

Cavitation and the state of stress in a flowing liquid

Daniel D. Joseph
University of Minnesota
Dept. of Aerospace Engineering & Mechanics
110 Union St.
Minneapolis, MN 55455
December, 1997

The problem of the inception of cavitation is formulated in terms of a comparison of the breaking strength or cavitation threshold at each point of a liquid sample with the principal stresses there. A criterion of maximum tension is proposed which unifies the theory of cavitation, the theory of maximum tensile strength of liquid filaments and the theory of fracture of amorphous solids. Liquids at atmospheric pressure which cannot withstand tension will cavitate when and where tensile stresses due to motion exceed one atmosphere. A cavity will open in the direction of the maximum tensile stress which is 45° from the plane of shearing in pure shear of a Newtonian fluid. New explanations of cavitation inhibition due to polymer additives are considered. Experiments which support these ideas are discussed and some new experiments are proposed.

1 Introduction

In previous papers (Joseph [1995], Joseph, Huang and Candler [1996]) I drew attention to the fact that the pressure in a flowing incompressible liquid is not a fundamental dynamic variable; at each point of the liquid the state of stress is determined by three principal stresses. In Newtonian fluids the pressure is the negative of the mean of these stresses (6); in non-Newtonian fluids the pressure is an unknown field variable whose relation to the principal stresses depends on the choice of a constitutive equation.

We may generally express the stress \mathbf{T} by a constitutive equation of the form

$$\mathbf{T} = -p\mathbf{1} + \boldsymbol{\tau}[\mathbf{u}] \quad (1)$$

where the part $\boldsymbol{\tau}$ of \mathbf{T} which is characterized by a constitutive equation can be regarded as functional of the velocity \mathbf{u} . For incompressible liquids, the conservation of mass is expressed by

$$\operatorname{div} \mathbf{u} = 0 \quad (2)$$

and the conservation of momentum by

$$\rho \left[\frac{\partial \mathbf{u}}{\partial \tau} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \text{div } \boldsymbol{\tau}[\mathbf{u}]. \quad (3)$$

Equations (2) and (3) are four equations for three components of velocity and the pressure p is an additional unknown which we need to close the system.

For Newtonian liquids

$$\boldsymbol{\tau}[\mathbf{u}] = 2\eta \mathbf{D}[\mathbf{u}] \quad (4)$$

where $\mathbf{D}[\mathbf{u}]$, the rate of strain, is the symmetric part of the gradient of velocity, η is the viscosity, and

$$\text{Trace } \mathbf{D}[\mathbf{u}] = \text{div } \mathbf{u} = 0 \quad (5)$$

As a consequence of (5),

$$p = -\frac{1}{3} \text{Trace } \mathbf{T} \quad (6)$$

More generally, $\text{Trace } \boldsymbol{\tau} \neq 0$ and

$$p = -\frac{1}{3} \text{trace } (\mathbf{T} - \boldsymbol{\tau}) \quad (7)$$

depends on the constitutive equation, the choice of the functional relating $\boldsymbol{\tau}$ to \mathbf{u} .

Though it is true that a liquid at rest, in which all the stresses are all equal to $-p$, can make sense of (6), a moving liquid cannot average the principal stresses as is required by (6), and (7) is even more a consequence how we choose to define $\boldsymbol{\tau}$ than a fundamental quantity which can be felt at a point by the liquid.

2 Cavitation index

The idea is that the state of stress at each point of a moving liquid is determined by the three principal stresses

$$T_{11} \geq T_{33} \geq T_{22} \quad (8)$$

and not by the pressure given by (7). Criteria for the inception of cavitation in liquids are here framed in terms of the principal stresses (8) rather than the pressure (7) used traditionally. Most of the traditional studies are framed in terms of a

cavitation index. The utility of a cavitation index based on pressure is not evident. In one formulation, the index is given by

$$k = \frac{\tilde{p} - p_c}{\frac{1}{2}\rho U^2} \quad (9)$$

where \tilde{p} is the static pressure in the main stream, U is the bulk velocity of the fluid and \tilde{p} and U are measured in the instant that cavitation commences.

Milne-Thompson [1965, Chap XII] considers the vapor cavity behind a moving cylinder and he forms an index which he attributes to Prandtl

$$k = \frac{\tilde{p} - p_c}{\frac{1}{2}\rho U^2} = \frac{V^2 - U^2}{U^2} \quad (10)$$

where \tilde{p} is the pressure at ∞ , p_c is the pressure in the cavity and V is the fluid speed on the cavity wall. By Bernoulli's theorem

$$\tilde{p} + \frac{1}{2}\rho U^2 = p_c + \frac{1}{2}\rho V^2 \quad (11)$$

This index shows that a cavity will form on the top of the cylinder where the flow is fastest.

Some limitations of the cavitation index are widely appreciated by the cavitation community. The important discovery (Arakeri & Acosta [1973]) has been that, even though viscous stresses are thought to have a negligible effect in cavitating water flows, viscosity has a major impact on flow structure, like separation points, which impact the pressure distribution as a consequence of which cavitation is also affected. Franc and Michel [1985] found that in the flow of water over circular and elliptic cylinder cavities do not detach from the body at the minimum pressure point, but behind a laminar separation. They noted a direct link between separation of the laminar boundary layer and the initial signs of cavitation both of which are located in the recirculation zone downstream of the detachment.

Other limitations of the cavitation index, like cavitation induced by high flow induced tensile stresses, have not been considered by the cavitation community. Such stresses, though typically small in water, could reach sensible values in more viscous liquids, and even in special flows of water. High tensile stresses on water threads stripped of a drop by high speed air may also cavitate (Joseph, Huang and Candler [1996]). The possibility of flow-induced tensile stresses due to stretching motions at a point of separation in cavitating flows at the inlet of holes in atomizers ought to be considered.

3 Principal stresses and cavitation

The state of stress rather than its average value is fundamental for all the motions of an incompressible fluid. Here, however we focus on the inception of cavitation and not on the shape and motion of an open cavity. Even though criteria for cavitation ought to be based on the principal stresses and not the pressure, it is useful to introduce a pressure as the mean normal stress as in a Newtonian liquid and to define it that way for Non-Newtonian liquids. If we write

$$\mathbf{T} = -\pi \mathbf{1} + \mathbf{S} = -p \mathbf{1} + \boldsymbol{\tau} \quad (12)$$

where p is given by (7) and \mathbf{S} is the stress deviator

$$\pi = -\frac{1}{3} \text{Trace } \mathbf{T}, \quad \text{Trace } \mathbf{S} = S_{11} + S_{22} + S_{33} = 0. \quad (13)$$

Since $S_{11} \geq S_{33} \geq S_{22}$ we have

$$S_{11} > 0 \text{ and } S_{22} < 0 \quad (14)$$

where

$$S_{11} - S_{22} > 0 \quad (15)$$

is largest in the coordinate system in which \mathbf{T} is diagonal.

