

# Foamy oil flow in porous media

Daniel D. Joseph

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## 1 Introduction

This model is motivated by the need to explain anomalous features associated with production from reservoirs of so-called foamy oils. These oils are described by some of their properties of response to pressure declines; it is noted that they nucleate dispersed gas bubbles and display obvious foaminess in well head samples produced by solution gas drive in which oil and gas is produced by the draw down of pressure.

When compared with the response of conventional oils, the response of foamy oils to draw down of pressure is more favorable; primary recovery factor (percentage of the oil in the reservoir which can be recovered) the rate of production, the ratio of oil to gas which is recovered and the length of time that a given pressure gradient can be maintained are all increased substantially; the reasons for the favorable response of foamy oils in solution gas drive are not well understood and tentative explanations which have been put forward are controversial (see Maini (1996) and Pooladi-Darvish and Firoozabadi (1997) for recent reviews).

One property of foamy oils which has not been emphasized in the literature on solution gas drive is that these oils carry copious amounts of dissolved gases in the condensed state. The relevant thermodynamic property for this is “gas solubility”; a function of temperature and pressure at equilibrium which gives the volume fraction of dispersed gas from the crude oil by outgassing. Tables of solubility of methane, carbon dioxide and other gases in various Canadian crude oils have been given by Svrcek and Mehorta (1982), Peng, Fu Bird and Hsi ( )<sup>1</sup> and others. Naturally, if there is a lot of dissolved gas in, there is potential for a lot of gas to come out. The oils considered to be foamy evidently cavitate small dispersed bubbles which move with the crude oil in which they are dispersed. Experiments done by Pooladi-Darvish and Firoozabadi (1997) have shown that bubbles which arise from depressurization of silicone oil and heavy crude of equivalent vis-

cosity are very different; the bubbles in the silicone oil are larger and much more mobile than the ones in crude oil. Viscosity alone is not enough to demobilize dispersed gas; it is necessary to look at other properties like surface tension and surface active agents. Obviously there are surface active agents which are present naturally in foamy crudes which allow them to foam, but the precise agents, their composition and the mechanics by which they are released apparently have not been studied.

Having said all this, we might test the validity of the notion that a foamy oil is a heavy crude with high gas solubility which cavitates (or nucleates) in small dispersed, relatively immobile gas bubbles which foam possibly under rapid depressurization. Subject to revision, such a definition does lend itself to laboratory tests which could allow one discriminate between the “foaminess” of different foamy oils. *Two tests are suggested: (1) solubility, (2) foaming threshold.*

In solution gas drive of foamy oil the depressurization of the sample leads to cavitation of small dispersed bubbles. The volume fraction of dispersed gas increases the volume of our composite fluid and it acts as a pump, gas coming out of solution pumps the fluid outward. This pumping action is well described by the continuity equation (16) which implies that in a closed volume  $v$  with boundary  $\partial v$  containing dispersed bubbles of volume fraction  $\phi$ .

$$\int_v \frac{1}{1-\phi} \frac{d\phi}{dt} dv = \oint_{\partial v} \mathbf{u} \cdot \mathbf{n} \quad (1)$$

Where  $\mathbf{n}$  is the outward normal on  $\partial v$  and  $\mathbf{u}$  is the velocity of our composite fluid.

The relative velocity of dispersed gas is important; if the bubbles coalesce and move relative to the oil more gas and less oil will be produced. Good recovery is sometimes described by a critical gas saturation value; this value is a volume fraction of the gas in the pores of a sand pack so that when a critical amount of dispersed gas is collected in the pores, the gas connects and moves out of the system. Maini [1996] identifies this critical saturation as a percolation limit, whilst Firoozabadi, Ottensen and Mikkelsen [1992] and Pooladi-Darvish and Firoozabadi [1997] identify this even by visual observation of bubbles in a viewing window. The values given by Firoozabadi et al are about 5 times smaller than those given by Maini and his coworkers.

When the gas percolates, the good news about recovery is over; it is no wonder that all authors find that the critical saturation values are about the

same as the primary recovery factors (which is the fraction of oil recovered by solution gas to oil in the reservoir) even when they disagree about definitions. The description of recovery and saturation might be usefully framed in terms of holdup

$$h = U_0/U_g \tag{2}$$

where  $U_0$  is the volume flow rate of oil over the cross sectional area of oil, similarly for gas. For dispersed gas bubbles moving with the oil  $h = 1$ , when the gas bubbles are trapped in the sand,  $U_g = 0$  and  $h = \infty$  and when the oil is trapped,  $h = 0$ . Trapped oil is not good for production, but the values of  $h$  greater than or equal to a number not too much smaller than one are interesting.

