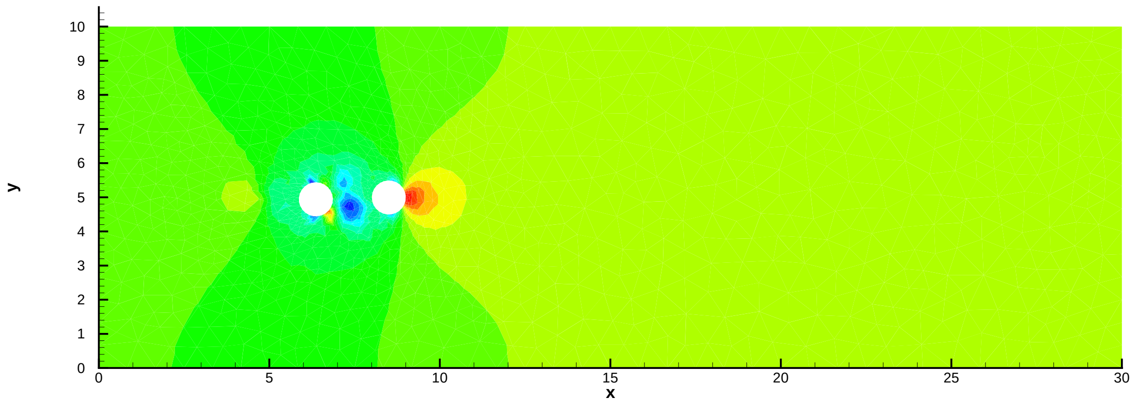

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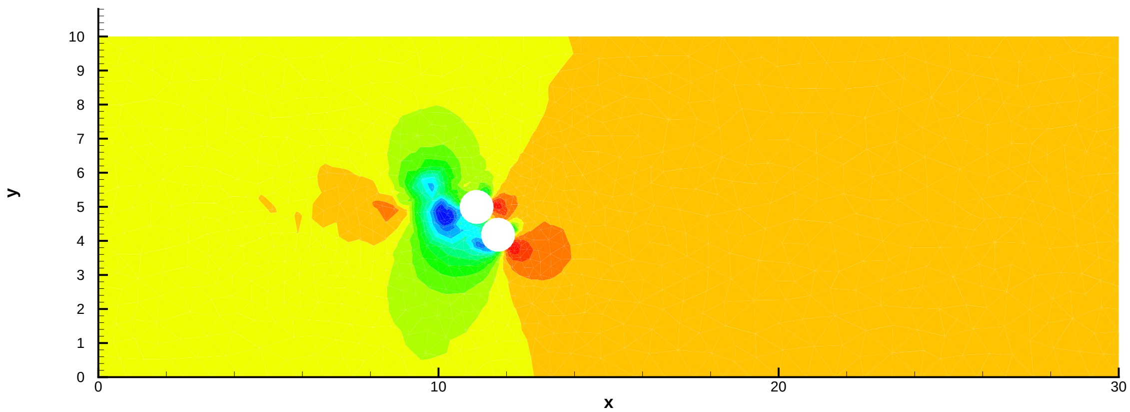

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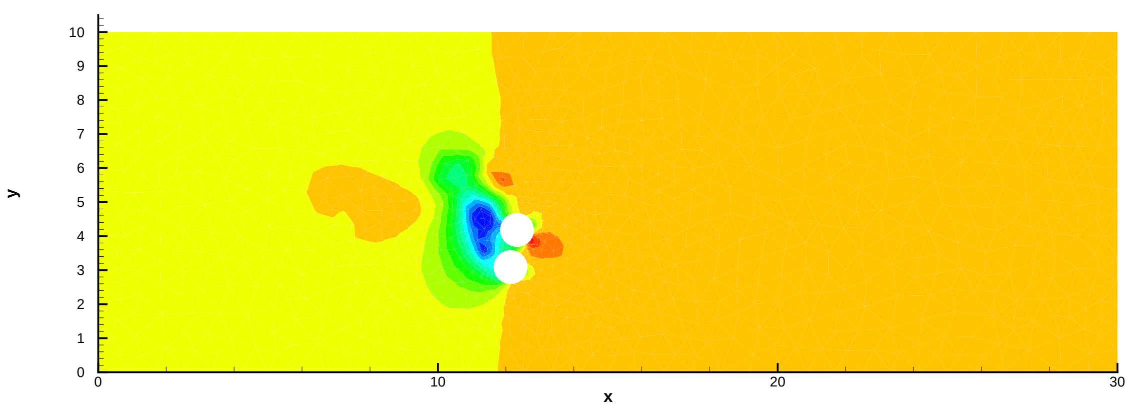