Consider now the opening of a small cavity. It is hard to imagine very large differences in the pressure of the vapor in the cavity so that the cavity should open in the direction where the tension is greatest. The idea that vapor cavities open to tension is endemic in the cavitation community, but it seems not to have been noticed before that this idea requires one to consider the state of stress at a point and, at the very least, to determine the special principal axes coordinates in which the tension is maximum. To remind us of this important point we shall call $\zeta(\theta)$ the special coordinate system in which the orthogonal transformation \mathbf{Q} diagonalizes \mathbf{T} (and \mathbf{S}):

$$\mathbf{Q}^T \mathbf{T} \mathbf{Q} = \text{diag}(T_{11}, T_{22}, T_{33}) \quad (16)$$

Here θ in $\zeta(\theta)$ stands for the direction cosines in the diagonalizing transformation, and θ is the diagonalizing angle for the two-dimensional rotation. The rotation of \mathbf{T} is an important part of the theory of cavitation which has not been considered before.

In two dimensions the components of the stress deviator in $\zeta(0)$ are given by

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & -S_{11} \end{bmatrix}; \quad (17)$$

The angle θ that diagonalizes \mathbf{S} , $S'_{12} = 0$ is given by

$$\begin{cases} \sin 2\theta = S_{12}/\sqrt{S_{12}^2 + S_{11}^2}, \\ \cos 2\theta = S_{11}/\sqrt{S_{12}^2 + S_{11}^2} \end{cases} \quad (18)$$

and

$$[\mathbf{S}] = \sqrt{S_{12}^2 + S_{11}^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (19)$$

The largest stress component in the principal value coordinate system is

$$T_{11} + \frac{1}{2}(T_{11} + T_{22}) = S_{11}; \quad (20)$$

the smallest component is

$$T_{22} + \frac{1}{2}(T_{11} + T_{22}) = -S_{11} \quad (21)$$

and

$$T_{11} - T_{22} = 2S_{11} \quad (22)$$

We call T_{11} the maximum tension and T_{22} is the minimum tension. If the maximum tension is negative, it is compressive; the minimum tension is even more compressive.

If the cavitation (outgassing) threshold p_c is above $\Pi - S_{11}$ but below Π the cavity will appear when and where the tension due to motion is large enough; if this threshold is greater than $\Pi - S_{22}$ ($S_{22} < 0$) then the cavity will open only at those points where no component of the total stress is larger than the cavitation threshold; this is the minimum tension criterion and in neither case is the criterion framed in terms of the pressure Π alone.

For Newtonian fluid $S_{11} = 2\eta\partial u_1/\partial x_1$, $S_{22} = -S_{11}$ and $S_{12} = \eta(\partial u_1/\partial x_2 + \partial u_2/\partial x_1)$ where η is the viscosity. In principal coordinates, $S_{12} = 0$.

If a cavitation bubble opens up, it will open in the direction of maximum tension. Since this tension is found in the particular coordinate system in which the stress is diagonal, the opening direction is in the direction of maximum extension, even if the motion is a pure shear. It may open initially as

an ellipsoid before flow vorticity rotates the major axis of ellipsoid away from the principal tension axis of stress, or it may open abruptly into a “slit” vacuum cavity perpendicular to the tension axis before vapor fills the cavity as in the experiments of Kuhl et al. [1994] (see figures 2 and 3).

The features in the two dimensional problem which were just discussed have an immediate and obvious extension to three dimensions.

4 Cavitation criteria

It is well known that cavitation occurs at weak spots (nucleation sites) in the fluid. It is not necessary to form vapor bubbles; outgassing may occur and in a severe cavitation even a vacuum cavity may open up (see section 8). It is nearly impossible to know where the nucleation sites are or what the outgassing or breaking stress p_c may be, especially in pure and carefully prepared liquids. In section 9, I argue that outgassing may be regarded as a phase change for condensed gas in solution under conditions for which the gas not in solution could not condense. The liquid can saturate with condensed gas; cavitation is apt to occur in the supersaturated case at pressures of the order of the vapor pressure. Here, we shall assume that the breaking stress is a given parameter which can be defined at each point of a liquid; we then compare the state of stress in a moving liquid at the point with p_c to form a cavitation criteria.

The cavitation threshold used in the prior literature is framed in terms of a mean stress

$$\pi = -\frac{1}{3}(T_{11} + T_{22} + T_{33}) \quad (23)$$

cavitation will occur when $\pi - p_c < 0$ and will not occur when $\pi - p_c > 0$. The mean stress may be a good estimate for breaking thresholds, but it does not enter into the criteria since it has no physical meaning in a moving fluid. The fluid cannot average its stresses.

Two cavitation thresholds based on the maximum tension T_{11} and minimum tension T_{22} in three dimensions can be considered, recall that the deviatoric stresses are such that

$$S_{11} > 0, S_{22} < 0 \quad (24)$$

so that $T_{22} = S_{22} - \pi$ is the minimum tension.

The **maximum tension** criterion is given by

$$B_{11} \stackrel{\text{def}}{=} T_{11} + p_c = S_{11} - \pi + p_c > 0 \quad (25)$$

In this case

$$\pi - p_c < S_{11} \quad (26)$$

and since the tension $S_{11} > 0$, $\pi - p_c$ could be larger than zero and the liquid would still cavitate. If (25) holds and

$$B_{22} \stackrel{\text{def}}{=} T_{22} + p_c < 0 \quad (27)$$

then relative to the threshold p_c , the stress B_{11} is in “tension” and $B_{22} < 0$ is a “compression”.

If (25) holds and

$$B_{22} > 0 \quad (28)$$

then all three of the relative principal stresses B_{11}, B_{22}, B_{33} are positive and a cavity will open. This is the **minimum tension criterion**. This criterion for cavitation is more severe than the classical one which requires that the average value of these relative stresses be positive.

The archival literature on cavitation allows only for breaking in tension, though the state of stress at a point which ought to be considered, has not been considered. Typically the discussion of cavitation is framed in the context of the breaking strength of liquids; the main conclusion is that liquids may withstand very large tensions if impurities and nucleation sites are suppressed. A convenient and readable discussion of this point has been given by Batchelor [1967]. There is a vast literature on the tensile strength of liquids some of which may be found in the book by Knapp, Daily & Hammitt [1970] who say that “... Measurements have been made by several different methods and are too numerous to report completely” and in other books on cavitation.

Knapp et al. [1970] have considered whether the cavitation threshold ought to be framed in terms of the vapor pressure or the tensile strength of liquids, concluding for the latter. They say that

... the elementary concept of inception is the formation of cavities at the instant the local pressure drops to the vapor pressure of the liquid. However, the problem is not so simple. Although experiments show inception to occur near the vapor pressure, there are deviations of various degrees with both water and other liquids that are not reconcilable with the vapor-pressure concept. We define the vapor pressure as the equilibrium pressure, at a specified temperature, of the liquid’s vapor which is in contact with an *existing* free surface. If a cavity is to be

created in a homogeneous liquid, the liquid must be ruptured, and the stress required to do this is not measured by the vapor pressure but is the *tensile strength* of the liquid at that temperature. The question naturally arises then as to the magnitudes of tensile strengths and the relation these have to experimental findings about inception.

