Viscous Potential Flow Analysis of Kelvin-Helmholtz Instability

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It is well known that the Navier-Stokes equations are satisfied by potential flow; the viscous term is identically zero when the vorticity is zero but the viscous stresses are not zero [Joseph and Liao 1994]. It is not possible to satisfy boundary conditions when the flow is given by a potential. Even for free surface problems, where one of the fluids is effectively a vacuum and is therefore "stressless," the condition of a zero shear stress cannot be exactly satisfied by potential flow [Lundgren 1989]. Nevertheless we propose to study interface problems by assuming potential flow completely neglecting continuity of tangential stresses (which gives too many boundary conditions for potential flow). This procedure leads to a very accurate approximation of the fully viscous Rayleigh-Taylor instability problem [Joseph, Belanger and Beavers 1999]. The viscous stresses enter the problem only through the normal stress at an interface with normal $\boldsymbol{n}$

$$T_{nn} = -p + 2\mu \boldsymbol{n} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{n}. \tag{1}$$

The balance of normal stresses at an interface requires $T_{nn1} = T_{nn2}$.

As an example we consider the classical combined Rayleigh-Taylor, Kelvin-Helmholtz instability when potential flow is assumed, but viscous stresses are not put to zero. We follow the analysis of Drazin and Reid [Drazin and Reid 1982] with two uniform parallel flows along the x axis. The upper fluid has velocity U_2, density ρ_2, viscosity μ_2 while the lower fluid has U_1, ρ_1, μ_1. We assume that the small perturbations in each fluid are irrotational. Then, following the analysis of Drazin and Reid except for replacing the pressure balance by a viscous normal force balance;

$$p_2' - 2\mu_2 \frac{\partial w_2'}{\partial z} = p_1' - 2\mu_1 \frac{\partial w_1'}{\partial z}. \tag{2}$$

Taking the perturbation in the form of functions of z (the normal direction) times $\exp(\sigma t + \imath k x)$ we find the dispersion relation

$$(\rho_2 + \rho_1)\sigma^2 + 2\sigma\left[\imath k(\rho_2 U_2 + \rho_1 U_1) + k^2(\mu_2 + \mu_1)\right] - k^2\left(\rho_2 U_2^2 + \rho_1 U_1^2\right) + 2\imath k^3(\mu_2 U_2 + \mu_1 U_1)$$
$$+ (\rho_1 - \rho_2) g k + \gamma k^3 = 0 \tag{3}$$

where in addition to the parameters already defined γ is surface tension and g the gravitational constant. Gravity acts downward from side 2 towards side 1. The real part of σ is the growth rate given as a function of the wavenumber k.

This may then be arranged as

$$A\sigma^2 + 2B\sigma + C = 0, \tag{4}$$

where

$$A = \rho_2 + \rho_1, \quad B = \imath k \left(\rho_2 U_2 + \rho_1 U_1\right) + k^2 \left(\mu_2 + \mu_1\right), \tag{5a, b}$$

$$C = \left(\rho_1 - \rho_2\right) gk - k^2 \left(\rho_2 U_2^2 + \rho_1 U_1^2\right) + 2\imath k^3 \left(\mu_2 U_2 + \mu_1 U_1\right) + \gamma k^3, \tag{5c}$$

for which the solution σ may be expressed as

$$\sigma = -\frac{B}{A} \pm \sqrt{\frac{B^2}{A^2} - \frac{C}{A}}. \tag{6}$$

0.1 Cut-off wavenumber

Equation (6) shows that all wavenumbers below a cut-off wavenumber lead to instability. To find this number we set $\sigma = \sigma_R + \imath\sigma_I$ and identify the real and imaginary parts of (4):

$$\left(\rho_2 + \rho_1\right)\left(\sigma_R^2 - \sigma_I^2\right) + 2\sigma_R k^2 \left(\mu_2 + \mu_1\right) - 2\sigma_I k \left(\rho_2 U_2 + \rho_1 U_1\right) - k^2 \left(\rho_2 U_2^2 + \rho_1 U_1^2\right)$$

$$+ \left(\rho_1 - \rho_2\right) gk + \gamma k^3 = 0, \tag{7a}$$

$$\sigma_I = -\frac{\sigma_R k \left(\rho_2 U_2 + \rho_1 U_1\right) + k^3 \left(\mu_2 U_2 + \mu_1 U_1\right)}{\left(\rho_2 + \rho_1\right)\sigma_R + k^2 \left(\mu_2 + \mu_1\right)}. \tag{7b}$$