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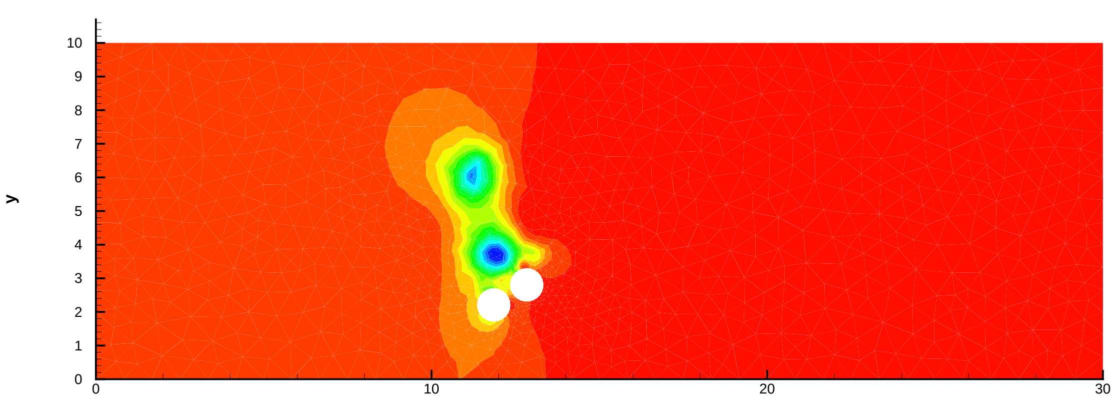

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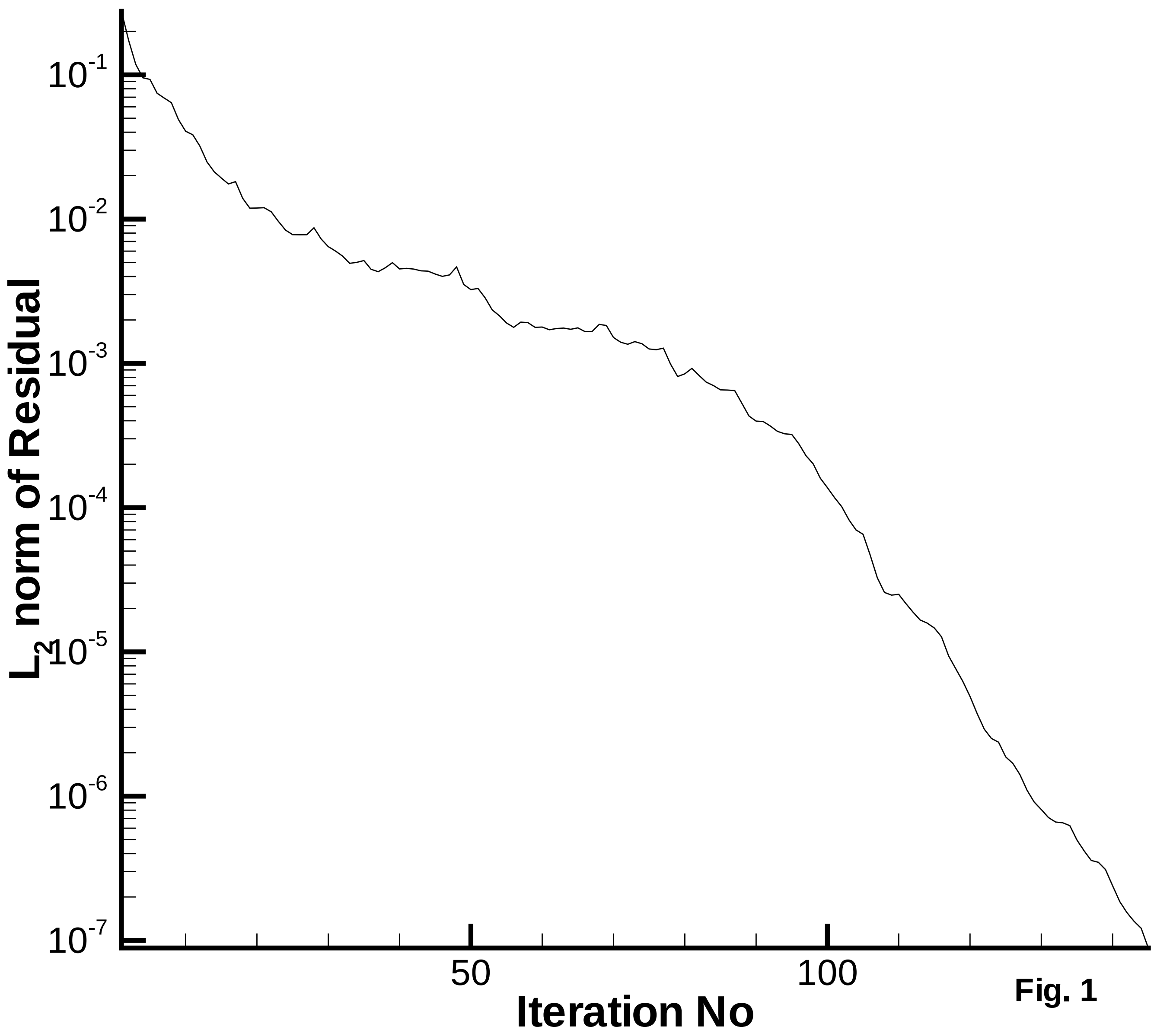