A similar point of view was expressed by Plesset [1969]

... A central problem in cavitation and boiling is how macroscopic vapor cavities can form when moderate tensions are applied to the liquid. The theory of the tensile strength of pure liquids predicts that a vapor cavity will form only when the liquid is under extremely large tensions; as an equivalent effect the theory also predicts that vapor bubbles appear in boiling only when the liquid has very large superheats. Since these large tensile strengths and superheats are not observed, the idea of nuclei has been introduced. These nuclei are in some sense holes in the liquid which are already beyond molecular dimensions and which may therefore grow into macroscopic bubbles under moderate liquid tensions.

Brennen [1995] notes that "... This ability of liquids to withstand tension is very similar to the more familiar property exhibited by solids and is a manifestation of the elasticity of a liquid." Of course the elasticity of liquids, solid-like behavior, could occur only in time so short that the configurations of molecules is not changed by flow, as could be expected in a cavitation event. Fischer [1948] notes that "... Glass and other undercooled liquids may fail by the nucleation and propagation of cracks, rather than of bubbles as do more mobile liquids." Nucleation and propagation of cracks have been realized in the experiments of Kuhl et al. [1994] discussed in section 8.

The theory of cavitation, the tensile strength of liquids and the fracture of amorphous solids may be framed in a unified manner in which the breaking strength of the material is defined in terms of tensile stresses along the principal axes of stress. The formation of cracks or bubbles is probably controlled by comparing rapidity of flow with the propagation speed of fracture. Glass at different temperatures is a perfect material for these considerations. At high temperatures the molten glass flows and under the right conditions, flow bubbles ought to open at a weak spot in the direction of the principal tension. Low temperature glass is an amorphous solid and we can imagine a crack to be initiated under tension at the same weak spot. Glass at intermediate temperatures may exhibit as yet unknown properties between cavity formation and crack propagation.

The nucleation of a cavity can occur as a sudden and not a continuous event. The fluid must first rupture; then it fills with vapor or gas and flows as in the experiments of Israelachvili and his collaborators described in section 8. To open a cavity, the liquid must be supersaturated; practically this supersaturation can be achieved by tensions created by flow. If the ambient pressure is atmospheric we might expect to nucleate vapor or gas bubbles at points at which the flow-induced tensions exceed 1 atmosphere $\approx 10^5$ Pa.

5 Cavitation in shear

Consider plane shear flow between parallel plates as in figure 1.

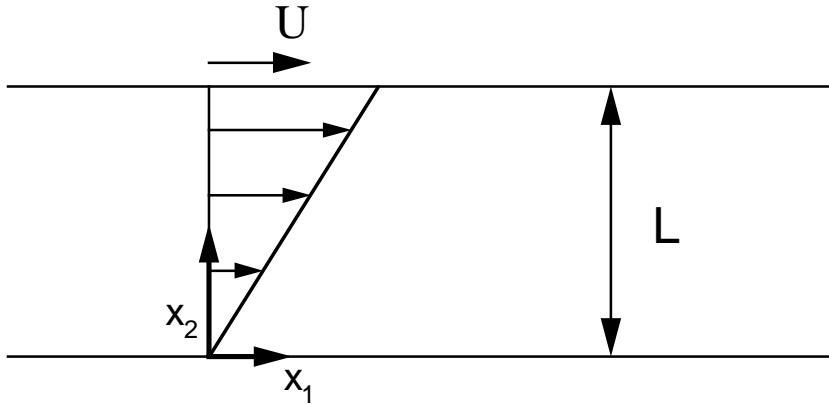


Figure 1: Plane Couette flow between walls

The stress in this flow is given by

$$\begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} = -\pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \eta \begin{bmatrix} 0 & \frac{U}{L} & 0 \\ \frac{U}{L} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (29)$$

where $\pi = \frac{1}{3}(T_{11} + T_{22} + T_{33})$ is determined by the “pressurization” of the apparatus. The angle which diagonalizes \mathbf{T} is given by (18) as $S'_{12} = 0$ or

$$\cos 2\theta = 0, \quad \theta = 45^\circ$$

(In the break-up of viscous drop experiments in plane shear flow done by G.T. Taylor [1934], the drops first extend at 45° from the direction of shearing.)

Then, using (19), in principal coordinates, we have

$$\begin{bmatrix} T_{11} + \pi & 0 & 0 \\ 0 & T_{22} + \pi & 0 \\ 0 & 0 & T_{33} + \pi \end{bmatrix} = \eta \frac{U}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (30)$$

and

$$B_{11} = p_c - \pi + \eta \frac{U}{L}, \quad (31)$$

$$B_{22} = p_c - \pi - \eta \frac{U}{L}, \quad (32)$$

The difference between the largest and smallest stresses is

$$B_{11} - B_{22} = 2\eta \frac{U}{L}. \quad (33)$$

This difference is of the order of one atmosphere of pressure if

$$2\eta \frac{U}{L} = 10^6 \frac{\text{dynes}}{\text{cm}^2} \quad (34)$$

If $\eta = 1000$ poise, $U = 10$ cm/sec and $L = 10^{-1}$ cm, we may achieve such a stress. It is possible to imagine such a shearing motion between concentric rotating cylinders filled with silicon oil, though the conditions are severe. If we could depressurize the system so that a threshold of pressure less than one atmosphere were required, we might see cavities appear in shear flow when $B_{11} > 0$ and $B_{22} < 0$. I am not aware of reports of cavities forming in shear flows, but the conditions required are at the border of realistic experiments and may have escaped detection. Experiments of this kind ought to be tried.

Note added in proof:

Cavities formed in shear flows have been reported recently in a paper by Archer, Ternet, and Larson [1997]: “Fracture” phenomena in shearing flow of viscous liquids.

Abstract: In startup of steady shearing flow of two viscous unentangled liquids, namely low-molecular-weight polystyrene and α -D-glucose, the shear stress catastrophically collapses if the shear rate is raised above a value corresponding to a critical initial shear stress of around 0.1-0.3 Mpa. The time-dependence of the shear stress during

this process is similar for the two liquids, but visualization of samples in situ and after quenching reveals significant differences. For α -D-glucose, the stress collapse evidently results from debonding of the sample from the rheometer tool, while in polystyrene, bubbles open up within the sample; as occurs in cavitation. Some similarities are pointed out between these phenomena and that of “lubrication failure” reported in the tribology literature.

The critical stress 0.1-0.3 Mpa = 1-3 atmospheres is just what might have been guessed for cavitation under shear.

6 Cavitation in extension

We have argued that cavities always appear in the extensional flows defined in principal axes coordinates even when the flow is pure shear. However, the direct creation of a pulling flow without rotation (vorticity) may lead to a higher level of dynamic stresses than could be otherwise achieved. Let us suppose that a small diameter thread open to the atmosphere is anchored at a solid wall at $x = 0$ and is being pulled out at a constant rapid rate \dot{S} in the direction x .

$$u = \dot{S}x, \quad \nu = -\frac{1}{2}\dot{S}y, \quad \omega = -\frac{1}{2}\dot{S}z. \quad (35)$$

The thread is in tension when \dot{S} is large enough

$$T_{11} = -\pi + 2\eta\frac{\partial u}{\partial x} \approx -p_a + 2\eta\dot{S} \quad (36)$$

where, for very thin threads $\pi \approx -p_a$ where p_a is atmospheric pressure. According to the maximum tension criterion (23) cavities will form in the thread, and the thread may actually break, when

$$B_{11} \approx p_a - p_c - 2\eta\dot{S} < 0 \quad (37)$$

The stretch rate \dot{S} for breaking can be estimated assuming that the thread cannot sustain a tension, by $p_c = 0$; then

$$\dot{S} > 10^6/2\eta(\text{sec}^{-1})$$

For very viscous threads, say $\eta = 500$ poise, the stretch rate for breaking

$$\dot{S} > 10^5(\text{sec}^{-1})$$

is rather large.

