A note on dissipation approximation and viscous potential flow

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Abstract

Dissipation approximation (using inviscid potential flow) and viscous potential flow calculations are compared in two cases: a rising spherical bubble in a liquid and the decay of gravity waves on water. The resulting drag force on the rising bubble by the dissipation approximation is $D = 12\pi a\mu U$. The viscous potential flow solution gives rise to: a shear stress, a viscous normal stress and a pressure. The drag due to the normal stress is $D_N = 8\pi a\mu U$; the drag due to the shear stress is $D_S = -8\pi a\mu U$. Hence, D-Alembert's paradox holds in viscous potential flow even though the dissipation $\mathcal{D}$ is not zero. The dissipation $\mathcal{D}$ is always equal to the power $\mathcal{P}$ of the traction vector; hence the dissipation approximation can be called a power of traction approximation. Following Kang and Leal (1988), we note that the viscous correction to the pressure gives rise to a drag $D = D_N + D_v = 12\pi a\mu U$ and find that $D_v = -D_S/2$ is the drag due to the viscous correction to the pressure; in this case $DU = \mathcal{D} = \mathcal{P}$. The decay rate of gravity waves on water is calculated using viscous potential flow which is embedded in the analysis of Kelvin-Helmholtz instability by Funada and Joseph (2001). The rate is one-half of the decay rate calculated using the dissipation approximation by Lamb (1924). Direct calculation of the effect of viscosity on water-waves, also given by Lamb (1924), agrees with the dissipation approximation for long waves and agrees with viscous potential flow for short waves.

Introduction

Inviscid potential flow is a special case of viscous potential flow in which the viscosity is put to zero. Viscous potential flow gives rise to better results than inviscid potential flow where "better" is relative to experiments or to solutions for which potential flow is not assumed. As a matter of principle vorticity will always be generated at an interface when the no-slip condition is applied. In many cases this vorticity is confined to a boundary layer whose effects are important in some cases or for some solutions properties and not important for others. An understanding of when and where these effects are important is of interest.

Dissipation approximations are one way in which the effects of vorticity layers can be determined without actually calculating the layers. As far as we know there are only two cases in which the dissipation approximation has been applied: to the rise velocity of a spherical gas bubble and to the decay of gravity waves on water. Both cases present situations in which the effects of vorticity layers are important as well as those for which it is unimportant. The analysis of the relation of the dissipation approximation to viscous potential flow given below will point to the kind of discriminations which ought to be better understood.

Rising bubbles in a liquid

The mechanical energy equation for the Navier-Stokes equations is

$$\frac{d}{dt}\int_V \frac{\rho u^2}{2}dV = \int_{A'} \mathbf{u}\cdot(\mathbf{T}\cdot\mathbf{n})\,\mathrm{d}A' + \int_A \mathbf{u}\cdot(\mathbf{T}\cdot\mathbf{n})\,\mathrm{d}A - \int_V 2\mu\,\mathbf{D}:\mathbf{D}\,\mathrm{d}V \qquad (1)$$

where

$$\mathbf{T} = -p\mathbf{1} + 2\mu\,\mathbf{D}[\mathbf{u}]$$

is the stress, $\mathbf{D}[\mathbf{u}]$ is the rate of strain tensor, V is the control volume, A is the interface between the sphere and the fluid, and A' is the outer boundary of V. Our reference frame is fixed on the far-field fluid and it can be shown that the kinetic energy flux across the boundaries is zero and $\int_{A'} \mathbf{u}\cdot(\mathbf{T}\cdot\mathbf{n})\,\mathrm{d}A' = 0$. For steady flow, equation (1) becomes,

$$\mathcal{P} = \int_A \mathbf{u}\cdot(\mathbf{T}\cdot\mathbf{n})\,\mathrm{d}A = \int_V 2\mu\,\mathbf{D}:\mathbf{D}\,\mathrm{d}V = \mathcal{D}$$

where $\mathcal{P}$ is the power of traction and $\mathcal{D}$ is the dissipation. The dissipation term was used by Levich (1949) to compute the rise velocity of a spherical gas bubble. He writes

$$DU = 2\mu\int_V \mathbf{D}:\mathbf{D}\,\mathrm{d}V = 2\mu\int_V \frac{\partial^2\phi}{\partial x_i\partial x_j}\frac{\partial^2\phi}{\partial x_i\partial x_j}\mathrm{d}V \tag{2}$$