The literature on solution gas drive of foamy oils focuses on analysis of nucleation and growth of gas bubbles; no satisfactory discussion of the forces which induce these bubbles to move relative to the oil has been given. One obvious mechanism for producing such a relative motion is buoyancy; if this mechanism is important for critical gas saturation and primary production we could expect to see large differences between critical values in horizontal and vertical sand packs.

The model proposed here does not require information about nucleation, bubble growth, compressibility or forces which produce relative velocity. We put up a one phase or mixture theory in which the dispersed gas is described by a gas fraction field in a single fluid in which the viscosity, density and mobility in Darcy's law all depend on the gas fraction. This fluid satisfies the usual Darcy law, and the continuity equation together with a kinetic (constitutive) equation required by the condensation and outgassing of methane (or other gases) in heavy crude. The theory depends only on parameters which can be measured in a PVT cell and sand pack. The virtue of the model is simplicity, but it can only work when the gas is dispersed. Certainly such a theory could not be expected to give rise to a percolation threshold or even to a critical gas fraction. We hope that it can describe many features of solution gas drive of foamy oils in the regimes when the bubbles in the mixture are dispersed and even when they are trapped in foam.

It is our idea the increased recovery and production are generated by the pumping of nucleating and growing gas bubbles embodied in (1). However recovery factors and production rates are not the same and we might test some ideas: if two foamy oils have the same viscosity, the one with higher solubility will have higher primary recovery and production rate; if two

foamy oils have the same solubility, the one with lower viscosity will have a higher rate of production but a lower primary recovery. If the oil foams in situ, the primary recovery factor increases and the rate of production decreases.

## 2 Solubility of gas in crude oil

In this model we avoid all constitutive equations regarding nucleation rates, bubble growth, and compressibility. In our model we have only foamy oil and dispersed gas, but the dispersed gas enters only through

$$\phi = \frac{V \text{dispersed gas}}{\text{Total Volume}}$$

We propose to describe the evolution of  $\phi$  by an evolution equation for uniform samples in a PVT cell of the form

$$\frac{\partial \phi}{\partial t} = f(p, \phi)$$

where  $f(p, \phi)$  is to be determined from experiments. We have equilibrium whenever

$$f(p, \phi) = 0$$

Svrcek and Mehrota (1982) give volumetric solubility curves (CO<sub>2</sub> and methane CH<sub>4</sub>, in figure 1 below). In these figures  $\hat{\phi}$  is the volume fraction of gas that can be evolved out of bitumen when the pressure is dropped to less than one atmosphere at a temperature of 100°C. We can assume that this tells you how much dispersed gas can come out of solution of condensed gas which is at a saturation value at any pressure and temperature. We are going to assume that this  $\hat{\phi}$  determines the dispersed gas fraction  $\phi$  following an argument put forward in section 4.

In the present approach we have no way to predict the size or size distribution of gas bubbles. This means that we are free to choose the size and distribution to measure  $\phi$  and the most convenient choice is when all the released gas is collected at the top of the cell. Figure 2 describes a depressurization experiment