L_2 norm of Residual
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
50
100
Iteration No

Fig. 1

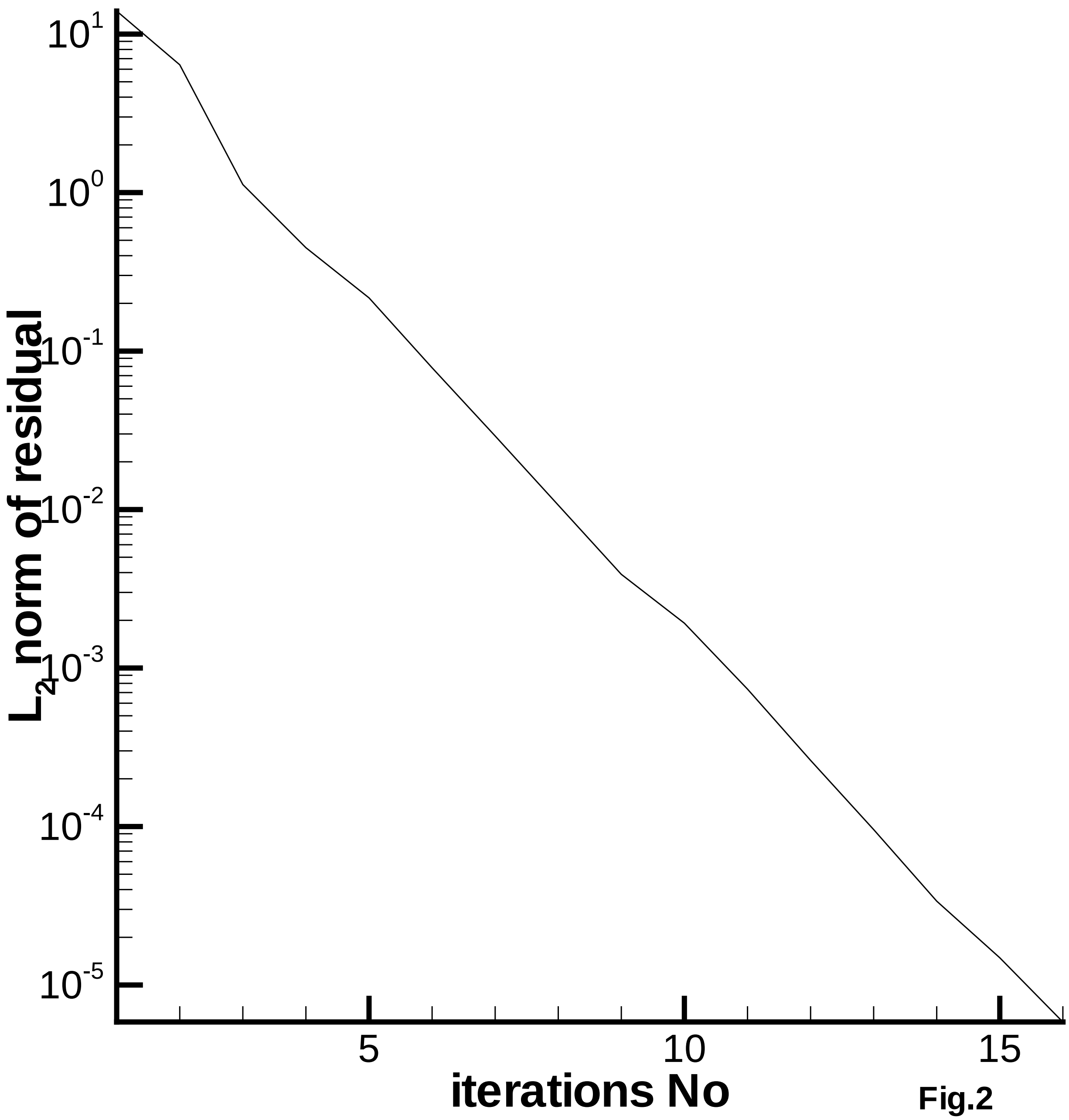
L_2 norm of residual
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
5
10
15
iterations No