where U is the rise velocity of the gas bubble and $D = \int \mathbf{e}_x\cdot\mathbf{T}\cdot\mathbf{n}\,\mathrm{d}A$ is the drag, A is the surface of the sphere and $\mathbf{u} = \nabla\phi$ where

$$\phi = -\frac{1}{2}Ua^3\frac{\cos\theta}{r^2} \tag{3}$$

is the potential for flow around a sphere. After putting (3) in (2), he found that

$$D = 12\pi a\mu U, \quad C_D = 48/R \tag{4}$$

where a is the radius of the bubble and $R=2a\rho U/\mu$ is the Reynolds number. Equation (4) is the drag result of the dissipation approximation.

Analysis of the rising bubble based on viscous potential flow is the same as for inviscid potential flow except that the viscous contribution to the normal stress balance must be included. Moore (1959) applied the normal stress boundary condition to a spherical bubble using (3) and found

$$T_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r} = -p_p - 6\left(\frac{\mu U}{a}\right)\cos\theta \tag{5}$$

where p_p is the pressure from the potential flow solution; he put the tangential stress of the potential flow on the bubble surface as zero, and computed

$$D = 8\pi\mu Ua, \; C_D = 32/\mathrm{R}. \tag{6}$$

Equation (6) neglects the contribution of the shear stress which is not zero for viscous potential flow, but is zero for an actual gas bubble. G. K. Batchelor pointed out that the discrepancy between (4) and (6) was due to the neglect of a pressure correction arising from viscous effect within the vorticity layer next to the surface of the bubble. In a later paper, Moore (1963) reported a complete boundary layer analysis including the viscous pressure correction and obtained C_D = 48/R using a momentum defect argument. Kang and Leal (1988) used another method to account for the viscous pressure correction and also obtained C_D = 48/R by direct integration of the normal viscous stress and pressure over the bubble surface.

The tangential stress corresponding to (5) does not vanish,

$$T_{r\theta} = -3\left(\frac{\mu U}{a}\right)\sin\theta . \tag{7}$$

We verify that the power of traction is equal to the dissipation $\mathcal{D} = -12\pi\mu a U^2$ in two ways. First using (5) and (7)

$$\int_A \mathbf{u}\cdot(\mathbf{T}\cdot\mathbf{n})\,\mathrm{d}A = \int_A u_r T_{rr}\,\mathrm{d}A + \int_A u_\theta T_{r\theta}\,\mathrm{d}A = -12\pi\mu a U^2 . \tag{8}$$

Second we put the tangential stress to be zero as it actually is for a gas bubble, but take the pressure correction term into account. Following Kang and Leal (1988), the dimensionless total normal stress is

$$\begin{aligned} T_{rr}^{*} &= -p_p - p_v + \frac{4}{R}\frac{\partial u_{pr}}{\partial r} + o(R^{-1}) \\ &= -p_p + \frac{18}{R}\cos\theta + \sum_{n\neq 1} C_n P_n(\cos\theta) + o(R^{-1}) \end{aligned} \tag{9}$$

where p_p is the pressure computed from the potential flow and p_v is the viscous pressure correction term. The power of traction is calculated solely from the normal stress,

$$\begin{aligned} \int_A \mathbf{u}\cdot(\mathbf{T}\cdot\mathbf{n})\,\mathrm{d}A &= \int_A u_r T_{rr}\,\mathrm{d}A \\ &= \int_A U\cos\theta \frac{9\mu U\cos\theta}{a}\,\mathrm{d}A = 12\pi\mu a U^2 \end{aligned} . \tag{10}$$

Hence, the two ways to calculate the power of traction both give consistent results as the dissipation method (note that the different signs of (8) and (10) are due to different reference frames). We could call the "dissipation approximation" a "power of the traction approximation".

Turning now to the direct calculation of the drag, using (5) and (7) from viscous potential flow, we get

$$\begin{aligned} D &= \int \mathbf{e}_x\cdot\mathbf{T}\cdot\mathbf{n}\,\mathrm{d}A = \int T_{rr}\cos\theta\,\mathrm{d}A - \int T_{r\theta}\sin\theta\,\mathrm{d}A \\ &= D_N + D_S = 8\pi\mu a U - 8\pi\mu a U = 0 . \end{aligned} \tag{11}$$

Equation (11) shows that the D-Alembert's paradox holds in viscous potential flow even though the dissipation is not zero. Following Kang and Leal (1988), we put tangential stress to zero and compute the drag with pressure correction,

$$\begin{aligned} D &= \int \mathbf{e}_x\cdot\mathbf{T}\cdot\mathbf{n}\,\mathrm{d}A = \int \tau_{rr}\cos\theta\,\mathrm{d}A + \int -p_v\cos\theta\,\mathrm{d}A \\ &= D_N + D_v = 8\pi\mu a U + 4\pi\mu a U = 12\pi\mu a U . \end{aligned} \tag{12}$$

It is noted the contribution to the drag from the pressure correction is half of that from the tangential shear stress obtained in viscous potential flow $D_v = -D_S/2$.

