed ω of the crank), the crank angle $\theta = \omega t$. The position of the plunger, measured from the center of the crank, is determined as

$$X(t) = R\cos\omega t + \sqrt{L^2 - \left(R\sin\omega t\right)^2} \tag{14}$$

which can be approximated as $X(t) = R\cos\omega t + L - \frac{R^2}{2L}(\sin\omega t)^2$ when $L >> R$, in agreement with the expression in Hau Pham's notes. In (14), R is the crank radius; L is the length of the connecting rod between the crank and the plunger (Figure 3).

The velocity of the plunger is then determined as,

$$V(t) = \frac{dX(t)}{dt} = -\omega R\sin\omega t - \frac{R^2\omega\sin\omega t\cos\omega t}{\sqrt{L^2 - (R\sin\omega t)^2}} \tag{14}$$

Or, $V(t) = -\omega R\left[\sin\omega t + (R/2L)\sin 2\omega t\right]$ when $L >> R$. When the crank angle $\theta = \omega t \ (0,\pi)$, the pump is at the suction phase; while $\theta = \omega t \ (\pi, 2\pi)$, the pump is at the discharge phase. In this dynamic model, since we will concentrate on the discharge phase, we introduce a phase shift of 180°(or $\omega t = \pi + \omega t'$ and drop the prime for convenience), such that the discharge phase starts at t=0. With this phase shift, the volume displacement rate Q_p of the plunger is obtained,

$$Q_p(t) = V(t)\frac{\pi D_p^2}{4} = \frac{\pi D_p^2}{4} R\omega\sin\omega t\left(1 - \frac{R\cos\omega t}{\sqrt{L^2 - (R\sin\omega t)^2}}\right) \tag{15}$$

where D_p is the diameter of the plunger.

Part of this volume of the fluid displaced by the plunger will be driven through the gap between the valve and its seat; and the rest be taken into the space created by the upward displacement of the valve. Assume the fluid is incompressible, the balance of the fluid volume leads to,

$$\begin{aligned}
&\frac{\pi D_p^2}{4} R\omega\sin\omega t\left(1 - \frac{R\cos\omega t}{\sqrt{L^2 - (R\sin\omega t)^2}}\right) \\
&= \pi\left(-R_2^2 + \frac{R_2^2 - R_1^2}{2\ln(R_2/R_1)}\right)\dot{h} + \frac{\pi}{6\mu}\frac{p_1 - p_2}{\ln(R_2/R_1)}h^3 + \dot{h}\pi R_2^2 \\
&= \frac{\pi}{2}\frac{R_2^2 - R_1^2}{\ln(R_2/R_1)}\dot{h} + \frac{\pi}{6\mu}\frac{(p_1 - p_2)}{\ln(R_2/R_1)}h^3
\end{aligned} \tag{16}$$

Equation (16) can be viewed as an equation for the pressure drop, $p_1 - p_2$, along the striking face, or explicitly,

$$(p_1 - p_2) = -3\mu\left(R_2^2 - R_1^2\right)\frac{\dot{h}}{h^3} + \frac{3\mu}{2h^3}\ln(R_2/R_1)D_p^2 R\omega\sin\omega t\left(1 - \frac{R\cos\omega t}{\sqrt{L^2 - (R\sin\omega t)^2}}\right) \tag{17}$$

This is the pressure difference created by the plunger motion. Substituting (17) into (11), we find the force acting on the valve, as a function of the plunger motion,

$$F_{flow} = \frac{3\pi}{4}\mu\left(R_2^2 - R_1^2\right)D_p^2 R\frac{\omega}{h^3}\sin\omega t\left(1 - \frac{R\cos\omega t}{\sqrt{L^2 - \left(R\sin\omega t\right)^2}}\right) - \frac{3\pi}{2}\mu\left(R_2^4 - R_1^4\right)\frac{\dot{h}}{h^3} \quad (18)$$

Finally, the equation of motion for the valve body takes the form of,

$$\left(M + M_f\right)\frac{d^2h}{dt^2} = -Mg\left(1 - \frac{\rho_{fluid}}{\rho_{valve}}\right) + F_{flow} - F_{spring} \quad (19)$$

where M is the mass of the valve; M_f is the added (virtual) mass due to the acceleration of the fluid (with the valve); ρ_{valve} is the density of the valve body; ρ_{fluid} is the density of the fluid (with the buoyancy force is accounted); the spring force is simply given by

$$F_{spring} = k\left(h + x_0\right) \quad (20)$$

where k is the spring constant, and x_0 is the spring reload (initial spring displacement).