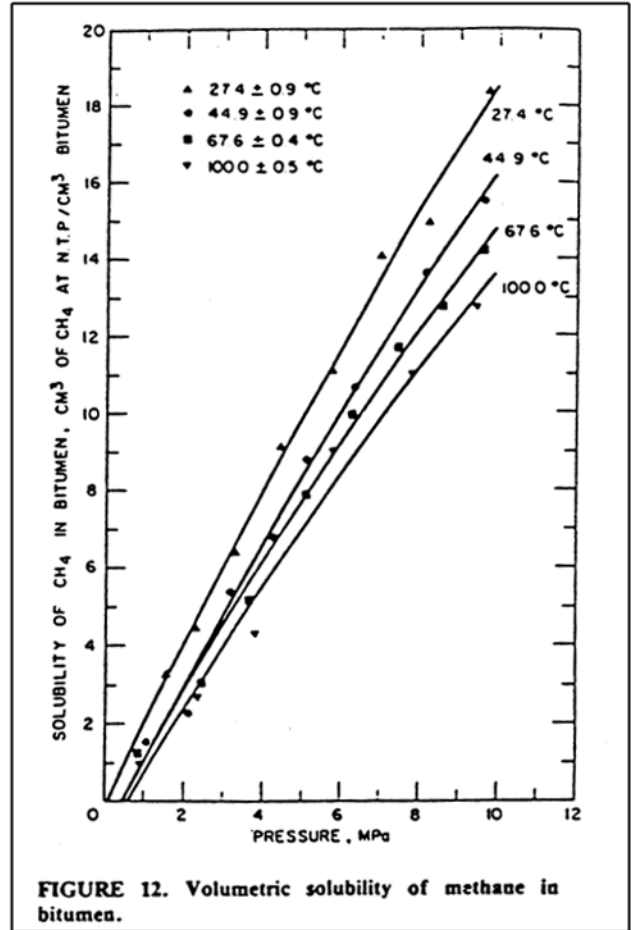
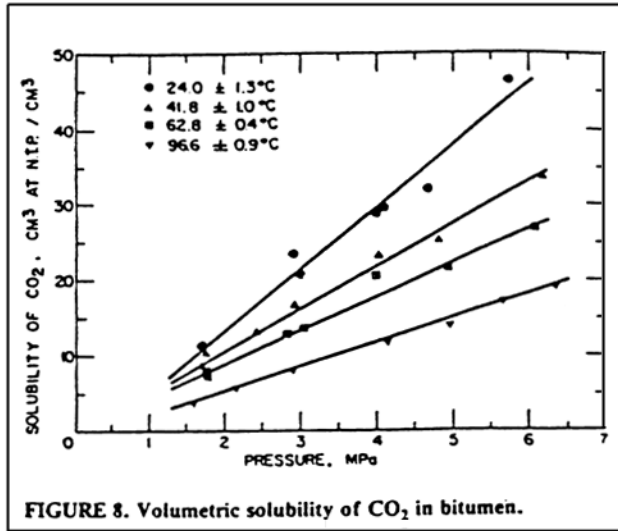
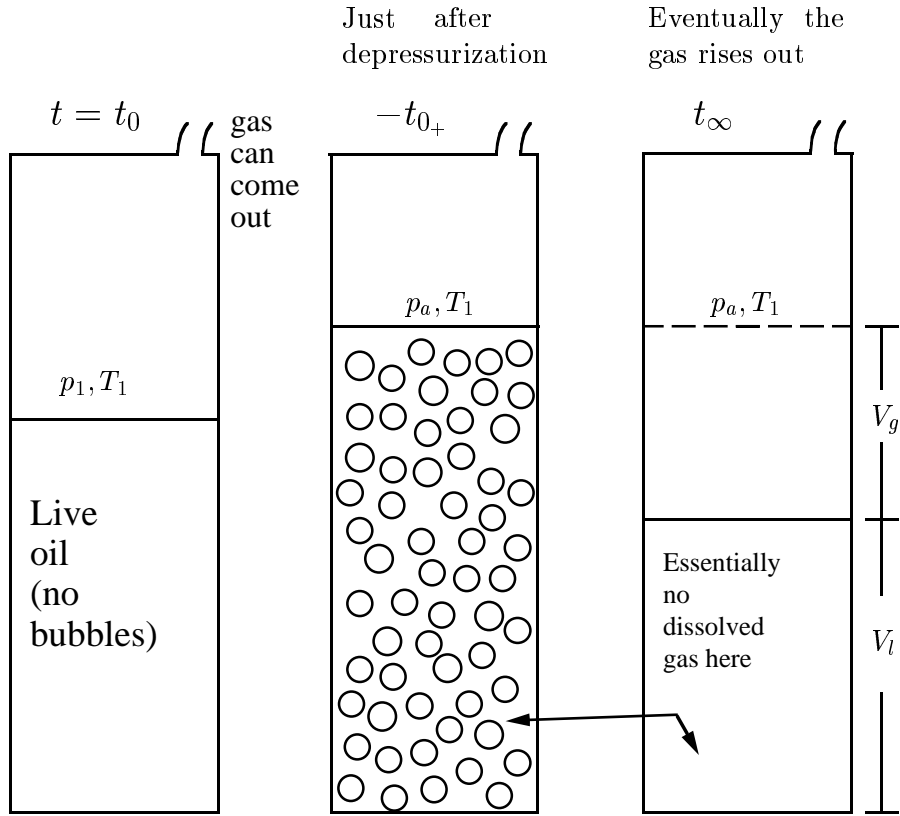


Figure 1: (Svrcek and Mehrota [1982]) Solubility curves:  $\hat{\phi}$  vs.  $p$

**Figure 2: Total depressurization experiment**



Gas is dissolved in the live oil at  $p_1, T_1$

$(p_1, T_1, \hat{\phi}_1)$  are a point on the solubility or saturation curve

The pressure is reduced to atmospheric. Assume that all the gas in the live oil comes out.

$V_g =$  volume of dispersed gas bubbles

$V_l$  is the volume of liquid for  $t > t_0$ . It was not conserved since gas came out.

The volume of gas which comes out is not the same as the volume of gas when condensed.

For  $t > 0$  the volumes  $V_g$  and  $V_l$  are conserved and

$$\hat{\phi}_1 = \frac{V_g}{V_g + V_l} \quad 1 - \hat{\phi}_1 = \frac{V_l}{V_g + V_l}$$

Now we do the same experiment at  $(p_2, T_1)$ . If  $p_2 > p_1$ , more gas is condensed in the liquid. Therefore  $V_g$  is larger and

$$\begin{aligned}\hat{\phi}_2 &= \frac{V_g}{V_g + V_l} \text{ at } T = T_1 \\ \hat{\phi}_2 &> \hat{\phi}_1\end{aligned}\tag{3}$$

These measurements give rise to the solubility isotherms shown in figure 1 with caveat that in the actual experiments of Svrcek & Mehorta, the temperature is put at 100°C instead of  $T_1$ . I don't think that makes a big difference.

