

# New ideas about flow induced cavitation of liquids

by Daniel D. Joseph  
University of Minnesota

## Abstract

The problem of the inception of cavitation is formulated in terms of a comparison of the breaking strength or cavitation threshold at each point of a liquid sample with the principal stresses there. A criterion of maximum tension is proposed which unifies the theory of cavitation, the theory of maximum tensile strength of liquid filaments and the theory of fracture of amorphous solids. It is argued that the liquid ruptures in tension at nucleation sites; the cavity then fills with gas and the liquid flows. Liquids at atmospheric pressure which cannot withstand tension will cavitate when and where tensile stresses due to motion exceed one atmosphere. A cavity will open in the direction of the maximum tensile stress which is  $45^\circ$  from the plane of shearing in pure shear of a Newtonian fluid. An analysis of capillary collapse based on viscous potential flow leads to the total collapse of a capillary filament in a finite time; before this the filament enters into tension and presumably would break under tension. For water the critical radius is about 1.5 microns.

## Conventional Cavitation

A fluid will cavitate when the local pressure falls below the cavitation pressure

- The cavitation pressure is the vapor pressure in a pure liquid
- Natural liquids have nucleation sites defined by impurities and may cavitate at higher pressures

## What is Pressure?

- In an incompressible Newtonian fluid “pressure” is the mean normal stress. A fluid cannot average its stresses, even though you can. The fluid knows its state of stress at a point.
- In non-Newtonian fluids the pressure is an unknown flow variable, usually not even the mean normal stress, and the definition of it is determined by the constitutive equation. This “pressure” has no intrinsic significance. The fluid doesn't recognize such a “pressure” and knows its state of stress.

## Nonconventional Cavitation Based on Principal Stresses

Look at the state of stress at each point in the fluid in principal axis coordinates. Identify the largest of the stresses. Suppose a static fluid cavitates at zero pressure. It will cavitate in flow wherever the maximum tensile stress is positive.

If it cavitates statically when the pressure falls below the vapor pressure, it will cavitate in flow even when the maximum tensile stress is only slightly negative.

Suppose you do an experiment in your lab where the ambient pressure is

$$\text{one atmosphere} = 10^5 \text{ Pa}$$

Then if you get tensile stresses due to flow larger than this, the fluid will cavitate.

## Stress, Principal Axes, Deviator

- Stress in two dimensions

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}$$

“pressure” cannot be recognized in a liquid; it sees a state of stress.

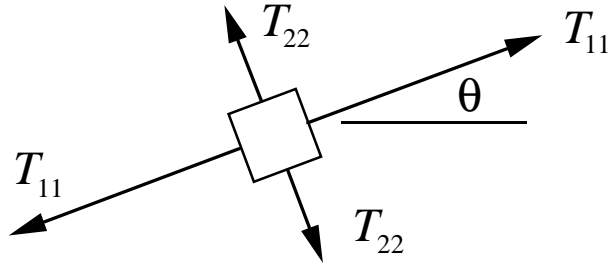


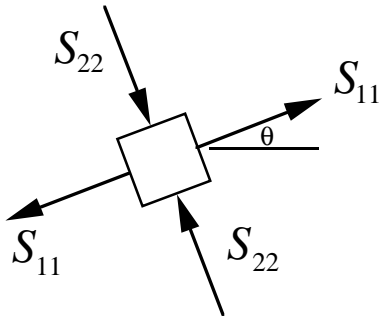
Figure 1: The direction of maximum tension. A cavitation must open in the direction  $\theta$ ; then it can rotate away.

- Principal coordinates (figure 1)
- Mean normal stress and deviator

$$\mathbf{T} = -p\mathbf{1} + \mathbf{S} \quad \mathbf{S} \text{ is the extra stress}$$