The extensional flow (33) with a time dependent \dot{S} may be used to model the motion emanating from a stagnation point at the center of the neck in a collapsing capillary filament. Lundgren & Joseph [1997] found that the neck is of parabolic shape and its radius collapses to zero in a finite time. During the collapse the tensile stress due to viscosity increases in value until at a certain finite radius which is about 1.5 microns for water in air, the stress in the throat passes into tension, presumably inducing cavitation there.

7 Breaking tension of polymer strands

Another example of breaking of viscous threads in tension has been documented in experiments by Wagner, Schulze, and Göttfert [1996] on the drawability of polymer melts.

In these experiments the tensile force needed to elongate an extruded polymer melt is measured as a function of the draw ratio $V = \nu/\nu_0$ where ν_0 is the velocity of the spinline at the die and ν is the velocity of the spinline at the takeup wheels. The tensile force F is measured at the wheel and the stress in the strand at the wheel is said to be given by

$$\sigma = FV/A_0$$

where A_0 is the area of the crosssection of the die hole. V and F increase together and at a certain critical F_B (and V_B) the strand breaks. The remarkable feature of this breaking is that the breaking stress σ_B is independent of the extrusion pressure (the wall shear stress) and temperature. Wagner et al. [1996] conclude that the breaking stress σ_B is a “pure material constant”.

The breaking stress in their LDPE sample A18 ($\eta_0 = 10^4$ PaS) is

$$\sigma_B \approx 10^6 Pa.$$

The breaking stress in the HDPE sample H50 ($\eta_0 > 4.8 PaS$) is

$$\sigma_B \approx 1.1 \cdot 10^6 Pa.$$

Atmospheric pressure is roughly

$$p_a \cong 1.1 \cdot 10^5 Pa;$$

the pressure in the thread is somewhat larger than this because of surface tension. The radius of the die is 1mm; if the thread thins by 10 or more the surface tension addition to pressure in the thread will be sensible. It is nevertheless certain that the strand is in tension when it breaks.

8 Cavitation experiments at the nanoscopic level

Chen and Israelachvili [1991] and Kuhl, Ruths, Chen and Israelachvili [1994] have done important direct visualization studies of cavitation of ultrathin nanometer liquid films using the surface forces apparatus technique. They are able to visualize cavitation between mica surfaces in approach-separation and shearing motions. They noticed that vapor cavities developed when two curved surfaces are moved away from each other faster than some critical velocity ν_c . In the experiments described by Kuhl et al. [1994], the liquid between 1cm radius hemispheres of mica was a low molecular weight, Newtonian, 180 poise polybutadiene and the separating motions can be thought to give rise to extensional motions like those described in (33).

Chen & Israelachvili say that

We have found that cavitation bubbles can occur either totally within the liquid, that is, away from the surfaces, or at the solid-liquid interfaces. The adhesion of untreated (polar) mica surfaces to the PBD liquid is stronger than the cohesion between the liquid molecules themselves (“wetting” conditions); hence, the cavities form totally within the liquid. In contrast, for surfaces coated with a surfactant monolayer, the nonpolar solid-liquid adhesion is weaker . . . and the cavities form at the interfaces.

A qualitative description of their observation for the case of strong adhesion is described in the caption for the cartoon in figure 2.

The experiments of Israelachvili and his associates show that cavities open in tension at a threshold value of the extensional stress and that the formation of cavities is analogous to the fracture of solids, with the added caveat that the liquid can flow into the crack immediately after fracture. In the words of Kuhl et al. [1994]

If the speed of separation is increased, the surfaces become increasingly more pointed just before they rapidly move apart. Then, above some critical speed ν_c (here about $100 \mu\text{m/s}$) a completely new separation mechanism takes over, as shown in Figure 3. Instead of separating smoothly, the liquid ‘fractures’ or ‘cracks’ open like a solid. It is known that when subjected to very high shear rates, liquids begin to behave mechanically like solids, for example, fracturing like a brittle solid. In our experiments, the point and time at which this ‘fracture’ occurred was just as the surfaces were about to separate from their most highly pointed configuration (Fig. 3C) - for had the separation

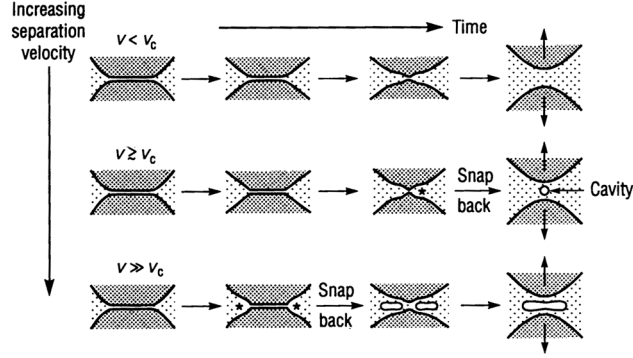


Figure 2: Schematic illustration of the separation of two observed mica surfaces at progressively increasing separation velocities as ascertained from the FECO fringe pattern and direct optical microscope visualization. The most likely places where recoil and damage occurred are shown by the starred points (*). **(Top)** $\nu < \nu_c$: smooth separation; no cavities. **(Middle)** $\nu \geq \nu_c$: abrupt separation; cavity and damage form at center. **(Bottom)** $\nu \gg \nu_c$: abrupt separation; cavities and damage form at rim (crater-like).

velocity been any smaller than ν_c they would have separated smoothly without fracturing. We consider that in the present case, the ‘fracturing’ or ‘cracking’ of the liquid between the surfaces must be considered synonymous with the “nucleation” or “inception” of a vapor cavity.

The stretch rate may be underestimated by ν_c/l where $2l$ is the shortest distance between the mica surfaces. To get cavitation it is necessary to cross a stress threshold which is consistent with the observation that “. . . The thicker the initial film thickness the higher the value of ν_c . . .”

Of course, the analysis of steady extension in section 6 does not apply to the highly unsteady cavitation being described here. An estimate of the stress level at cavitation can nevertheless be composed as

$$2\eta\dot{S}$$

with $\dot{S}(t)$ the maximum value of the stress rate between $t = 10.00$ sec when there is no cavity and $t = 10.01$ sec when a cavity has definitely opened. It may

be optimistic, but certainly possible, that the distance the bump on the top mica surface and the bottom surface changes by 1 nm in 10^{-4} to 10^{-5} sec. Then, with $2\eta = 36PaS$ we get

$$3.6 \times 10^5 < 2\eta \dot{S} < 3.6 \times 10^6 \text{ Pa}$$

which is greater than atmospheric pressure. A tension of this magnitude could open up a vacuum cavity. According to Kuhl et al. [1994] "... When a cavity initially forms and grows explosively, it is essentially a vacuum cavity since dissolved solute molecules or gases have not had time to enter into the rapidly growing cavity." The final collapse of the cavity is slower because the cavity fills with vapor.