The two figures in figure 1 show that the solubility isotherms are in the form

$$p = \gamma(T)\hat{\phi} + C = \gamma(T)\hat{\phi} + p_0\tag{4}$$

where  $C$  is a constant which could depend on  $T$ . The constant  $C$  is the zero  $\hat{\phi}$  crossing  $p = p_0$  at  $\hat{\phi} = 0$ . This is a low pressure threshold below which no gas can be dissolved.

### 3 Soluble gas.

I think of a gas as condensed and miscible in bitumen. Let  $\Psi$  be the volume fraction of soluble gas in bitumen. Then the liquid saturates with volume fraction  $\Psi_s$  at  $p_s$ .  $\Psi_s$  increases with  $p_s$ ; you get more gas into solution at high pressure and low temperature. The viscosity goes down rapidly as  $\Psi_2$  (and  $p_s$ ) goes up. You need only a little  $\Psi$  to make a lot of  $\hat{\phi}$ , because a phase change is involved.

The continuity equation for a miscible binary mixture of incompressible liquids  $g$  and  $b$  is

$$\frac{d\rho}{d\Psi} \frac{d\Psi}{dt} + \rho \operatorname{div} \mathbf{u} = 0$$

where

$$\rho = \rho_g \Psi + \rho_b(1 - \Psi)\tag{5}$$

$$\frac{d\rho}{d\Psi} = \rho_g - \rho_b\tag{6}$$

We can think that  $\rho_g$  here is dissolved gas and  $\rho_b$  is heavy crude. Experiments indicate that dissolved CO<sub>2</sub> has the same density as bitumen.

$$\rho_g = \rho_b$$

Dissolved methane has a density only slightly different than bitumen (see figures 6 and 11). Hence

$$\operatorname{div} \mathbf{u} = 0 \tag{7}$$

We can treat oil and dissolved gas as a one phase fluid and need not write a separate equation for dissolved gas.

In our one phase model

$$\rho \text{ given by (5) is constant} \tag{8}$$

(see figure 6 and 11 of Svrcek & Mehorta [1982] in figure 3 below). Since  $\rho(\hat{\phi})$  is a constant, the dissolved gas doesn't effect the dynamics of the mixture.

I have developed a good theory for mixtures of incompressible miscible liquids; for example, see Chapter 10 of my Springer book with Y. Renardy, "Fundamentals of two-fluid dynamics" or T. Liao & D.D. Joseph, "Sidewall effects in the smoothing of an initial discontinuity of concentration", *J. Fluid Mech.*, **342**, 37-51, (1997).



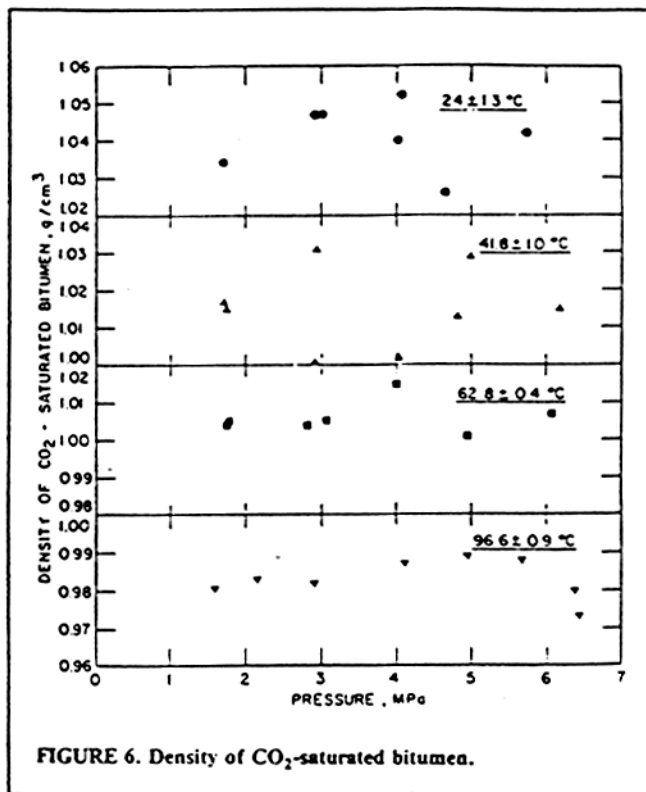


FIGURE 6. Density of CO<sub>2</sub>-saturated bitumen.