Fig.2

Splitting method for particle mover code II

H.G Choi

Absract

In this report , the symmetric pressure equation for the splitting method of the two phase flow problem is presented. Through numerical experiment , it is found that the newly proposed splitting method works well with matrix-free formulation for some bench mark problems avoiding erroneous pressure field, which appears when using the conventional pressure equation. In the typical pressure equation of the splitting method (the old version tested by author), the motion of solid particle is treated in an explicit way, so that the particle moves by the known form drag(pressure drag) that is calculated from the pressure equation in the previous step. In other words, in obtaining the pressure field, the motion of solid particle is approximated by the 'intermediate velocity' instead of treating it as unknowns. From the numerical experiment, it was shown that this method gives erroneous pressure field even for the very small time step size as a particle velocity increases. Therefore, coupling the particle velocity unknowns in the pressure equation is proposed, where the resulting matrix is reduced to the symmetric one by applying the projector of combined formulation.[Knepley et al. ,Parallel simulation of particulate flows, 1998]. It has been tested over some bench mark problems and gives a reasonable pressure field.

Fractional step method

For the typical single phase flow problems, the SPD(symmetric positive definite) pressure equation is easily derived in the splitting formulation by imposing divergence free condition as follows:

The momentum equation is solved with the known pressure(or dropping the pressure gradient term) in the First step:

$$\begin{aligned} &\frac{\hat{u}_i - u_i^n}{\Delta t} + \hat{u}_j^g \hat{u}_{i,j} = \hat{\sigma}_{ij,j} + S_i \\ &M \frac{dU_p}{dt} = G_p \end{aligned} \qquad (1)$$

where $S_i =$ **external forces** , $\hat{\sigma}_{ij,j} = \nu D_{ij}(\mathbf{u})$ and combined formulation by Hesla is used .

Unlike conventional split method, all velocity components are calculated in a coupled manner due to the combined formulaiton. The Galerkin formulation of the above equation can be written as follows:

Find $u_i^h(\mathbf{x},t) \in H_h^1(\Omega)$ such that

$$\int_\Omega [w_i^h(\frac{\hat{u}_i - u_i^n}{\Delta t} + \hat{u}_j^g \hat{u}_{i,j}) + \nabla w_i^h : \hat{\sigma}_{ij}^h] d\Omega - \int_\Omega w_i^h \hat{\sigma}_{ij}^h n_j d\Gamma + \sum_p \delta U_p (M \frac{dU_p}{dt} - G_p) = \int_\Omega w_i^h S_i^h d\Omega$$

for all admissible weight function $w_i^h \in V_h$ with the following boundary conditions