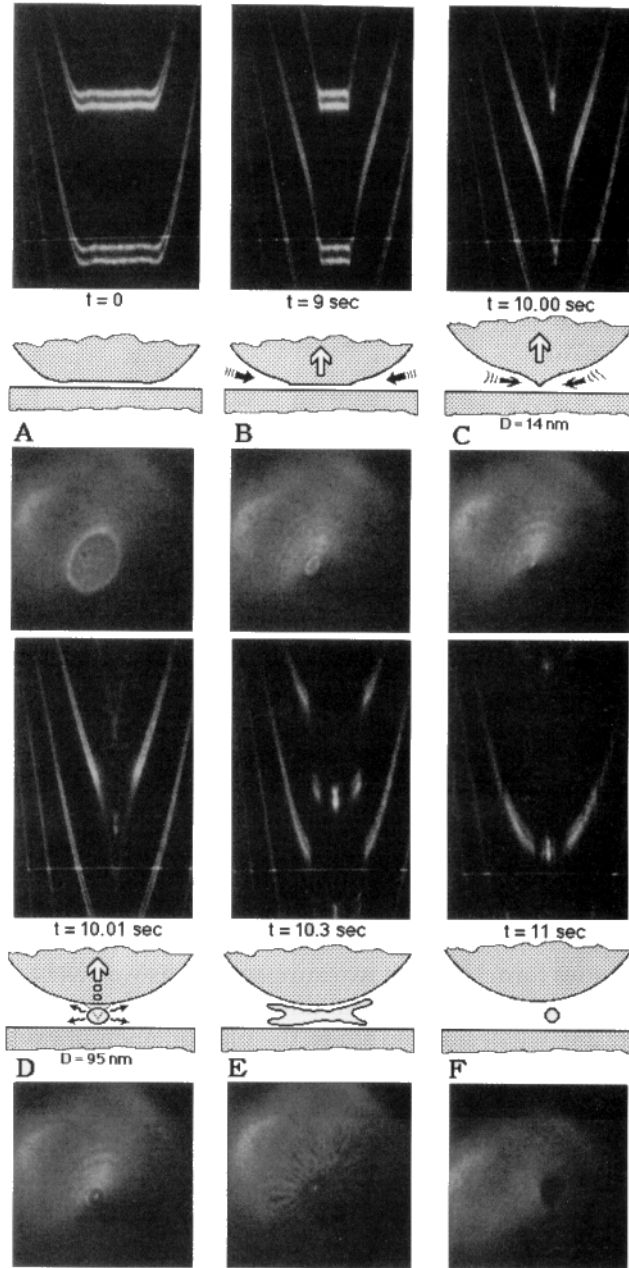


Figure 3: Surfaces separating at high speed, $\nu > \nu_c$, showing FECO fringes (top), schematic side-view (middle), and optical microscope view (bottom) of the surfaces. Homogeneous nucleation of a vapor cavity is shown in D ($t = 10.01 \text{ sec}$), after which the cavity grows rapidly and then collapses (D to F). Note that in picture F (1 sec after inception) the cavity has still not totally disappeared (evaporated or collapsed).

9 Outgassing and cavitation

It is generally believed that vapor bubbles may be initiated from small gas nuclei (Plesset & Prosperetti [1977]). Here we are looking at the situation in which a gas (say air) rather than vapor of the solvent goes in and comes out of solution. The cavitation of pure liquid into vapor is a phase change. When the pressure in the liquid drops below the saturation pressure at a given temperature the liquid is superheated (say, supersaturated) and vaporizes; if the pressure is raised above the saturation pressure, the vapor condenses. Cavitation bubbles are found at places where the liquid is supersaturated.

We may also model the dissolution and cavitation of foreign gases in the solvent as phase changes. Dissolved gases also pass into solution when the pressure is raised above a saturation value; the dissolved gas has condensed into a liquid and the condensed liquid is miscible.

Gases dissolved in liquids experience intermolecular forces between liquid and gas molecules. We could think of a gas molecule as a bridge between two liquid molecules. Of course, you can liquefy gases, but the thermodynamic conditions under which a substance is gas or liquid in isolation need not, indeed should not, apply when the substance is dissolved in another liquid.

Unlike miscible liquids, condensed gas cannot be mixed into the solvent in all proportions; there is a saturation concentration given as “solubility” which is expressed as a weight or volume fraction of gas in solution. Solubility is a thermodynamic variable which depends on temperature and pressure. In the case of condensed gas, saturation conditions are expressed in terms of three quantities, temperature, pressure and gas fraction, rather than the two quantities, temperature and pressure which define saturation in a pure liquid. Naturally, if a solution is degassed, it is starved of gas and will not cavitate.

The thermodynamic of solubility is meant to apply to pure liquids. Such liquids must be carefully prepared; natural liquids are “dirty” and contain dust particles and other impurities which act as nucleation sites for cavitation. Tap water and crude oil are impure in this sense. A precise characterization of the impurities in “dirty” liquids, especially water is problematic. According to Batchelor [1967]

Tests on liquids at rest show that the tendency to form cavities when the pressure is reduced nearly to zero is associated with the continual presence of nuclei which are believed to be tiny pockets of undissolved gas; some liquid vapour is certain to exist also in any small pockets, but the gas, usually air, appears to be the more essential element for cavity formation. It is not known with certainty how these pockets of gas are able to persist in a liquid under normal conditions.

The inward force at the boundary of a small spherical bubble due to surface tension is very strong, much too strong to be balanced by vapour pressure, and gas subjected to this pressure would quickly pass into solution in the liquid. A common postulate is that the pockets of gas and vapour are able to persist in equilibrium under normal conditions by being trapped in crevices in small hydrophobic (non-wetting) solid particles such as dust particles, which are usually present in liquids; the liquid surface in such cracks and crevices can be concave outwards, in which case the direction of the surface tension force is outward. Then when the pressure in the surrounding liquid is reduced sufficiently below the vapour pressure (which for water at 15°C is 1.704×10^4 dyn/cm², or about 0.017 atmosphere), the gaseous pocket grows and, despite the fact that for a cavity larger than the host solid particle the surface tension force is inward, will not be able to find a new equilibrium radius. For tap water and sea water, the critical steady ambient pressure, below which cavities grow indefinitely in size, is found to be different from the vapour pressure of the water by only a small margin which is usually neglected. On the other hand, water which has been compressed for a few minutes at about 700 atmospheres and which is saturated with air can withstand *tensions* of about 25 atmospheres . . . presumably because all except the smallest pockets of undissolved air have been eliminated. Water which has been ‘degassed’ in this way likewise does not boil at atmospheric pressure until the temperature is considerably above 100°C; the phenomena of boiling and growth of cavities in liquids at low pressures are of course mechanically similar.