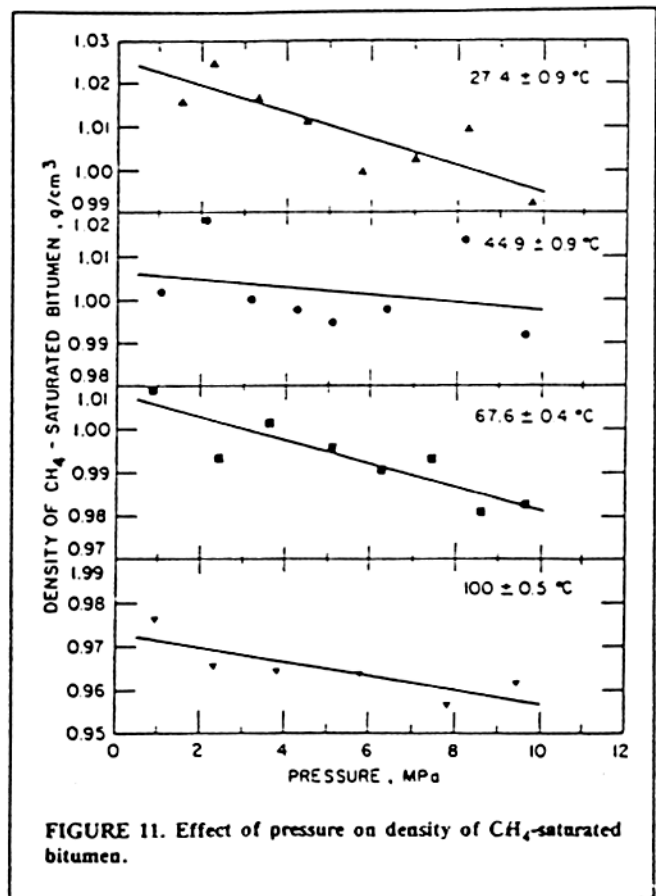


FIGURE 11. Effect of pressure on density of CH<sub>4</sub>-saturated bitumen.

Figure 3: The density of bitumen & condensed gas mixtures. The density is nearly constant over very wide ranges of pressure. We can imagine that the mixture density is independent of pressure. Since gas solubility is a strong function of pressure; the density is also more or less independent of the fraction  $\hat{\phi}$  of dissolved gas at constant temperature.

## 4 Gas evolution at equilibrium when the pressure is dropped.

$\hat{\phi}$  is the volume fraction of gas that comes out of solution when the pressure in a gas saturated bitumen is dropped to one atmosphere at constant temperature. This  $\hat{\phi}$  is the solubility of the gas given as a volume fraction and it is plotted in figure 1. The saturated bitumen is called “live oil”.

$\phi$  is the volume fraction of dispersed gas which comes out of solution.

### Major Assumption:

We suppose that all the gas which comes out of solution is disperse and does not percolate. In the experiments in figure 1 we must suppose that the nucleation, growth and compressibility of gas bubbles are working, but these microstructural features are suppressed in these experiments; embedded all in the solubility fraction  $\hat{\phi}$ . This is also what we do in the mathematical model.

Now I am going to slightly alter a solubility isotherm in the cartoon of figure 4 below.

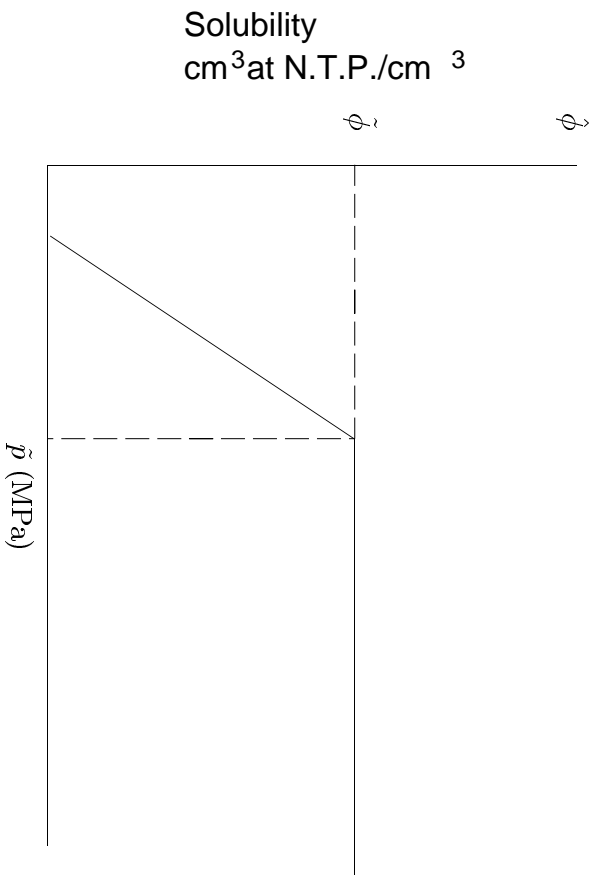


Figure 4: Solubility isotherm used in the model.