Combining (18), (20) into (19), we have,

$$\begin{aligned}&\left(M + M_f\right)\frac{d^2h}{dt^2} + \frac{3\pi}{2}\mu\left(R_2^4 - R_1^4\right)\frac{1}{h^3}\frac{dh}{dt} + kh \\ &= \frac{3\pi}{4}\mu\left(R_2^2 - R_1^2\right)D_p^2 R\frac{\omega}{h^3}\sin\omega t\left(1 - \frac{R\cos\omega t}{\sqrt{L^2 - \left(R\sin\omega t\right)^2}}\right) - kx_0 - W\left(1 - \frac{\rho_{fluid}}{\rho_{valve}}\right)\end{aligned} \quad (21)$$

where $W=Mg$ is the weight of the valve. This is a non-linear ordinary differential equation (ODE) for h(t), and needs to be solved numerically. The parameters in the model are: spring $\left(k, x_0\right)$, plunger $\left(R, L, D_p, \omega\right)$, valve $\left(R_1, R_2\right)$ and fluid $\left(\rho_{fluid}, \mu\right)$.

In the Dowell frac pumps, the actual parameters have the following properties:

1. Crank radius R=2.5 in., and the connecting rod length L=16 in., such that R/L=0.15625 <<1.

2. The spring preload $kx_0 = 34.62$ lbf, and the weight of the valve W=5 lbf. Thus the effective weight of the valve can be neglected in (21). As a matter of fact, the weight of the valve only increases the spring preload as indicated by the equation (21).

Therefore, equation (21) can be simplified to,

$$\left(M+M_f\right)\frac{d^2h}{dt^2}+\frac{3\pi}{2}\mu\left(R_2^4-R_1^4\right)\frac{1}{h^3}\frac{dh}{dt}+kh=\frac{3\pi}{4}\mu\left(R_2^2-R_1^2\right)D_p^2R\frac{\omega}{h^3}\sin\omega t-kx_0 \quad (22)$$

Clearly, the driving force of the valve motion is the first term on the right-hand-side of (22) corresponding to the plunger motion. When the magnitude of this term overcomes the preload of the valve spring (the second term on the right-hand-side), the valve starts to move $dh/dt>0$. The magnitude of this driving force is amplified by $1/h^3$ when h is small due to the fact that when the valve is nearly closed the pressure under the valve increases sharply due to the impressibility of the fluid.

Equation (22) can be organized into a non-dimensional form. Let's introduce $\tau=\omega t$ as the non-dimensional time, and $H=h/h_m$ as the non-dimensional valve height, where h_m is a characteristic valve height to be defined later. Therefore, the ODE (22) can be rewritten as,

$$\begin{aligned}&H^3\frac{d^2H}{d\tau^2}+\frac{3\pi}{2}\frac{\mu\left(R_2^4-R_1^4\right)}{\left(M+M_f\right)h_m^3\omega}\frac{dH}{d\tau}+\frac{k}{\left(M+M_f\right)\omega^2}H^4\\&=\frac{3\pi}{4}\mu\frac{\left(R_2^2-R_1^2\right)D_p^2R}{\left(M+M_f\right)h_m^4\omega}\sin\tau-\frac{kx_0}{\left(M+M_f\right)h_m\omega^2}H^3\end{aligned} \quad (23)$$

The characteristic valve height can defined as the height of the valve when the plunger is assumed to move steadily at its maximum velocity ($\sin\tau=1$ or $\cos\tau=0$), which is to set H=1 and $dH/d\tau=d^2H/d\tau^2=0$ in (23),

$$k\left(x_0+h_m\right)h_m^3=\frac{3\pi}{4}\mu\left(R_2^2-R_1^2\right)D_p^2R\omega \quad (24)$$

or simply set,

$$h_m=\left[\frac{3\pi}{4}\mu\left(R_2^2-R_1^2\right)D_p^2R\omega/kx_0\right]^{1/3} \quad (25)$$

since the displacement in the spring preload, x_0, is usually larger than h_m. Substituting (25) back into (23), we have

$$KH^3\frac{d^2H}{d\tau^2}+A\frac{dH}{d\tau}+H^4=B\left(\sin\tau-H^3\right) \quad (26)$$

where

$$K=\left(\frac{\omega}{\omega_n}\right)^2 \text{ and } \omega_n=\sqrt{k/\left(M+M_f\right)} \quad (27)$$

is the natural frequency of the valve spring system. Furthermore,

$$A = 2\frac{R_2^2 + R_1^2}{D_p^2 R}x_0, \; B = \frac{x_0}{h_m} \tag{28}$$

Therefore, three independent non-dimensional parameters completely describes this system: K is the non-dimensional frequency of the system; A relates the driving of the crank-plunger system; and B is the parameter describing the spring preload.

In (26) the asymptotic behavior of $H(\tau)$ leads to,

$$H(\tau) = \frac{B}{2A}\left(\tau^2 - \frac{1}{12}\tau^4 + \frac{1}{360}\tau^6\right) + O\left(\tau^7\right) \text{ as } \tau \to 0 \tag{29}$$