$$\begin{aligned} &\boldsymbol{u} = \mathbf{g} && \text{on } \Gamma_{\mathbf{g}} \\ &\sigma \cdot \mathbf{n} = \boldsymbol{h} && \text{on } \Gamma_{\mathbf{h}} \\ &\boldsymbol{u} = \boldsymbol{U}_p + \Omega_p \times (\mathbf{x} - \mathbf{X}_p) && \text{on } \Gamma_{\mathbf{p}} \end{aligned}$$

where, $\mathbf{n}$ is the normal vector to the boundary Γ_{h} and the $\mathbf{x}$ is the coordinate of a node on the particle surface and $\mathbf{X}_p$ is the coordinate of the center of the particle and

$$\mathbf{V}_h = \{\boldsymbol{w}_h | \boldsymbol{w}_h \in H_h^1(\Omega), \boldsymbol{w}_h = 0 \textbf{ on } \Gamma_{\mathbf{g}},\ \boldsymbol{w}_h = \delta\mathbf{U}_p + \delta\boldsymbol{\Omega}_p \times (\mathbf{x} - \mathbf{X}_p)$$
$$\text{on } \Gamma_{\mathbf{p}} \text{ for } \mathrm{p} = 1, \ldots., \mathrm{N}_{\mathrm{p}}\}$$

In the first step, the resulting matrix is $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ [1].

A: mass matrix + convection matrix + diffusion matrix.

D: diagonal matrix

B: sparse matrix from the kinematic constraint

C: matrix from combined formulation.

Note:
Matrix [1] is well preconditioned even by the simple diagonal preconditioner. <u>Please note that in the present splitting code, most of CPU time is used for solving symmetric pressure equation. So, we need to focus on finding a effective preconditioner for the symmetric pressure equation</u>.

<u>Pressure Equation</u>

In the next step of splitting method, the pressure equation has to be obtained from divergence free condition and the velocity is updated accordingly by the new pressure.

$$\frac{u_i^{n+1} - \hat{u}_i}{\Delta t} = -\frac{1}{\rho} p_{,i}^{n+1} \qquad (2-1)$$

$$M \frac{dU_p}{dt} = F_p \text{(force from pressure gradient, but need not be calculated} \qquad (2-2)$$

explicitly due to combined formulation)

In the old splitting code, the pressure equation was derived by imposing divergence-free condition on Eq.(2-1), where the particle velocity(U^{n+1}) is approximated by intermediate particle velocity($\hat{U}$), as follows:

$$\frac{\Delta t}{\rho} \int_{\Omega} w_{,i} p_{,i}^{n+1} d\Omega = \int_{\Omega} w_{,i} \hat{u}_i d\Omega - \int_{\Gamma} w u_i^{n+1} n_i d\Gamma \qquad (3)$$

After solving Eq.(3) by Conjugate Gradient, the Eq.(2-1,2) is solved by combined formulation. In the previous numerical experiment, it was found to give an erroneous pressure field. The main reason is that the divergence free is not exactly satisfied after solving Eq.(2) with the known pressure obtained from Eq.(3) since the unknown particle velocity(U^{n+1}) has to be approximated by intermediate particle velocity($\hat{U}$) on the particle surface.

In the new splitting code, the particle velocity is solved with the fluid variables at the same time so that the motion of particle can be linked with pressure equation implicitly. The governing equation is the coupled equation for u_i, U_p, p at time step n+1.

$$\frac{u_i^{n+1} - \hat{u}_i}{\Delta t} = -\frac{1}{\rho} p_{,i}^{n+1} \qquad (3)$$

$$M \frac{U_p^{n+1} - \hat{U}_p}{dt} = F_p$$

$$u_{i,i}^{n+1} = 0$$

Applying weak formulation, the resulting matrix is

$$\begin{bmatrix} A & -B & C \\ B^t & 0 & 0 \\ D & E & F \end{bmatrix} \begin{bmatrix} u_i \\ p \\ U \end{bmatrix} = \begin{bmatrix} f_u \\ 0 \\ f_U \end{bmatrix} \qquad (4)$$

where A is the mass matrix.Note that the resulting matrix is the unsymmetric one due to the combined formulation. Here, matrix (4) can be transformed into symmetric one using the projector [Knepley et al.] (for details refer Knepley et al., Parallel simulation of particulate flows,1998) .