Suppose that at some high pressure  $p = \tilde{p}$ ,  $\hat{\phi} = \tilde{\phi}$ , there is no more gas

available to go into solution; then  $\hat{\phi} = \tilde{\phi}$  when  $p > \tilde{p}$ .

Now drop the pressure; gas nucleates, bubbles grow and expand, but they stay dispersed and don't percolate out. Then

$$\tilde{\phi} - \phi(p) = \hat{\phi}(p) \quad (9)$$

Equation (9) says that when  $p < \tilde{p}$ , the dispersed gas fraction is just the right amount to keep the outgassed liquid at its saturation solubility  $\hat{\phi}(p)$ . Then, combining (4) and (9) we get

$$\begin{aligned} p &= \dot{\gamma}(\tilde{T})\hat{\phi} + c = \gamma(\tilde{T})\hat{\phi} + p_0 - \phi \\ &= \tilde{c}(\tilde{T}) - \phi \end{aligned} \quad (10)_1$$

where

$$\tilde{c}(\tilde{T}) = \tilde{p}$$

because  $\phi = 0$  at  $p = \tilde{p}$ .

Equation (4)

$$p + \gamma\phi = \tilde{p} \quad (10)_2$$

is our equilibrium isotherm for dispersed gas.

## 5 Constitutive equation relating $p$ and dispersed gas fraction

I will assume that the evolution of  $p$  and  $\phi$  is governed by a rate equation of Maxwells type

$$\tau \frac{dp}{dt} + p = -\gamma\phi + \tilde{p} \quad (11)$$

$\frac{dp}{dt} = 0$  when  $p$  and  $\phi$  are at equilibrium and satisfy (10)<sub>2</sub>. When  $p$  is below this value, say, and  $\phi$  at equilibrium, then  $\frac{dp}{dt} > 0$  and the pressure will increase to its equilibrium value. All in all, we get stress ( $p$ ) relaxation to equilibrium from (11). When  $\phi = 0$ ,  $p - \tilde{p}$  relax to zero exponentially.

Logically, in a first order constitutive theory, I should also include a term proportional to  $d\phi/dt$ . I haven't figured out a way to eliminate this possibility, and we could include it in the future and see what it predicts. Inclusion of such a term would lead to an Oldroyd B type of model with a retardation as well as relaxation time  $\tau$ . Let us proceed with the Maxwell model; **the relaxation time  $\tau$  is the only unknown, to-be-determined parameter in this theory.**

## 6 Governing equations:

We have to introduce porosity for porous media.

$$\left\{ \begin{array}{l} \alpha \text{ porosity} \\ \lambda \text{ mobility} \\ \frac{d(\circ)}{dt} = \frac{\partial \alpha(\circ)}{\partial t} + \mathbf{u} \cdot \nabla(\circ) \end{array} \right. \quad (12)$$

Equations for porous media are

$$\begin{aligned} -\lambda \nabla p = \mathbf{u} & \quad \text{Darcy law} \\ \lambda = \lambda(\phi) & \end{aligned} \quad (13)$$

and the continuity equation

$$\frac{d\rho}{d\phi} \frac{d\phi}{dt} + \rho(\phi) \operatorname{div} \mathbf{u} = 0 \quad (14)$$

where

$$\rho(\phi) = \rho_g \phi + \rho_o(1 - \phi) \approx \rho_o(1 - \phi) \quad (15)$$

because  $\rho_g \ll \rho_o$  where  $\rho_o$  is the unique density of bitumen plus dissolved gas which figure 3 shows can be taken as constant. Hence (14) reduces to

$$-\frac{1}{1 - \phi} \frac{d\phi}{dt} + \operatorname{div} \mathbf{u} = 0 \quad (16)$$

or

$$\frac{d \log(1 - \phi)}{dt} + \operatorname{div} \mathbf{u} = 0 \quad (17)$$

We have 5 equations

$$\left\{ \begin{array}{l} \mathbf{u} = -\lambda \nabla p \\ \frac{d \log(1 - \phi)}{dt} + \operatorname{div} \mathbf{u} = 0 \\ \tau \frac{dp}{dt} = -\gamma \phi - p + C \end{array} \right. \quad (18)$$

for 5 unknowns  $\mathbf{u}, p, \phi$ .