In Batchelor’s cartoon of impure water just cited we have condensed air and very small pockets of undissolved gas attached to hydrophobic particles. The pockets of undissolved gas are postulated as nucleation sites. A degree of supersaturation is required for these pockets to open “. . . when the pressure in the surrounding is reduced sufficiently below the vapor pressure.” The required supersaturation implies that the pocket opening would occur suddenly, with rapid filling of the gas bubble with gaseous air vaporizing from liquid air in solution.

The same discussion of cavitation of impure water can be applied to flowing water with the added caveat that cavitation is expected when one of the principal values of the stress at the gas pocket falls below the vapor pressure. Since the pocket is attached to a crack or crevice of hydrophobic dust and since supersaturation out of equilibrium is required, the cavitation event is to a degree like the fracture of impure solids with points of weakness at the boundaries of foreign inclusions.

The postulate of gas pockets on hydrophobic particles leaves certain questions unanswered. What is the nature of cavitation in water with no particles or hydrophilic particles? What kind of cracks and crevices can be expected on hydrophobic dust? What does it take to keep hydrophobic dust with gas pockets from floating out of solution? Can hydrophobic particles be floated by bubbling water?

Impure liquids may be regarded as just another kind of (multiphase) liquid with its own thermodynamic properties. Solubility measurements can be carried in impure liquids and gas bubbles will nucleate when one of the principal stresses puts the liquid into supersaturation at a nucleation site.

Solubility measurement for impure liquids have been carried out extensively by Canadian researchers (see Peng et al. [1991]) in studies of “foamy oils” in which copious amounts of gases (Methane, Carbon dioxide, Nitrogen) may be dissolved in bitumen from the Alberta Oil Sands. These oils are a good laboratory for testing the idea that even in impure liquids the dissolved gases are condensed and miscible.

Svrcek and Mehrotra [1982] studied gas solubility, viscosity and density for mixtures of Athabasca bitumens and dissolved CO₂, CH₄ (methane) and N₂ (Nitrogen) gases at saturation. They found that the viscosity and density of saturated oils depends on the gas, bitumen, temperature and pressure and is such that more gas is dissolved at higher pressures and lower temperatures. The viscosity of the solution drops precipitously with the amount of dissolved gas. The viscosity of bitumen without gas is more or less independent of pressure and drops more rapidly with temperature than does the saturated bitumen. The saturation concentration of CO₂, CH₄, and NO₂ in bitumen increases with pressure at each fixed temperature; the increase is linear for pressures below 5 MPa and is almost linear above. The solubility and viscosity are greater at lower temperatures; at room temperature the viscosity of dissolved CO₂ in bitumen drops nearly two orders of magnitude from the value of degassed bitumen, apparently because more condensed CO₂ is added as the pressure is increased. The solubility and reduction in viscosity is greatest for carbon dioxide and least for nitrogen.

The reduction of viscosity of bitumen with increasing concentrations of liquefied gas and the fact that different condensed gases lead to different reductions is consistent with the miscible liquid model. The parameters governing solubility are not so easily modeled.

The miscible liquid model may also be applied to density measurements; assuming that the solution is an ideal mixture the density of the solution ought to be equal to $f_g \rho_g + f_b \rho_b$ where ρ_b and ρ_g are densities of the bitumen and the condensed gas and ϕ is the volume fraction of gas. The volume fraction of condensed gas can be obtained from the measured increase in the solution volume with concentration (the swelling factor in figure 5 of Peng et al. [1991]). The measurements

of Svrcek & Mehrotra [1982] show that the density of carbon dioxide and nitrogen in bitumen is independent of pressure, hence concentration, whereas the methane in bitumen solution decreases linearly with pressure; that is, with concentration. The ideal mixture theory then implies that the density of carbon dioxide and nitrogen condensed in bitumen is nearly the same as the density of the bitumen, but the density of condensed methane is smaller.

10 The effect of polymeric additive on cavitation

It is generally thought that the effects of polymer additives on cavitation are due to the large-scale modification of flow structure due to polymers, as in the suppression of turbulence, rather than to a direct effect on the formation of cavities. However, consideration has not been given to effects that might be associated with the deviatoric part of the stress (rather than the “pressure”) due to motion in such non-linear fluids. Since these direct effects of motion on the formation of cavities is the precise focus of this paper, and they have not been considered before, it is of value to carry this further.

Unfortunately, most of the studies of cavitation in non-Newtonian fluids are restricted to the same extremely dilute solutions of polymer in water which reduce drag. Reviews of these studies have been given by Arndt, et al. [1976], Hoyt & Taylor [1981], Arndt [1981] and Trevena [1987]; they show that the polymer reduces the cavitation index (suppresses cavitation) in some flows, whereas in others the results are ambiguous and even depend on the material of construction of the flow obstacle used to create low pressure. The suppression of cavitation with polymers might be expected to coincide with a reduction in the breaking strength of the solution; this does not appear to be the case. Sedgewick and Trevena [1978] studied the effects of polyacrilamide additives on the breaking tension of water using a static Berthelot tube method and a dynamic bullet piston method. In the static case, the presence of the polymer additive did not noticeably change the breaking tension, but in the dynamic tests the breaking tension was *reduced*; the greater the concentration of polymer the greater was the reduction. To understand these contradictory results it is necessary to better understand how polymer additives modify the state of stress in a flowing fluid.

Mathematical studies of the effects of polymer additives on cavitation have not been published. There are two difficulties: the choice of the constitutive equation and the calculation of the pressure; the “pressure” in an Oldroyd B fluid and in a second order fluid are not in the same relation to the mean normal stress. The composition of the stress into a pressure and a part related a-priori to deformation is not unique, but the state of stress a point given by different constitutive equations

applied to the same problem can be compared.

Ellis and Ting [1974] constructed an argument based on the second order fluid noting that this fluid "... appears capable of explaining the observed cavitation pressure due to flow." Their idea is that cavitation suppression is due to an overpressure in the region of the obstacle where the pressure would ordinarily be low enough to permit cavitation in the case of water. Further, they attribute the difference to the presence of a correction term in the Bernoulli equation for a second order fluid.

Lumley in an appendix to the aforementioned paper agreed with their arguments and conclusions but he objected to using a second order model "... since it includes only in a rudimentary way the effect of molecular extension..." which Lumley argued was responsible for drag reduction. He gets a similar result using a centerline Bernoulli equation for an Oldroyd B model. (Lumley [1972]).

Here, I am going to construct a mathematical analysis based mainly on the second order theory in which the idea of Ellis and Ting is put to test; we find there is a basis for the overpressure they mention, at the boundary of a rigid solid, and there is a Bernoulli equation in potential flow with an extra term. However, the extra term contributes to the promotion rather than to the suppression of cavitation. It is certain that the second order fluid is far from an ideal model for the fast flows usually studied in experiments on cavitation. On the other hand, the second order fluid is the only model of a viscoelastic fluid, other than the linear viscoelastic model, which admits a pressure function in potential flow (Joseph & Liao [1996]). Moreover, the second order fluid admits a pressure function of an entirely different origin, for perturbations of Stokes flow, which allows one to draw conclusions following from the no-slip or incomplete slip condition of real obstacles on which cavitation bubbles appear that cannot be studied in potential flows.