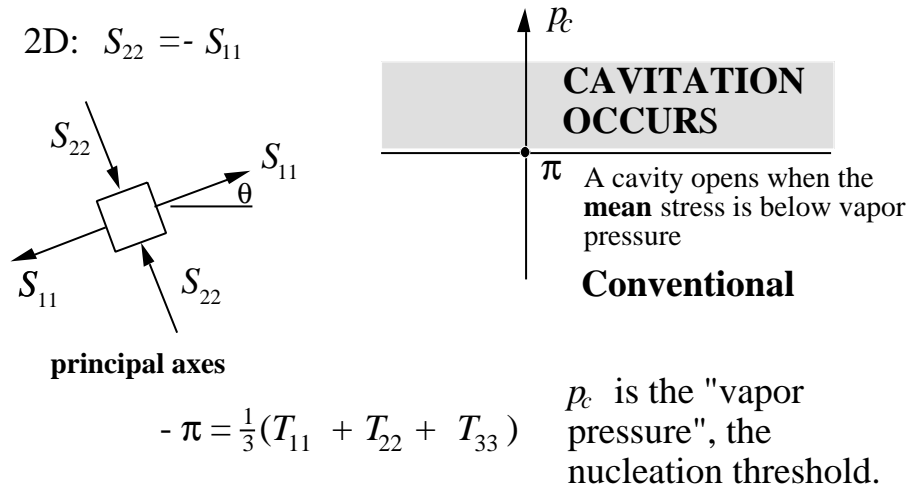
$$p = -\frac{T_{11} + T_{22}}{2} \quad \text{the fluid cannot average stresses}$$

$$\mathbf{S} = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}, \quad S_{11} + S_{22} = 0$$



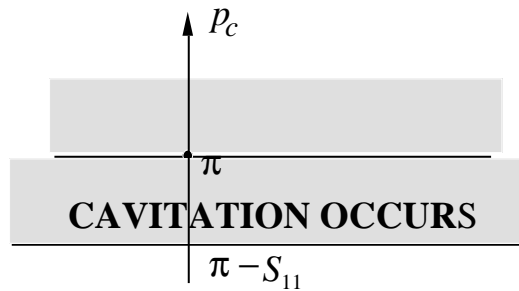
The extra stress is good because it has positive and negative components.

## Cavitation Criteria



### Maximum Tension:

A cavity opens when **one** of the principal stresses is below vapor pressure



### Minimum Tension:

A cavity opens when **all** of the principal stresses are below vapor pressure

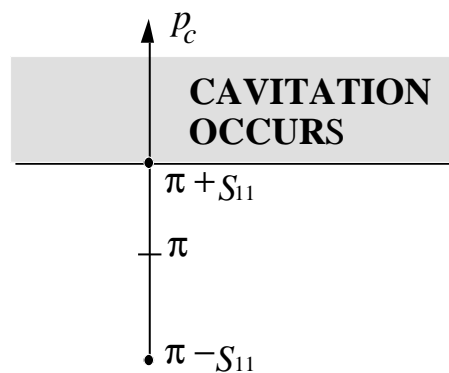


Figure 2: Three criteria for cavitation could be proposed, but the one based on maximum tension is the only one consistent with fracture of solids and solid-like liquids.

## Maximum Tension

*All books on cavitation have sections on “tensile strength of liquids.” A stress tensor is **never** introduced.*

“If a cavity is to be created in a homogeneous liquid, the liquid must be ruptured, and the stress required to do this is not measured by the vapor pressure but is the tensile strength of the liquid at that temperature.” (Knapp et al. 1970)

**FIRST THE FLUID RUPTURES  
THEN VAPOR FILLS THE CAVITY**

## Cavitation in Shear

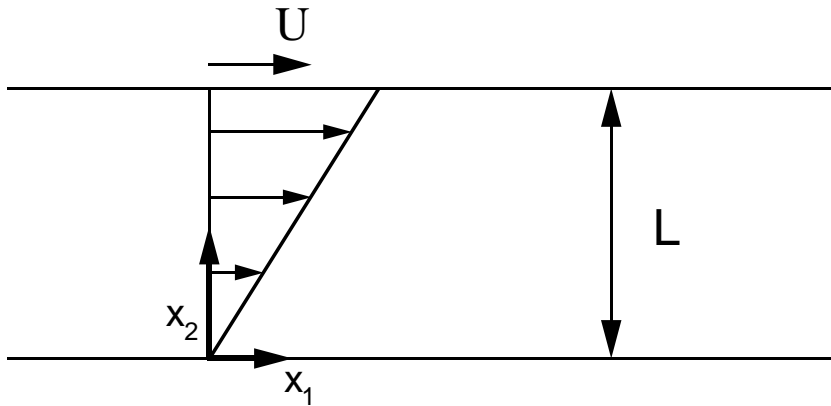


Figure 3: Simple shear between walls.

