# CRITICAL MACH NUMBER

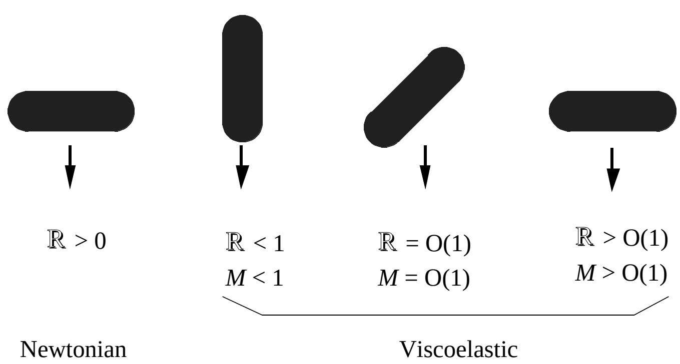
Viscous effects and normal stress effects are suppressed when the relative speed $U$ between particles and fluid is greater than the speed of propagation for duffusion

$$U > \frac{\nu}{d}, \quad \mathbb{R} = U\frac{d}{\nu} > 1$$

and is greater than the speed $c$ of propagation of shear waves

$$U > c (= \sqrt{\frac{\nu}{\lambda}}), \quad M = \frac{U}{c} > 1.$$

The motion is dominated by inertia when the Reynolds and viscoelastic Mach numbers are greater than one.




Orientation of cylinders falling in Newtonian and viscoelastic liquids.

# PERTURB STOKES FLOW WITH INERTIA AND VISCOELASTICITY

All the fluid models reduce to a 2nd order fluid

$$\begin{aligned}
\mathbf{T} &= -p\mathbf{1} + 2\eta\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \\
\mathbf{A}_1 &= \nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}} \quad \text{Newtonian} \\
\mathbf{A}_2 &= (\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)\mathbf{A}_1 + \mathbf{A}_1\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\mathbf{A}_1
\end{aligned}$$

The equations of motion are

$$\rho(\frac{\partial\mathbf{u}}{\partial \mathrm{t}} + \mathbf{u}\cdot\nabla\mathbf{u}) = \mathrm{div}\ \mathbf{T}$$

We consider separate and independent perturbations of Stokes flow

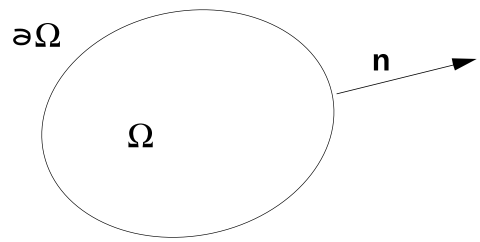
$$0 = -\nabla p^{\mathrm{stokes}} + \mathrm{div}\ 2\eta\mathbf{A}_1$$

with inertia and viscoelasticity.

# NORMAL STRESS ON BOUNDARY OF A RIGID BODY

$$T_{nn} = \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n}$$

The viscous part of the normal stress vanishes on every point of the boundary $\partial\Omega$ of a rigid body




$$\mathbf{n} \cdot \mathbf{A_1} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0}$$

2nd order fluid

$$T_{nn} = -p + \mathbf{n} \cdot (\alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2) \cdot \mathbf{n}$$

$$T_{nn}^{\text{stokes}} = -p^{\text{stokes}} \begin{cases} \text{linear in velocity} \\ \text{does not turn symmetric bodies} \\ \text{does not produce lateral drift} \end{cases}$$

$$T_{nn}^{\text{inertia}} = -p^{\text{inertia}} \begin{cases} \text{quadratic in velocity} \\ \text{turns bodies broadside on} \\ \text{Bernoulli like} \end{cases}$$

$$T_{nn}^{\text{viscoelastic}} = -\frac{\Psi_1}{4} Tr \mathbf{A_1^2} \begin{cases} \text{quadratic in velocity gradient} \\ \text{compressive: one sign} \\ \text{opposes } p^{\text{inertia}} \\ \text{turns long bodies into} \\ \text{the stream} \end{cases}$$

# STRESS IN PLANE FLOW OF A SECOND ORDER FLUID, GRESEKUS-TANNER THEOREM

$$a = \frac{\partial u}{\partial x} = -\frac{\partial V}{\partial y}, c = \frac{\partial V}{\partial x}, \quad b = \frac{\partial u}{\partial y} = \overset{\circ}{\gamma}$$

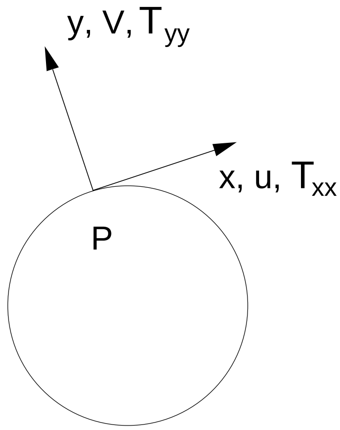
shear rate

$$\begin{bmatrix} T_{xx} & T_{xy} \\ T_{xy} & T_{yy} \end{bmatrix} = \quad -\big[p^{\text{stokes}} + \tfrac{\alpha_1}{\eta}\tfrac{dp}{dt}^{\text{stokes}}$$

$$+\tfrac{\alpha_1}{2}\{4a^2 + (b+c)^2\}\big] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$+(\eta + \alpha_1 \tfrac{d}{dt}) \begin{bmatrix} 2a & b+c \\ b+c & 2a \end{bmatrix}$$

$$+\alpha_1(b-c) \begin{bmatrix} -b-c & 2a \\ 2a & b+c \end{bmatrix}$$




$$\text{at } P \subset \partial\Omega \ : \ a = c = 0$$

$$b = \overset{\circ}{\gamma}$$

local coordinates at $P$ :

$$T_{yy} + p^{\text{stokes}} + \frac{\alpha_1}{\eta}\frac{dp}{dt}^{\text{stokes}} = \frac{\alpha_1}{2}\overset{\circ}{\gamma}^2$$