It should be noted that to compute the drag on a sphere by the dissipation approximation, one needs to use the total dissipation inside and outside the sphere. In the case of a solid sphere or a gas bubble, the dissipation comes completely from the flow outside the sphere. In the case of a Hadamard-Rybczynski droplet, the dissipation comes from the flow outside and inside of the sphere,

$$DU = 6\pi\mu a U^2 \frac{1+(2/3)\sigma}{1+\sigma} = \mathcal{D}_t \tag{13}$$

$$\mathcal{D}_t = \mathcal{D}_o + \mathcal{D}_i = 6\pi\mu a U^2 \left[\frac{1+(2/3)\sigma}{1+\sigma} - \frac{\sigma/3}{(1+\sigma)^2}\right] + 6\pi\mu a U^2 \frac{\sigma/3}{(1+\sigma)^2} \tag{14}$$

where $\sigma \equiv \mu/\mu_i$ is the viscosity ratio, and $\mathcal{D}_t$, $\mathcal{D}_o$ and $\mathcal{D}_i$ are total dissipation, dissipation outside the sphere and dissipation inside the sphere.

The discrepancy of the drag force computed by viscous potential flow and by dissipation approximation indicates that vorticity layer has a strong effect on rising spherical bubbles. Large gas bubbles do not stay spherical; instead they take the lenticular shape of a spherical cap bubble. The vorticity layers do not appear to strongly affect spherical cap bubbles. Davies and Taylor (1950) showed that the rise velocity of such a bubble could be obtained from a local analysis without using a drag balance, noting that the nose of the bubble is spherical as a result of the pressure generated by motion, without surface tension. Joseph (2002) generalized their inviscid potential flow result to include effects of viscosity, surface tension and the deviation of the bubble nose from sphericity using viscous potential flow and he obtained a hyperbolic drag law

$$C_D = 6 + 32/R. \tag{15}$$

Davies and Taylor (1950) result, and the drag formula (15) of viscous potential flow, are in excellent agreement with experiments reported by Bhaga and Weber (1981) after (15) is scaled so that the effective diameter used in the experiments and the spherical cap radius of Taylor are the same (see Figure 1).

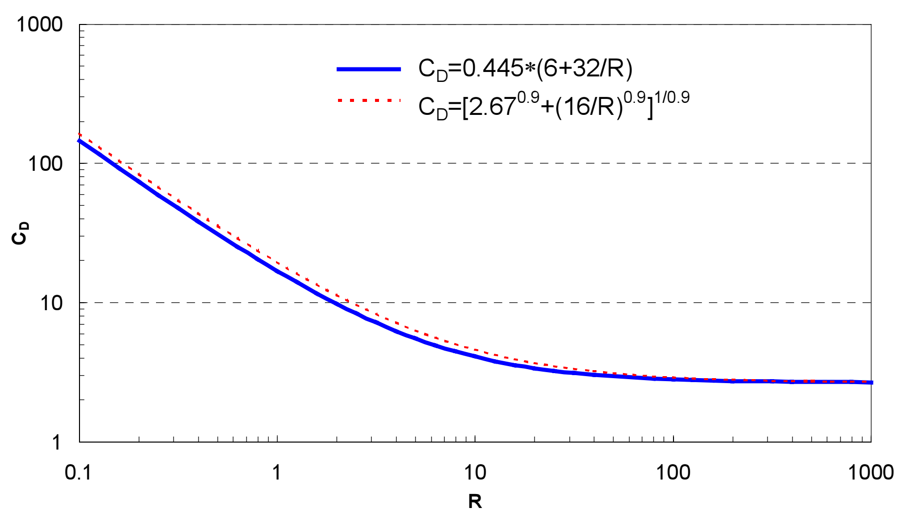

Figure 1. Comparison of the empirical drag law with the theoretical drag law (15) scaled by the factor 0.445 required to match the experimental data reported by Bhaga and Weber (1981) with the experiments of Davies and Taylor (1950) at large R.