The stress in this flow is given by

$$\begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} = -\pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \eta \begin{bmatrix} 0 & \frac{U}{L} & 0 \\ \frac{U}{L} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where  $\pi = \frac{1}{3}(T_{11} + T_{22} + T_{33})$  is determined by the “pressurization” of the apparatus. The angle which diagonalizes  $\mathbf{T}$  is given by

$$\theta = 45^\circ$$

(In the break-up of viscous drop experiments in plane shear flow done by G.T. Taylor [1934], the drops first extend at  $45^\circ$  from the direction of shearing.)

In principal coordinates, we have

$$\begin{bmatrix} T_{11} + \pi & 0 & 0 \\ 0 & T_{22} + \pi & 0 \\ 0 & 0 & T_{33} + \pi \end{bmatrix} = \eta \frac{U}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$T_{11} + \pi = S_{11} = \eta \frac{U}{L}$$

is a tension.

This tension is of the order of one atmosphere of pressure if

$$\eta \frac{U}{L} = 10^6 \frac{\text{dynes}}{\text{cm}^2} = 10^5 \text{Pa}$$

If  $\eta = 1000$  poise,  $U = 10$  cm/sec and  $L = 10^{-1}$ cm, we may achieve such a stress. *A shear stress of this magnitude is enough to put the liquid into tension.*

## The production of cavitation in pure shear appears to have been realized recently (1997)

Abstract, “Fracture” phenomena in shearing flows of viscous liquids, L.A. Archer, D. Ternet and R. Larson:

In startup of steady shearing flow of two viscous unentangled liquids, namely low-molecular-weight polystyrene and  $\alpha$ -D-glucose,

The shear stress catastrophically collapses if the shear rate is raised above a value corresponding to a critical initial shear stress of around 0.1 – 0.3 Mpa. The time-dependence of the shear stress during this process is similar for the two liquids, but visualization of samples in situ and after quenching reveals significant differences. **For  $\alpha$ -D-glucose, the stress collapse evidently results from debonding of the sample from the rheometer tool, while in polystyrene, bubbles open up within**

**the sample; as occurs in cavitation.** Some similarities are pointed out between these phenomena and that of “lubrication failure” reported in the tribology literature.

We have adhesive and cohesive fracture, 0.1–0.3 Mpa = 1–3 atm. This is enough to put the sample into tension 45° from the direction of shearing.

## Breaking Strength of Polymer Strands

The strand breaks at the thinnest cross section of the strand when the tensile stress

$$\sigma = \frac{FV}{A_0} \approx 10^6 \text{ Pa} = 10 \text{ atmospheres}$$

for many kinds of polymeric liquids. They say that the breaking stress is a material constant.

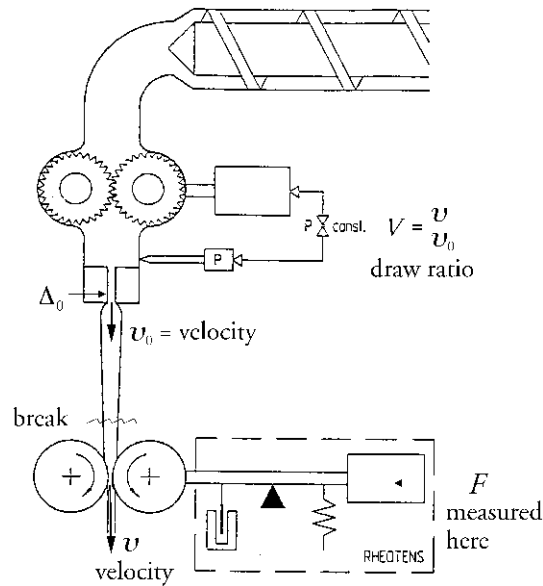


Figure 4: (Wagner, Schulze and Gottfert [1996]) Drawdown apparatus.

## Breaking Time & Flow Time Vacuum Cavities

Experiments of Israelachvili and coworkers (Chen & Israelachvili [1991], Kuhl et al. [1994]) on ultrathin (nanometer) films show that cavities open in tension at a threshold value of the extensional stress  $2\eta\dot{S}$

$$\left( \begin{array}{c} \text{which I estimate as} \\ 3.6 \times 10^5 \text{Pa} < 2\eta\dot{S} < 3.6 \times 10^6 \text{Pa} \end{array} \right)$$

and that the formation of cavities is analogous to the fracture of solids except *after* fracture, vapor flows into the cavity “...When a cavity initially forms and grows explosively, it is essentially a VACUUM CAVITY since dissolved solute molecules or gases have not had time to enter the rapidly growing cavity.”