Eliminating σ_I between (7a) and (7b) we get the quartic

$$a_4 \sigma_R^4 + a_3 \sigma_R^3 + a_2 \sigma_R^2 + a_1 \sigma_R + a_0 = 0, \tag{8}$$

where the coefficients are given by

$$a_4 = \left(\rho_2 + \rho_1\right)^3, \quad a_3 = 4k^2 \left(\rho_2 + \rho_1\right)^2 \left(\mu_2 + \mu_1\right), \tag{9a, b}$$

$$a_2 = 5k^4 \left(\rho_2 + \rho_1\right)\left(\mu_2 + \mu_1\right)^2 + k^2 \left(\rho_2 + \rho_1\right)\left(\rho_2 U_2 + \rho_1 U_1\right)^2$$

$$+ \left(\rho_2 + \rho_1\right)^2 \left[\left(\rho_1 - \rho_2\right) gk + \gamma k^3 - k^2 \left(\rho_2 U_2^2 + \rho_1 U_1^2\right)\right], \tag{9c}$$

$$a_1 = 2k^6 \left(\mu_2 + \mu_1\right)^3 + 2k^2 \left(\rho_2 + \rho_1\right)\left(\mu_2 + \mu_1\right)\left[\left(\rho_1 - \rho_2\right) gk + \gamma k^3 - k^2 \left(\rho_2 U_2^2 + \rho_1 U_1^2\right)\right]$$

$$+ 2k^4 \left(\rho_2 U_2 + \rho_1 U_1\right)^2 \left(\mu_2 + \mu_1\right), \tag{9d}$$

$$a_0 = -k^6 \left(\rho_2 + \rho_1\right)\left(\mu_2 U_2 + \mu_1 U_1\right)^2 + k^4 \left(\mu_2 + \mu_1\right)^2 \left[\left(\rho_1 - \rho_2\right) gk + \gamma k^3 - k^2 \left(\rho_2 U_2^2 + \rho_1 U_1^2\right)\right]$$

$$+ 2k^6 \left(\rho_2 U_2 + \rho_1 U_1\right)\left(\mu_2 U_2 + \mu_1 U_1\right)\left(\mu_2 + \mu_1\right). \tag{9e}$$

The cut-off wavenumbers are those for which $k = k_C$ and $a_0(k_C) = 0$. One root is $k_C = 0$. Other roots are non-zero roots of $a_0(k_C) = 0$, for which $a_0(k)/k^5$ leads to a quadratic

$$-k\frac{\left(\rho_2 \mu_1^2 + \rho_1 \mu_2^2\right)\left(U_2 - U_1\right)^2}{\left(\mu_2 + \mu_1\right)^2} + \left(\rho_1 - \rho_2\right) g + \gamma k^2 = 0, \tag{10}$$

whose solutions are given by

$$k_C = \frac{1}{2\gamma}\frac{\left(\rho_2 \mu_1^2 + \rho_1 \mu_2^2\right)\left(U_2 - U_1\right)^2}{\left(\mu_2 + \mu_1\right)^2} \pm \frac{1}{2\gamma}\sqrt{\frac{\left(\rho_2 \mu_1^2 + \rho_1 \mu_2^2\right)^2 \left(U_2 - U_1\right)^4}{\left(\mu_2 + \mu_1\right)^4} - 4\gamma\left(\rho_1 - \rho_2\right) g}, \tag{11a, b}$$

when $\rho_1 - \rho_2 \geq 0$ and the term in the square root is not negative. When $\rho_1 = \rho_2$, we have

$$k_C = \frac{1}{\gamma}\frac{\left(\rho_2\mu_1^2 + \rho_1\mu_2^2\right)\left(U_2 - U_1\right)^2}{\left(\mu_2 + \mu_1\right)^2}, \quad k_C = 0, \tag{11c, d}$$

and when $\rho_1 - \rho_2 < 0$, the cut-off wavenumber is

$$k_C = \frac{1}{2\gamma}\frac{\left(\rho_2\mu_1^2 + \rho_1\mu_2^2\right)\left(U_2 - U_1\right)^2}{\left(\mu_2 + \mu_1\right)^2} + \frac{1}{2\gamma}\sqrt{\frac{\left(\rho_2\mu_1^2 + \rho_1\mu_2^2\right)^2\left(U_2 - U_1\right)^4}{\left(\mu_2 + \mu_1\right)^4} - 4\gamma\left(\rho_1 - \rho_2\right)g}. \tag{11e}$$

When $\gamma = 0$ and $\rho_1 - \rho_2 > 0$ in (10), we have the cut-off wavenumber

$$k_C = \frac{\left(\mu_2 + \mu_1\right)^2}{\left(\rho_2\mu_1^2 + \rho_1\mu_2^2\right)\left(U_2 - U_1\right)^2}\left(\rho_1 - \rho_2\right)g. \tag{12}$$

Special case of $(\mu_2 U_2 + \mu_1 U_1) = 0$ If $(\mu_2 U_2 + \mu_1 U_1) = 0$, we have from (10) the quadratic equation:

$$\left(\rho_1 - \rho_2\right)g + \gamma k^2 - k\left(\rho_2 U_2^2 + \rho_1 U_1^2\right) = 0. \tag{13}$$