Decay of gravity waves on water

We turn next to the dissipation approximation for the decay of water waves. Lamb (1924, p. 624) considered the effect of the viscous dissipation of a free traveling wave given by the potential

$$\phi = ace^{ky} \cos k\,(x\text{-}ct) \tag{16}$$

where c is the wave-velocity and $c = \sqrt{g/k}$ for inviscid potential flow. He found that the mean value of the dissipation per unit area is given by $2\mu k^3 a^2 c^2$. The kinetic energy per unit area is ¼ $\rho k a^2 c^2$, and the total energy (kinetic plus potential) is therefore double of this. Hence in the absence of surface forces $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho k c^2 a^2\right) = -2\mu k^3 a^2 c^2$, it follows that $\frac{\mathrm{d}\,a}{\mathrm{d}\,t} = -2\nu k^2 a$, or

$$a = a_0 e^{-2\nu k^2 t}\,. \tag{17}$$

Equation (17) gives the rate of decay of a free wave computed by dissipation approximation.

Lamb (1924, p. 625) also did direct calculation of the effect of viscosity on water-waves. He solved the 2D water-wave problem utilizing a potential and a stream function,

$$\phi = (Ae^{ky} + Be^{-ky})e^{ikx+nt}\,,\ \psi = (Ce^{my} + De^{-my})e^{ikx+nt}\,. \tag{18}$$

Lamb gave the decay rate n, which is the solution of the following equation:

$$(n + 2\nu k^2)^2 + gk + \gamma' k^3 = 4\nu^2 k^3 \sqrt{k^2 + n/\nu}\,. \tag{19}$$

where $\gamma' = \gamma / g$, γ is the surface tension coefficient. When $\nu k^2 << \sqrt{gk + \gamma' k^3}$ (long waves), n reduces to

$$n = -2\nu k^2 \pm i\sqrt{gk + \gamma' k^3} \tag{20}$$

which agrees with the dissipation approximation result (17). When $\nu k^2 >> \sqrt{gk + \gamma' k^3}$ (short waves) and with γ' ignored, n reduces to

$$n = -\frac{g}{2k\nu} \tag{21}$$

which agrees with the viscous potential flow result which will be derived next.

Lamb's problem, the dissipation of energy of water waves below a vacuum, is a special case of Kevin-Helmholtz instability considered by Funada and Joseph (2001) in which the gas density, viscosity and the relative velocity between stratified fluid and gas are put to zero. The Kevin-Helmholtz instability is very strongly influenced by the gas even though the gas density and viscosity are negligible relative to the liquid; the important physical parameter is the kinematic viscosities of the fluid and gas which are of the same order in many case.

An analysis of the stability of gravity waves using viscous potential flow is embedded in the analysis of Kelvin-Helmholtz instability by Funada and Joseph (2001). The linearized governing equations for the water-wave problem are:

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p - \rho g \mathbf{e_y} + \mu \nabla^2 \mathbf{u}\ ,\ \mathrm{div}\,\mathbf{u} = 0\,. \tag{22}$$

Using the viscous potential flow $\mathbf{u} = \nabla\phi$, $\nabla^2\phi = 0$, the momentum equation becomes

$$\rho\frac{\partial\phi}{\partial t} = -p - \rho g\eta \tag{23}$$

where η is the surface elevation. The potential ϕ is in the form:

$$\phi = Ae^{ky}e^{ikx+nt}\,. \tag{24}$$

With negligible surface tension, normal stress balance gives:

$$-p + 2\mu\frac{\partial^2\phi}{\partial y^2} = 0 \tag{25}$$

We find, after eliminating the pressure in the normal stress balance, that

$$\frac{\partial\phi}{\partial t} + g\eta + 2\nu\frac{\partial^2\phi}{\partial y^2} = 0 \tag{26}$$

and from the kinematic condition we get

$$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} \tag{27}$$

on $y = 0$. After eliminating η in (26) using (27) and applying normal modes proportional to exp $k(y - ict)$ we find

$$c = -i\nu k \pm\sqrt{\frac{g}{k} - \nu^2 k^2} \tag{28}$$

Hence the normal modes solution for viscous potential flow is proportional to

$$e^{-\nu k^2 t}e^{ik\left(x \pm t\sqrt{\frac{g}{k}-\nu^2k^2}\right)}. \tag{29}$$