Reduction to 2 equations in  $p$  and  $\phi$ :

After writing

$$\pi = p - \tilde{p},$$

taking  $\alpha$  constant, we get two equations in two unknowns

$$-\alpha \frac{\partial \phi}{\partial t} + \lambda \nabla \pi \cdot \nabla \phi = (1 - \phi) \operatorname{div} \lambda \nabla \pi \quad (19)$$

$$\tau \left[ \alpha \frac{\partial \pi}{\partial t} - \lambda |\nabla \pi|^2 \right] + \gamma \varphi + \pi = 0. \quad (20)$$

**(19) and (20) are the fundamental equations of my model.**

- We can eliminate  $\phi$  from these equations; this gives rise to a very nonlinear PDE in the pressure  $\pi$ .
- **Equilibrium solutions.** I define these to be those for which

$$\tau = 0$$

In this case

$$\mathbf{u} = -\lambda \nabla \pi = \lambda \gamma \nabla \phi \quad (21)$$

The fluid moves from high to low pressure; at the same time there are more dispersed gas bubbles when  $p$  is smaller. So the fluid will flow to the region of higher  $\phi$  (lower  $p$ ). The bubbles are a barometer.

If  $\tau = 0$ , we can eliminate  $\varphi$ , even  
 $\lambda(\phi) = \lambda(-\frac{\pi}{\gamma})$

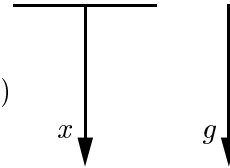
and

$$\alpha \frac{\partial \pi}{\partial t} - \lambda |\nabla \pi|^2 = (\gamma + \pi) \operatorname{div} \lambda \nabla \pi. \quad (22)$$

This is an interesting non-linear diffusion equation, worthy of analysis.

- Effects of Gravity

$$\begin{aligned} \mathbf{u} &= -\lambda \nabla (p + \rho g x) \\ \rho &= \rho_g \phi + \rho_l (1 - \phi) \cong \rho_l (1 - \phi) \\ \mathbf{u} &= -\lambda \nabla p - \lambda \rho_l g \nabla (1 - \phi) x \\ \text{For equilibrium } \nabla p &= \gamma \nabla \phi \end{aligned}$$



- Perturbation of constant state solutions

The basic equations (19) and (20) admit constant state solutions  $\phi = c_1, \pi = c_2$  where  $c_1$  and  $c_2$  are independent of  $\mathbf{x}$  or  $t$ . Of course

$$\gamma c_1 + c_2 = 0 \tag{23}$$

is our solution. We can perturb this and neglect terms which are quadratic in the perturbation variables

$$\begin{aligned} \phi &= c_1 + \boldsymbol{\phi} \\ \pi &= c_2 + \boldsymbol{\pi} \end{aligned} \tag{24}$$

then

$$-\alpha \frac{\partial \boldsymbol{\phi}}{\partial t} = (1 - c_1) \lambda_1 \nabla^2 \boldsymbol{\pi} \tag{25}$$

$$\tau \alpha \frac{\partial \boldsymbol{\pi}}{\partial t} + \gamma \boldsymbol{\phi} + \boldsymbol{\pi} = 0 \tag{26}$$

where  $\lambda_1 = \lambda(c_1)$ . We may eliminate  $\boldsymbol{\pi}$  or  $\boldsymbol{\phi}$  from (25) and (26); in both cases we find a telegraph equation

$$\frac{\partial^2 \boldsymbol{\pi}}{\partial t^2} + \frac{1}{\alpha \tau} \frac{\partial \boldsymbol{\pi}}{\partial t} = \frac{(1 - c_1) \lambda \gamma}{\tau \alpha^2} \nabla^2 \boldsymbol{\pi}. \tag{27}$$

The same equation is satisfied by  $\boldsymbol{\phi}$ . This equation gives rise to a wave equation with a wave speed

$$V = \sqrt{\frac{(1 - c_1) \lambda \gamma}{\tau \alpha^2}} \tag{28}$$

The waves are damped; this is an effect of the second term in (27). If the relaxation time  $\tau \rightarrow 0$ , then the effects of wave propagation gives way to diffusion

$$\frac{\partial \boldsymbol{\pi}}{\partial t} = \frac{(1 - c_1) \lambda \gamma}{\alpha} \nabla^2 \boldsymbol{\pi}. \tag{29}$$

Let us consider the solution of the telegraph equation (27) for ‘‘Stokes 1st problem’’ (see Joseph [1990], pgs. 582-584) to the sudden pressurization of our fluid at the boundary  $y = 0$  of a semi-infinite region  $y \geq 0$ . We must solve (27) satisfying

$$\pi(y, t) = 0; \quad t \leq 0, y \geq 0 \tag{30}$$

The pressure at  $y = 0$  is suddenly raised and held at value  $\pi_0$ .

$$\pi(y, t) = \pi_0 H(t) \quad (31)$$

where  $H(t)$  is Heaviside's step function and

$$\pi(y, t) \rightarrow 0 \text{ as } y \rightarrow \infty \quad (32)$$

Equation (27) in one dimension is

$$\frac{\partial^2 \pi}{\partial t^2} + \frac{1}{\alpha \tau} \frac{\partial \pi}{\partial t} = \frac{(1 - C_1) \lambda \gamma}{\tau \alpha^2} \nabla^2 \pi \quad (33)$$