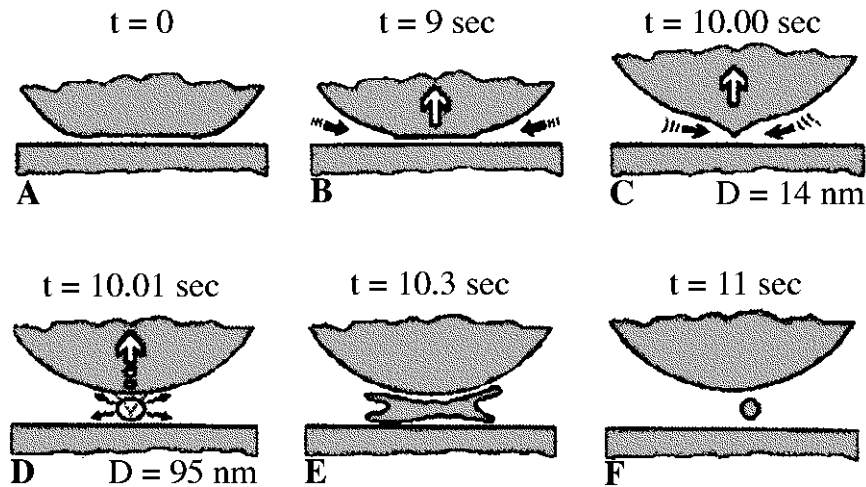


Figure 5: Surfaces separating at high speed,  $v = v_c$

Kuhl et al. 1994 describe the experiment shown in figure 5 as

“If the speed of separation is increased, the surfaces become increasingly more pointed just before they rapidly move apart. Then, above some critical speed  $\nu_c$  (here about  $100 \mu\text{m/s}$ ) a completely new separation mechanism takes over, as shown in Figure



5. Instead of separating smoothly, the liquid ‘fractures’ or ‘cracks’ open like a solid. It is known that when subjected to very high shear rates, liquids begin to behave mechanically like solids, for example, fracturing like a brittle solid. In our experiments, the point and time at which this ‘fracture’ occurred was just as the surfaces were about to separate from their most highly pointed configuration (Fig. 5C) - for had the separation velocity been any smaller than  $\nu_c$  they would have separated smoothly without fracturing. We consider that in the present case, the ‘fracturing’ or ‘cracking’ of the liquid between the surfaces must be considered synonymous with the “nucleation” or “inception” of a vapor cavity.”

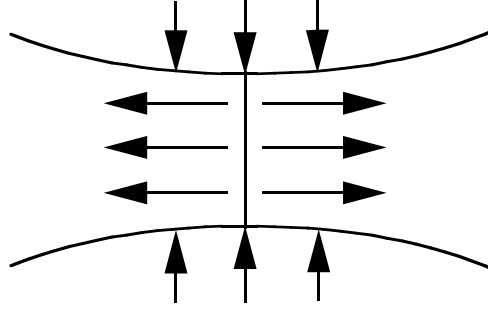
## Capillary Collapse and Rupture

It might be thought that flow stress induced cavitation is restricted to rather viscous liquids, where high stress levels can be achieved. However such high stresses can be reached even in water, as the following analysis of Lundgren & Joseph [1997] shows:

“The breakup of a liquid capillary filament is analyzed as a viscous potential flow near a stagnation point on the centerline of the filament towards which the surface collapses under the action of surface tension forces. We find that the neck is of parabolic shape and its radius collapses to zero in a finite time. During the collapse the tensile stress due to viscosity increases in value until at a certain finite radius, which is about 1.5 microns for water in air, the stress in the throat passes into tension, presumably inducing cavitation there.”

Potential flows satisfy the Navier-Stokes equations, though they slip at solid boundaries, the viscosity of the fluid never has to be and should not be, put to zero (see Joseph and Liao [1994], Liao and Joseph [1994] for a complete discussion). The flow at the stagnation point of a collapsing capillary can be treated as a viscous potential flow.