In their approach, the fluid velocity variables are divided into the internal variables and the variables on the particle surface, $u = [u_I, u_\Gamma]$..
Then, the resulting matrix for the combined formulation can be rewritten as follows:

$$\begin{bmatrix} A_I & A_{I\Gamma} & -B_I & 0 \\ A_{\Gamma I} & A_\Gamma & -B_\Gamma & 0 \\ B_I^t & B_\Gamma^t & 0 & 0 \\ 0 & 0 & 0 & F \end{bmatrix} \begin{bmatrix} u_I \\ u_\Gamma \\ p \\ U \end{bmatrix} = \begin{bmatrix} f_u \\ f_\Gamma \\ 0 \\ f_U \end{bmatrix}$$

Here, applying the kinematic constraint on the particle surface, $u_\Gamma = PU$, the following matrix is obtained:

$$\begin{bmatrix} F + P^t A_\Gamma P & P^t A_{\Gamma I} & -P^t B_\Gamma & 0 \\ A_{I\Gamma} P & A_I & -B_I & 0 \\ B_\Gamma^t P & B_I^t & 0 & 0 \\ -P & 0 & 0 & I \end{bmatrix} \begin{bmatrix} U \\ u_I \\ p \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} f_U + P^t f_\Gamma \\ f_u \\ 0 \\ 0 \end{bmatrix} \qquad (5)$$

where, projector P is defined as

$$P^u = \begin{bmatrix} 1 & 1 & \ldots\ldots & 1 & 1 & 1 \\ -dy_1 & -dy_2 & \ldots\ldots & & -dy_{n-1} & -dy_n \end{bmatrix}$$

$$P^v = \begin{bmatrix} 1 & 1 & \ldots\ldots & 1 & 1 & 1 \\ dx_1 & dx_2 & \ldots\ldots & & dx_{n-1} & dx_n \end{bmatrix} \text{ n is the No. of node on the particle surface}$$

Note that P is set up on the element basis since in the present study matrix-free approach is used. Rewriting the matrix (5), we get

$$\begin{bmatrix} \tilde{A} & -\tilde{B} \\ \tilde{B}^t & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ p \end{bmatrix} = \begin{bmatrix} \tilde{f}_u \\ 0 \end{bmatrix} \qquad (6) \qquad \text{with } u_\Gamma = PU$$

$$\tilde{A} = \begin{bmatrix} F + P^t A_\Gamma P & P^t A_{\Gamma I} \\ A_{I\Gamma} P & A_I \end{bmatrix} : \text{SPD matrix(diagonally dominant)}$$

$$\tilde{B} = \begin{bmatrix} P^t B_\Gamma \\ B_I \end{bmatrix} \qquad : \text{transformed gradient matrix}$$

Matrix (6) is the special case of the saddle point problem $\begin{pmatrix} A & B \\ B^t & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$, where c=0 and *A* is the SPD matrix…

Now, we focus on how to effectively solve the Matrix (6). In the present study, from (6) the following symmetric pressure equation is derived:

$$\tilde{B}^t \tilde{A}^{-1} \tilde{B} p = -\tilde{B}^t \tilde{A}^{-1} \tilde{f}_u \qquad (7)$$

In order to solve Eq.(7), presently CG is used without any preconditioner (Incomplete factorization of $\tilde{B}^t \tilde{B}$ can be considered for future study ..or Could you suggest any other good preconditioner for Eq. (7) ?) Fig.1 shows the convergence history of solving Eq.(7) with CG for 2 particles' sedimentation. The number of pressure unknown is about 1,200 and the Reynolds number is about 1,000.

When solving Eq.(7), $\tilde{A}x = y$ is to be solved …In the present study, it is well solved with diagonal preconditioned CG since $\tilde{A}$ is highly diagonally dominant matrix. Fig.2 shows the corresponding convergence history at the 1st CG iteration of pressure equation.

For the following iterations, it converges within about 5 iterations.

After solving Eq.(7), the fluid and particle velocity is obtained in the post-processing part by solving the following equation with the
given boundary condition.

$$\tilde{A}\tilde{u} = \tilde{f}_u + \tilde{B}p$$

In the present numerical experiment, most of CPU time is consumed in solving Eq.(7)…
Therefore, we need to focus on how to effectively solve Eq. (7) …Preconditioned Uzawa method or Arrow-Hurwicz ….Any suggestion?