This gives the cut-off wavenumber as, for $\rho_1 > \rho_2$ and $\left(\rho_2 U_2^2 + \rho_1 U_1^2\right)^2 > 4\gamma\left(\rho_1 - \rho_2\right)g$

$$k_C = \frac{\left(\rho_2 U_2^2 + \rho_1 U_1^2\right)}{2\gamma} \pm \frac{1}{2\gamma}\sqrt{\left(\rho_2 U_2^2 + \rho_1 U_1^2\right)^2 - 4\gamma\left(\rho_1 - \rho_2\right)g}, \tag{14a, b}$$

for $\rho_1 = \rho_2$

$$k_C = \frac{\left(\rho_2 U_2^2 + \rho_1 U_1^2\right)}{\gamma}, \quad k_C = 0, \tag{14c, d}$$

and for $\rho_1 < \rho_2$

$$k_C = \frac{\left(\rho_2 U_2^2 + \rho_1 U_1^2\right)}{2\gamma} + \frac{1}{2\gamma}\sqrt{\left(\rho_2 U_2^2 + \rho_1 U_1^2\right)^2 - 4\gamma\left(\rho_1 - \rho_2\right)g}. \tag{14e}$$

Special case of $U_2 = U_1$ In (10), we recall that for $U_2 = U_1$ and $\rho_2 > \rho_1$ we have the following equation for the cut-off wavenumber:

$$\left(\rho_1 - \rho_2\right)g + \gamma k_C^2 = 0, \tag{15}$$

as was seen in the R-T instability.

When there is $k_C > 0$ such that $\sigma_R = 0$, then since $\sigma_R = 0$ when $k = 0$ we have, at least, a maximum growth rate.

0.2 Asymptotic forms of σ for large k

Hadamard instability refers to situations where the positive growth rate tends to infinity as the wavelength $\lambda = 2\pi/k$ tends to zero. The amplitude of the perturbation tends to infinity at any fixed time no matter how small. Hadamard instabilities raise havoc in numerical analysis since improving the numerical resolution makes perturbations grow faster. Examples of Hadamard instabilities are the R-T instability ($\mu_2 = \mu_1 = 0$, $U_2 = U_1 = 0$, $\gamma = 0$, $\rho_1 > \rho_2$) which has $Re\{\sigma\} \propto \sqrt{k}$ and the K-H instability ($\mu_2 = \mu_1 = 0$, $U_2 \neq U_1$, $\gamma = 0$) which has $Re\{\sigma\} \propto k$. Both R-T and K-H are stable for large wavenumbers when surface tension is included; both are unstable for some wavenumbers but the growth rate goes to zero as $k \to \infty$. In the case under consideration, the highest power of k occurs in the viscous terms, for which it is to be expected that they have a remarkable effect on stability when the wavenumber is large.

Seeking asymptotic forms of σ for large k, we have from (4)

$$\sigma_+ = -k\left(\frac{\gamma}{2\left(\mu_2+\mu_1\right)} + \imath\frac{\mu_2 U_2 + \mu_1 U_1}{\mu_2+\mu_1}\right)$$

$$-\frac{\left(\rho_2+\rho_1\right)\gamma^2}{8\left(\mu_2+\mu_1\right)^3} + \frac{\left(\rho_2\mu_1^2+\rho_1\mu_2^2\right)}{2\left(\mu_2+\mu_1\right)^3}\left(U_2-U_1\right)^2 - \imath\gamma\frac{\left(\rho_1\mu_2-\rho_2\mu_1\right)}{2\left(\mu_2+\mu_1\right)^3}\left(U_2-U_1\right), \tag{16a}$$

$$\sigma_- = -2k^2\frac{\mu_2+\mu_1}{\rho_2+\rho_1} + \frac{\gamma k}{2\left(\mu_2+\mu_1\right)} - \imath k\left[2\frac{\left(\rho_2 U_2+\rho_1 U_1\right)}{\left(\rho_2+\rho_1\right)} - \frac{\left(\mu_2 U_2+\mu_1 U_1\right)}{\left(\mu_2+\mu_1\right)}\right]$$

$$+\frac{\left(\rho_2+\rho_1\right)\gamma^2}{8\left(\mu_2+\mu_1\right)^3} - \frac{\left(\rho_2\mu_1^2+\rho_1\mu_2^2\right)}{2\left(\mu_2+\mu_1\right)^3}\left(U_2-U_1\right)^2 + \imath\gamma\frac{\left(\rho_1\mu_2-\rho_2\mu_1\right)}{2\left(\mu_2+\mu_1\right)^3}\left(U_2-U_1\right). \tag{16b}$$

The stabilizing effect of the surface tension is confirmed here. Even when $\gamma = 0$, $Re\{\sigma_+\}$ is found to be a positive constant. This is due to the viscous potential flow.

A typical case As a typical case, we may set as

$$\rho_1 = \rho_2 = \rho, \quad \mu_1 = \mu_2 = \rho\nu, \quad U_2 = -U_1 = -U, \quad \gamma = 0, \tag{17}$$

then we have from (16a,b) the solutions

$$\sigma_+ = \frac{U^2}{2\nu}, \quad \sigma_- = -2\nu k^2 - \frac{U^2}{2\nu}. \tag{18}$$

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