The amplitude of the wave decays at a rate

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\nu k^2 a\,, \tag{30}$$

one-half of the rate given by (17) and (20). The wave speed c is given by

$$c = \sqrt{\frac{g}{k} - \nu^2 k^2}\,, \tag{31}$$

which is slower than $\sqrt{g/k}$ for $k^3 < g/\nu^2$. For very large values of k, short standing waves do not propagate but simply decay at a rate given by

$$a = a_0\exp\left(-\frac{1}{2}\frac{g}{\nu k}t\right). \tag{32}$$

Equation (32) is in agreement with equation (21) which is the direct calculation result in the short waves limit.

The discrepancy between viscous potential flow and the direct calculation for long waves is caused by the viscous pressure correction in a vorticity layer at the free surface. Prosperetti

(1976) studied small-amplitude standing water waves and derived the viscous pressure correction p_v. We apply his correction for the progressive waves considered here (written in our notation)

$$p_v = -2\mu ack^2 e^{ik(x-ct)} e^{ky} . \tag{33}$$

The pressure from potential flow is given by

$$p_p = -\rho g\eta - \rho \frac{\partial\phi}{\partial t} . \tag{34}$$

The normal stress balance gives:

$$-p + 2\mu\frac{\partial^2\phi}{\partial y^2} = -p_p - p_v + 2\mu\frac{\partial^2\phi}{\partial y^2} = 0 . \tag{35}$$

Inserting (33) and (34) into (35), we get

$$\rho g\eta + \rho\frac{\partial\phi}{\partial t} - p_v + 2\mu\frac{\partial^2\phi}{\partial y^2} = 0 ,$$

$$-g + c^2 k + 2i\mu ck^2 + 2i\mu ck^2 = 0 . \tag{36}$$

It can be seen that the contribution from the pressure correction term is the same as that from the viscous stress. We find that

$$c = -2i\nu k \pm \sqrt{\frac{g}{k} - 4\nu^2 k^2} .$$

Hence, the normal modes solution for viscous potential flow is proportional to

$$e^{-2\nu k^2 t} e^{ik\left(x \pm t\sqrt{\frac{g}{k} - 4\nu^2 k^2}\right)} .$$

The amplitude of the wave decays at a rate

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -2\nu k^2 a . \tag{37}$$

Equation (37) agrees with the equations (17) and (20). For very large values of k, the waves become short standing waves and decay at a rate given by

$$a = a_0 \exp\left(-\frac{1}{4}\frac{g}{\nu k}t\right). \tag{38}$$

The decay rate is one-half of the rate computed from viscous potential flow without the pressure correction; it is also one-half of the rate by direct calculation at short waves limit.

The pressure correction of viscous potential flow does not give rise to the decay rate from the direct calculation at short waves limit, but it does reduce to the decay rate given by the dissipation approximation.

Conclusions and Discussion

We list the results in this paper:

- In the case of a rising spherical bubble in a liquid, the dissipation $\mathcal{D}$ is equal to the power of traction $\mathcal{P}$. The dissipation approximation can be called a power of traction approximation.
- D-Alembert's paradox holds in viscous potential flow even though the dissipation is not zero, $D = D_N + D_S = 0$, for $\mathcal{D} = \mathcal{P} \neq 0$.
- For a gas bubble, putting tangential shear stress to zero but using pressure correction, $DU = \mathcal{D} = \mathcal{P}$, $D = D_N + D_v$ and $D_v = -D_S/2$.

- The decay of the amplitude *a(t)* in the linear theory of progressive water waves below a vacuum was calculated by Lamb (1924). He obtained,

$$\frac{da(t)}{dt} = \text{Real}[n(k)].$$

When k is large (short waves), this rate reduces to the rate calculated from viscous potential flow (Funada and Joseph (2001)). When k is small, it reduces to the rate given by the dissipation approximation.

$$\lim_{k\to 0} \text{Real}[n(k)] = -2\nu k^2 \text{ and } \lim_{k\to\infty} \text{Real}[n(k)] = -\frac{g}{2k\nu}.$$

The role of pressure corrections in the theory of viscous potential flow is not well enough understood. It is certainly true that the irrotational shear stress will always generate vorticity at a gas-liquid interface, but the idea that the pressure forces in a vorticity layer always give rise to a correction comparable to forces generated by the viscous normal stress does not seem to accord to the facts. Apparently, a viscous correction to the irrotational pressure was suggested by G.K. Batchelor (Moore 1962) as the way to resolve the discrepancy between the drag $8\pi a\mu U$ which Moore (1959) computed from potential flow, neglecting the contribution from the irrotational viscous shear stress and the dissipation value $12\pi a\mu U$. Moore (1962) did a boundary layer analysis, which does give rise to $12\pi a\mu U$ plus smaller terms, but his solution is not valid near the rear stagnation point and has other problems specified in the paper by Kang and Leal (1987).