11 Potential flow cavitation of viscoelastic fluids

The constitutive equation for a second order fluid is usually given as

$$\mathbf{T} = -p\mathbf{1} + \eta\mathbf{A} + \alpha_1\mathbf{B} + \alpha_2\mathbf{A}^2, \quad (38)$$

$$\mathbf{B} = \frac{\partial\mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{A} + \mathbf{A}\mathbf{L} + \mathbf{L}^T\mathbf{A},$$

$$\mathbf{L} = \nabla\mathbf{u} \left(L_{ij} = \frac{\partial u_i}{\partial x_j} \right),$$

$$\mathbf{A} = \mathbf{L} + \mathbf{L}^T,$$

$$\alpha_1 = -\Psi_1/2, \quad \alpha_2 = \Psi_1 + \Psi_2 \quad (39)$$

where $\Psi_1 > 0$ and $\Psi_2 < 0$ are the coefficients of the first and second normal stress differences. (A typical value for estimates is $\Psi_2 = -\Psi_1/10$)

The “pressure” p in (38) is not the mean normal stress; it is better for cavitation studies to write (38) as

$$\mathbf{T} = -\tilde{p}\mathbf{1} + \mathbf{S} \quad (40)$$

where, since $\text{Tr}\mathbf{A} = \text{div } \mathbf{u} = 0$ and $\text{Tr}\mathbf{B} = \text{Tr}\mathbf{A}^2$,

$$\tilde{p} = -\frac{1}{3} \text{Tr}\mathbf{T} = p - \frac{1}{3}(\alpha_1 + \alpha_2) \text{Tr}\mathbf{A}^2 \quad (41)$$

and

$$\mathbf{S} = \alpha_1\mathbf{B} + \alpha_2\mathbf{A}^2 - \frac{1}{3}(\alpha_1 + \alpha_2) \text{Tr}\mathbf{A}^2, \text{Tr}\mathbf{S} = 0 \quad (42)$$

The reader has already been alerted several times to the fact that the use of p in cavitation studies has no rational foundation.

Joseph [1992] has shown that equations of motion for a second order fluid admit all potential flow solutions; that is, velocity fields of the form

$$\mathbf{u} = \nabla\phi, \nabla^2\phi = 0 \quad (43)$$

give rise to solutions of

$$\rho \frac{d\mathbf{u}}{dt} = \text{div}\mathbf{T}, \text{div}\mathbf{u} = 0 \quad (44)$$

where \mathbf{T} is given by (38) and p has a Bernoulli equation with an extra term:

$$p + \rho \frac{|\mathbf{u}|^2}{2} - \frac{\hat{\beta}}{4} \text{Tr}\mathbf{A}^2 + \rho \frac{\partial\phi}{\partial t} = 0 \quad (45)$$

The extra term

$$-\frac{\hat{\beta}}{4} \text{Trace } \mathbf{A}^2 = \hat{\beta} \nabla\mathbf{u} : \nabla\mathbf{u} < 0 \quad (46)$$

is positive whenever the climbing constant $\hat{\beta} = 3\alpha_1 + 2\alpha_2$ is positive and it acts in opposition to inertia $\rho|\mathbf{u}|^2/2$. Using the estimate $\Psi_2 = -\Psi_1/10$ we get

$$\hat{\beta} = 2\Psi_2 + \Psi_1/2 = \frac{3}{10}\Psi_1 > 0.$$

Since nearly all polymeric solutions climb rotating rods we may assume that $\hat{\beta} > 0$. Looking at p in (45) we might conclude that the compression due to the extra

term in the Bernoulli equation is increased, suppressing cavitation as was supposed by Ellis and Ting [1970]. However, the mean normal stress \tilde{p} is another (more relevant) pressure given by

$$\tilde{p} + \rho \frac{|\mathbf{u}|^2}{2} - \frac{1}{3}(5\alpha_1 + 2\alpha_2)\nabla\mathbf{u} : \nabla\mathbf{u} + \rho \frac{\partial\varphi}{\partial t} = 0 \quad (47)$$

gives rise to a different result. Since

$$5\alpha_1 + 2\alpha_2 = 2\Psi_2 - \Psi_1/2 < 0 \quad (48)$$

the extra term in the mean normal stress promotes rather than suppresses cavitation.

I have argued that cavitation at any point in a flowing liquid depends on the state of stress and breaking stress there. For potential flow

$$\mathbf{B} = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\mathbf{A} + \mathbf{A}^2 \quad (49)$$

and the deviatoric part \mathbf{S} of the stress \mathbf{T} is given by

$$\mathbf{S} = \left(\eta + \alpha_1 \frac{d}{dt}\right)\mathbf{A} + (\alpha_1 + \alpha_2)\left[\mathbf{A}^2 - \frac{1}{3}\mathbf{1}\text{Tr } \mathbf{A}^2\right] \quad (50)$$

and the mean normal stress $-\tilde{p}$ is given by (47) as

$$-\tilde{p} = +\rho \frac{\partial\varphi}{\partial t} + \rho \frac{|\mathbf{u}|^2}{2} - \frac{1}{12}(2\Psi_2 - \Psi_1/2)\text{Tr } \mathbf{A}^2 \quad (51)$$

since $-2\Psi_2 + \Psi_1/2 > 0$ the term with $\text{Tr } \mathbf{A}^2$ is positive, tensile promoting cavitation. Therefore the suppression of cavitation must arise from deviatoric part of the stress. However, the deviator has both positive and negative eigenvalues (see Joseph [1992]) with the positive ones corresponding to tension, promoting cavitation.

The progressive reduction of the breaking strength with increasing concentration which was observed by Sedgewick and Trevena [1978] is consistent with the analysis just given, but the use of a second order fluid model to study cavitation in potential flows is not justified, too many effects are left out of the model.

12 Flow cavitation on solids

We can study flow cavitation for nearly steady flow of a viscoelastic fluid over bodies on which the no-slip condition applies as a viscoelastic perturbation of Stokes flow. The second order fluid arises universally as the asymptotic form taken by all

the special models in common use. Though the motions to which such an analysis can rigorously apply are restricted, the rheology for those motions is completely general; every special model has η, α_1, α_2 as limiting values of model parameters in nearly steady slow motion.

The perturbation of Stokes flow with a second order fluid gives rise to a pressure function for motions in two-dimensions (Tanner [1966]) and for motions in three dimensions when $\alpha_1 = -\alpha_2$ (Giesekus [1963]). The implications for forces on solid particles of the two-dimensional reduction were studied by Joseph [1996] and Joseph & Feng [1966] who showed that the normal stress on each and every point on the boundary of a rigid solid is given by

$$T_{nn} = -p_s + \frac{\Psi_1}{2\eta} \left(\frac{\partial p_s}{\partial t} + \mathbf{u} \cdot \nabla p_s \right) - \frac{\Psi_1}{4} \overset{\circ}{\gamma}^2 \quad (52)$$

where n is a coordinate along the outward \mathbf{n} on the boundary $\Psi_1 = -2\alpha_1, \overset{\circ}{\gamma}$ is the shear rate of the Stokes flow at that point and p_s is the Stokes flow pressure obtained from solving $\nabla p_s = \eta \nabla^2 \mathbf{u}$ for no-slip boundary condition. For steady flows over stationary bodies like those used in cavitation studies we get

$$T_{nn} = -p_s - \frac{\Psi_1}{4} \overset{\circ}{\gamma}^2 \quad (53)$$

which is positive in tension; the viscoelastic contribution $-\Psi_1 \overset{\circ}{\gamma}^2 / 4$ is a compression. In the three dimensional case with $\alpha_1 + \alpha_2 = 0$, the Stokes flow problem can be solved and $p_s = p_\infty - \frac{3}{2} \eta \frac{U}{a} \cos \theta$ and $\overset{\circ}{\gamma} = \frac{5}{2} \frac{U}{a} \sin \theta$ where θ is the polar angle.