# THE VISCOELASTIC PART OF THE NORMAL STRESS $T_{yy}$ IS A COMPRESSION

For steady flow with $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ (and more generally)

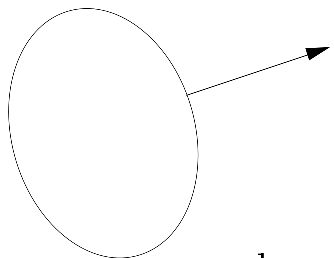
$$T_{yy} = -p^{\text{stokes}} - \frac{\Psi_1}{4}\overset{\circ}{\gamma}{}^2$$

where $\Psi_1 = -2\alpha_1 > 0$ is the coefficient of the first normal stress

This result shows that bodies are pushed by normal stress with pushing proportional to the square of local shear rate. This informs intuition about how bodies move in viscoelastic liquids.

## THEORY

For slow steady flow of a viscoelastic fluid in two dimensions, the normal stress due to viscoelasticity is *compressive* and given by



$$T_{nn} = \frac{-\Psi_1(0)}{4}\mathring{\gamma}^2$$

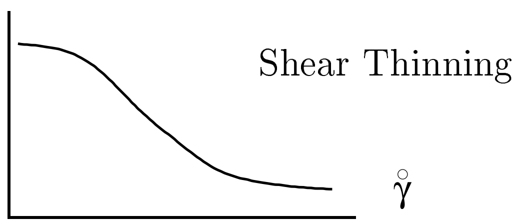
where

$\Psi_1(0) > 0$ is the coefficient of the first normal stress difference,
$\mathring{\gamma}$ is the shear rate at the wall

Bodies are *pushed* by normal stresses proportional to the square of the shear rate at each point on the surface of the body. This informs intuition about how bodies move in viscoelastic liquids.

# AMPLIFICATION OF NORMAL STRESSES BY SHEAR THINNING

$$
\begin{aligned}
T_{nn} + p^{\text{stokes}} &= \frac{1}{4}\Psi_1(0)\mathring{\gamma}_w^2 \quad \text{slow and steady} \\
\Psi_1(0) &= \eta(\lambda_1 - \lambda_2) > 0 \quad \text{Oldroyd B} \\
\eta &= \eta(\mathring{\gamma}_w) \quad \text{Assume, without justification}
\end{aligned}
$$




$$\tau_w = \eta(\mathring{\gamma}_w)\mathring{\gamma}_w \quad \text{wall shear stress}$$

Putting all together, we get

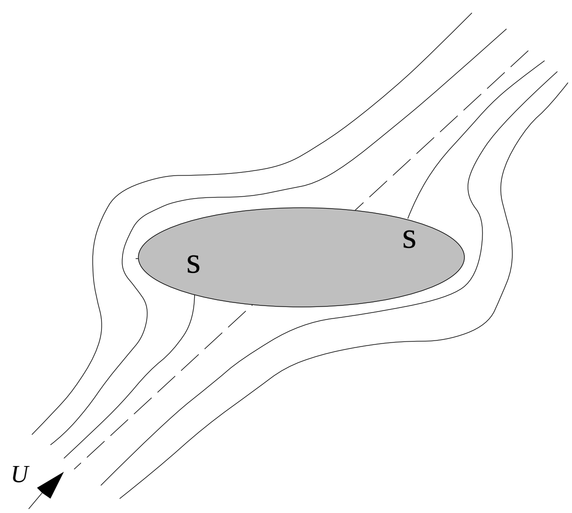
$$T_{nn} + p^{\text{stokes}} = \frac{1}{4}\eta(\mathring{\gamma}_w)(\lambda_1 - \lambda_2)\mathring{\gamma}_w^2 = \frac{\tau_w}{4}\mathring{\gamma}_w(\lambda_1 - \lambda_2)$$

Compare flows with the same shear stress $\tau_w$ (same pressure gradient). The more they shear thin, the greater is the normal stress.

# REVERSAL OF THE NORMAL STRESS AT A POINT OF STAGNATION

A point of stagnation on a stationary body in potential flow is a unique point at the end of a dividing streamline at which the velocity vanishes. In a viscous fluid all the points on the boundary of a stationary body have a zero velocity but the dividing streamline can be found and it marks the place of zero stress near which the velocity is small. The stagnation pressure makes sense even in a viscous fluid where the high pressure of the potential flow outside the boundary layer is transmitted right through the boundary layer to the body.

# POTENTIAL FLOW PAST A CYLINDER




The pressure at stagnation points $S$ will turn the broadside of the cylinder into the stream. If the extensional stress at $S$ were *reversed*, as is possible in a viscoelastic liquid, the cylinder would align itself with the stream. The Bernoulli equation for a second order fluid

$$\rho\phi_{,\tau} + \frac{\rho|\mathbf{u}|^2}{2} + p - \hat{\beta}|\nabla\mathbf{u}|^2 = \mathbf{c}$$

has an extra $\hat{\beta}|\nabla\mathbf{u}|^2 > 0$ term which works against inertia.