No other boundary layer analysis of Moore's problem has appeared. A boundary layer analysis for weak viscous effects on the oscillations of drops in zero gravity was given by Lundgren and Mansour (1988). They give an excellent discussion of the underlying issues and they introduce a novel method of including the effects of small viscosity in the form of modified surface boundary conditions which produce higher-order corrections to the potential flow. They do not derive an explicit pressure correction. The pressure corrections of greatest interest in Moore's problem are not the higher order ones and they are not restricted to weak viscous effects. Moreover, the analysis of Lundgren and Mansour does not lead to solution giving the flow in a boundary layer; instead they get a new set of boundary conditions.

We have already cited results taken from Kang and Leal (1987). They get the drag $12\pi a\mu U$ computed by the dissipation approximation by direct integration of the normal stress including the corrected pressure. They do not do a boundary layer analysis but instead obtain the pressure correction from a general relationship between the viscous correction and the vorticity distribution in axisymetric flow in which the irrotational shear stress is made to vanish. A vorticity layer does not appear in their final result; the pressure correction is obtained from the vorticity distribution *on the bubble surface independent of the vorticity distribution in the fluid* (which is irrotational). We have noted their pressure correction adds just the $4\pi a\mu U$ to $8\pi a\mu U$ computed by Moore (1959) needed to obtain the dissipation value.

Joseph (2003) presented a derivation of Prandtl's boundary layer equations based on the idea that the outer flow is a viscous potential flow. He finds that the argument which usually leads to replacing the pressure on the wall with the irrotational pressure leads now to replacing the pressure with the irrotational pressure plus ½ the viscous normal stress of the irrotational flow, as is true of the Kang-Leal result in a very different situation.

The fact that the pressure correction gives a contribution which is just ½ the contribution of normal stress of the irrotational flow is very curious; it does not look like a boundary layer result in the usual sense. The pressure correction is apparently a functional determined completely by the viscous irrotational flow.

The pressure correction for the water wave problem which was computed by Prosperetti (1976) leads to a contribution to the decay rate constant which is exactly equal to the contribution of the normal stress of the irrotational flow (Funada and Joseph 2002). The gold standard here is the exact solution of Lamb (1924) which reduces to the value of decay constant computed by the dissipation method when the waves are long, and to the viscous potential flow value of Funada and Joseph when the waves are short. The addition of the pressure correction raises the value of the decay rate to the dissipation value, which is correct for long waves but incorrect for short waves.

There are many cases in which pressure corrections are not important. One example is the short wave case just cited. Another is the drag on a spherical cap bubble computed by Davies and Taylor (1956) for large Reynolds numbers and by Joseph (2003) for finite Reynolds numbers. The formula $C_D = 6 + 32/R$ is in very good agreement with experiments tending to the Davies-Taylor value 6 at large R. A pressure correction is not used and is not needed. Examples from calculations of growth rates and neutral curves classical interfacial stability problems can also be cited. The results of computation of the forgoing quantities for Rayleigh-Taylor instability by Joseph, Belanger and Beavers (1999) from viscous potential flow, and by Joseph, Beavers and Funada (2002) for viscoelastic potential flow are so close to exact values that a correction due to the pressure could not be important. On the other hand, the viscous potential flow analysis of capillary instability of Funada and Joseph (2002) gives a good approximation to exact results in most cases but might be improved, especially for long waves, by a pressure correction.

The role of pressure corrections in the theory of irrotational flow of viscous fluids is not well enough understood. To make this theory more useful it is desirable to identify *a priori* the conditions under which viscosity is important but vorticity is unimportant. In many cases the flows of viscous fluids are basically irrotational; in some of these flows a pressure contribution due to vorticity on the gas-liquid interface is important, and in others it is not important. The *a priori* identification of the conditions under which pressure corrections ought to be computed is an important open problem in the theory of irrotational flow of viscous fluids.

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