$$[u_z, u_r] = a(t)[z, -\frac{r}{2}] \quad \text{Stagnation flow}$$



$$\varphi = \frac{1}{2}az^2 - \frac{1}{4}ar^2 \quad \text{Potential flow}$$

Bernoulli Equations:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(u_r^2 + u_z^2) + \frac{p}{\rho} = \frac{p_0}{\rho}$$

$$\frac{p - p_0}{\rho} = - \left( \frac{1}{2}\dot{a} + \frac{1}{2}a^2 \right) z^2 + \left( \frac{1}{4}\dot{a} - \frac{1}{8}a^2 \right) r^2$$

$$T_{zz} = -p + 2\mu \frac{\partial u_z}{\partial z} = -p + 2\mu a, \quad \text{Stresses}$$

$$T_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} = -p - \mu a$$

On  $r = R(z, t) = R_0(t) + R_2(t)z^2 + O(z^4)$

$$\begin{aligned} -T_{nn} - p_a &= \sigma \kappa \\ T_{nn} &= n_r^2 T_{rr} + n_z^2 T_{zz} \end{aligned} \quad \left| \begin{array}{l} \text{Normal} \\ \text{Stress} \\ \text{balance} \end{array} \right.$$

$$\begin{aligned} \kappa &= - \frac{\frac{\partial^2 R}{\partial z^2}}{\left(1 + \left(\frac{\partial R}{\partial z}\right)^2\right)^{\frac{3}{2}}} + \frac{1}{R \left(1 + \left(\frac{\partial R}{\partial z}\right)^2\right)^{\frac{1}{2}}} \\ &= \frac{1}{R_0} - 2R_2 + O(Z^2) \end{aligned}$$

It's zero at  $z = 0$   
because  $\partial u_z / \partial r = 0$  | shear stress

$$\begin{array}{l|l}
u_r = \frac{\partial R}{\partial t} + u_z \frac{\partial R}{\partial z} & \text{Kinematic Condition} \\
-\frac{1}{2}aR = \frac{\partial R}{\partial t} + az \frac{\partial R}{\partial z} & \\
-\frac{1}{2}aR_0 = \overset{\circ}{R}_0 & \text{Solve for } a \\
-\frac{5}{2}aR_2 = \overset{\circ}{R}_2 & R_2 = CR_0^5
\end{array}$$

The parabola flattens as it collapses

$$a = -\frac{2\overset{\circ}{R}_0}{R_0}$$

To lowest order in  $z^2$

$$\frac{p_a - \sigma\kappa}{\rho} = \frac{T_{nn}}{\rho} = \frac{T_{rr}}{\rho} = -\frac{p}{a} - va$$

$$\begin{array}{l}
\frac{p_a}{\rho} + \frac{\sigma}{\rho} \left( \frac{1}{R_0} - 2R_2 \right) = -\frac{p_0}{\rho} - \left( \frac{1}{4}\overset{\circ}{a} - \frac{1}{8}a^2 \right) R_0^2 - va \\
R_2 = CR_0^5 \qquad a = 2\overset{\circ}{R}_0/R_0
\end{array}$$

“Rayleigh Plesset” type of equation:

$$\frac{p_0 - p_a}{\rho} - \frac{1}{2}R_0\ddot{R}_0 - \frac{2v\dot{R}_0}{R_0} = \frac{\sigma}{\rho} \left( \frac{1}{R_0} - 2CR_0^5 \right).$$

$$\overset{\circ}{R}_0 = \frac{1}{2} \frac{\sigma}{\rho v} \quad \begin{array}{l} \text{Small } R_0, \text{ balances} \\ \text{viscosity against inertia} \end{array}$$

$$\begin{array}{l|l}
\text{to leading} & R_0 = \frac{\sigma}{2\rho v}(t_* - t) \\
\text{order in} & a = \frac{2}{t_* - t} \\
t_* - t & \frac{p}{\rho} = \frac{p_0}{\rho} - \frac{3z^2}{(t_* - t)^2}
\end{array}$$

$R_0$  collapses to zero in finite time

## AXIAL STRESS (to leading order)

$$\frac{T_{zz}}{\rho} = -\frac{2p_0 - p_a}{\rho} + \frac{4v}{t_* - t} = -\frac{2p_0 - p_a}{\rho} + \frac{2\sigma}{\rho R_0(t)}$$

$T_{zz}$  turns positive for small  $R_0$

$$R_{ocr} = \frac{2\sigma}{2p_0 - p_a}$$

Estimating  $p_0 = p_a = 10^6$  dynes/cm<sup>2</sup>

$$\sigma = 75 \text{ dyne/cm}$$

$$R_0 = 1.5 \text{ microns}$$

At the collapse condition

the Reynolds number is

about 55 based on  $\dot{R}_0$

**The capillary thread cavitates before it collapses to zero.**

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