This problem may be solved by Laplace transforms. The discontinuity propagates into the fluid with a speed  $V$  and attenuation given by

$$\pi(y, y/V) = \pi_0 \exp \left[ \frac{\alpha \tau y}{2V} \right] \quad (34)$$

## 7 Special Solutions

Our basic equations are (19) and (20). We can investigate some special types of solutions:

- Steady solutions
- Traveling wave solutions in one space dimension  $x$  in which  $x$  and  $t$  appear only in the combination  $\chi = x - vt$  for a constant  $V$ .
- Primary depletion

Here we treat the same one dimensional model considered by Sheng et al. (1996). At  $x = 0$  there is no flow,  $u = 0$  at  $x = 0$ ; hence

$$\frac{\partial p}{\partial x} = 0 \text{ at } x = 0. \quad (35)$$

The pressure decline is prescribed at  $x = L$

$$p = \tilde{p}(t) \text{ at } x = L. \quad (36)$$

Initially, at  $t = 0$ , the pressure is  $C_4$ , a constant

$$p = C_4 \text{ for } 0 \leq x \leq L \text{ at } t = 0_+.$$

I do not know if we have enough conditions to solve this problem in the general case (19) and (20). However, the problem may be solved in the equilibrium theory governed by (22) in one dimension:

$$\alpha \frac{\partial \pi}{\partial t} - \lambda \left( \frac{\partial \pi}{\partial x} \right)^2 = (\gamma + \pi) \frac{\partial}{\partial x} \left( \frac{\partial \pi}{\partial x} \right) \quad (37)$$

This is a two-point boundary initial-value problem appropriate for the 2nd order PDE (37).

## References

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## 8 Appendix: A different constitutive equation

For constitutive law we assume

$$\tau\gamma\frac{d\phi}{dt} + \gamma\phi = c - p \quad , \quad (38)$$

which is a statement that the volume fraction relaxes toward the equilibrium value, with time constant  $\tau$ . In addition we have Darcys law,

$$-\lambda\nabla p = \mathbf{u} \quad , \quad (39)$$

and conservation of mass (in the approximation that gas density is much less than liquid density),

$$\frac{d\log(1-\phi)}{dt} + \nabla \cdot \mathbf{u} = 0 \quad . \quad (40)$$

The material derivative  $d/dt$  is defined by

$$\frac{d}{dt} = \alpha\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad . \quad (41)$$

Combining (41) and (39), and expanding the material derivative, we obtain

$$\alpha\frac{\partial\phi}{\partial t} - \lambda\nabla p \cdot \nabla\phi + (1-\phi)\nabla \cdot \lambda\nabla p = 0 \quad (42)$$

as before. Combining (38) and (39), and expanding the material derivative, we obtain

$$\tau\gamma\frac{\partial\phi}{\partial t} - \tau\gamma\lambda\nabla p \cdot \nabla\phi + \gamma\phi = c - p \quad . \quad (43)$$

Equations (42) and (43) are the basic governing equations in this form of the theory. Clearly, when  $\tau = 0$ , we recovery exactly the same equilibrium theory as was derived above. It is also clear that this version of the theory supports steady uniform states as solution, again exactly as found above. The governing equations for small perturbations around these steady states are however different, as we shall now show. As before, introduce small perturbation quantities, setting

$$\phi = c_1 + \phi \quad (44)$$

$$p = c_2 + \pi \quad (45)$$

where equilibrium demands  $\gamma c_1 + c_2 = 0$ . Substituting these quantities into (42) and (43), neglecting quantities of second order in perturbations and treating the mobility as a constant, we obtain

$$\alpha \frac{\partial \phi}{\partial t} + (1 - c_1) \lambda \nabla^2 \pi = 0 \quad (46)$$

$$\tau \gamma \frac{\partial \phi}{\partial t} + \gamma \phi = -\pi \quad (47)$$

If we now take the time derivative of (47), and use (46) to replace  $\partial \phi / \partial t$  by the Laplacian of  $\pi$ , we find

$$(\alpha - \tau(1 - c_1)\gamma\lambda\nabla^2) \frac{\partial \phi}{\partial t} = (1 - c_1)\gamma\lambda\nabla^2 \pi \quad (48)$$

The perturbation volume fraction satisfies an identical equation. Initial and boundary conditions on the perturbation pressure are the same here as for the model presented above.

The evolution equation for the perturbation pressure, (48), is very different in character from the telegraph equation found for the pressure relaxation form of the constitutive law. It is lower order in time, and, from Laplace transform solutions, appears not to exhibit wave propagation; rather, the solutions are diffusive in character although similarity solutions in  $x/t^{1/2}$  do not exist (except as an asymptotic state at large  $t$ ). The rate of advance of the pressure perturbation front is found from these Laplace transform solutions to be faster in early time than would be the case for classical diffusion.