The shear rate $\overset{\circ}{\gamma}$ is greatest where the streamlines of the Stokes flow are most crowded; these are the “high” velocity points on the body. Since compressive stresses are generated at “high” velocity points, we get additional “pressure” due to viscoelasticity precisely at the “cavitating” points where the pressure is lowest in potential flow. The compressive contribution $-\Psi_1 \overset{\circ}{\gamma}^2 / 4$ is largest, for example, near the equator of a sphere where cavitation bubbles would appear if no polymers were present.

Van der Meulen [1973] has shown that cavitation inception on a hemispherical-nosed stainless steel body in a water tunnel is greatly reduced by the presence of polyethylene oxide, while a teflon coating showed a much smaller effect. If gas is nucleated at the surface of the hydrophilic stainless steel body it will be replaced by water. The additional pressure due to (52) is effective in suppressing this nucleation of gas. The teflon coated sphere is hydrophobic. Since it likes gas, the liberation of cavitation bubbles is not so evident and not readily replaced by

water. The additional pressure should also work here, but its overall effect could be less evident due to the adhesion of nucleated gas on the teflon.

The formula (53) should not be expected to apply strictly to the high speed flows in which cavitation is suppressed in experiments. Moreover, Ψ_1 is not known in the dilute solutions used in these experiments. It is nevertheless of interest to use (53) for an estimate, to see if it gives rise to a pressure level large enough to effect cavitation.

To construct such an estimate we first use data from figure 3 of Arndt et al. [1976] for flow at $U = 18.3\text{m/s}$ of a 20ppm solution in water of Polyox WSR-301 over a $2R=7.62\text{cm}$ diameter hemispherical nosed body. Estimating $\dot{\gamma} = O(U/R)$ we get $\dot{\gamma} \approx 500\text{sec}^{-1}$, $\dot{\gamma}^2 \approx 2.5 \times 10^5\text{sec}^{-2}$. To get pressure increases of the order of the vapor pressure; i.e., one hundredth of an atmosphere we would need to satisfy

$$\frac{\Psi_1}{4}(2.5 \times 10^5\text{sec}^{-2}) = 10^4 \frac{\text{dynes}}{\text{cm}^2} \quad (54)$$

or

$$10\Psi_1 = O(1) \frac{\text{gm}}{\text{cm}} \quad (55)$$

This is a value of Ψ_1 smaller than those which can be measured on rheometers but it is possibly larger than the value of Ψ_1 which could be expected from solutions so dilute as 20ppm.

A second estimate taken from data in Table 2 in the paper by Ellis, Waugh and Ting [1970] for flow of different fluids over a $1/4$ in radius hemispherical nose body; typical shear rates are of the order of

$$\frac{U}{r} \approx \frac{500}{1/4} = 2000\text{sec}^{-1}$$

For fluids, like aqueous 50 ppm polyox, for which cavitation suppression was observed we need Ψ_1 larger than

$$100\Psi_1 = O(1) \frac{\text{gm}}{\text{cm}}. \quad (56)$$

Ting [1978] studied cavitation on flat top posts protruding from a disk rotating in aqueous polyox FRA solutions with concentrations of 100, 250 and 500 ppm. The 0.29cm diameter posts were 0.29cm high and cavitation suppression was observed for values of $Re = \rho U d / \eta > 10^5$ where, for water $\rho/\eta = 100^{-1}$ and $d = 0.29$. Hence

$$\dot{\gamma} \cong \frac{U}{d/2} \cong 5000\text{sec}^{-1}$$

corresponding to estimates like (54) giving rise to

$$\Psi_1 \approx 1.6 \times 10^{-3} \frac{\text{gm}}{\text{cm}}. \quad (57)$$

Dilute polymer solutions might have normal stress coefficients larger than (57) but nothing is known. In section 12 we will argue that these viscoelastic effects, with very small $\Psi_1 = 2\eta\lambda_1$ where λ_1 is the relaxation time of the fluid, may be seen on small but not large bodies.

The estimates following from (53) are perhaps not so outrageous as to eliminate consideration of the mechanism of inhibition of cavitation inception by shear-induced normal stress. This mechanism has not much in common with mechanisms involving extensional effects due to the irrotational stretching of polymer coils which are popular in theory of drag reduction. Some of the problems encountered in estimating the viscoelastic effects in an irrotational strain field were discussed by Arndt et al. [1976] and they appear to support an alternate explanation presented first by Arakeri and Acosta [1973] which is discussed next.

Arakeri and Acosta [1973] have shown that polymer additives influence the point of laminar separation and inhibit cavitation inception. Van der Meulen [1976] did holographic studies in which he shows that when polymers are added, separation of the boundary layer does not occur and the boundary layer exhibits a turbulent character. This may be compatible with the inhibition of cavitation inception by shear induced normal stresses. In a Newtonian fluid, the position of separation is controlled by the pressure variation on the solid surface. In a viscoelastic fluid the normal stress T_{nn} is not given by pressure alone and terms contributed by normal stresses due to shearing do not vanish; obviously the fluid at the boundary of a solid can sense T_{nn} only and various decompositions of the stress giving rise to different expressions for the pressure are not relevant. To their credit, Ellis et al. [1970] recognize this when they say (p. 463) that "... cavitation bubble inception is inhibited by an actual change of principal stress (or less precisely, pressure)..." The effects of viscoelastic contributions to the variation of T_{nn} on the boundary of a solid which determines the position of points of separation are not clear. Arndt [1981] notes that "... All that can be said at the moment is that available experimental and theoretical information do not provide any clear cut criteria for the determination of early transition in the flow of polymer solutions."

13 Size effects and the elasticity number

A very interesting and robust effect in cavitation inception inhibition is that the inhibition is large for small bodies and is negligible for large bodies. Arndt et al.

[1976] studied cavitation in two recirculating water tunnels using four hemispherical nosed bodies ranging in size from 6.3mm to 203mm. They found that the desinent value of the cavitation index was reduced as much as 50% on the smaller bodies but the effect on the 203mm was negligible. The effect persisted even in well degraded polymer solution. Huang [1971] noted that the cavitation inception reduction was much smaller when a larger model was used in a water tunnel. A small propellor was observed by White [1971] to exhibit greatly reduced cavitation, in agreement with the diameter effect hypothesis.

A size effect scaling with the square of the radius is consistent with the equations governing the motion of viscoelastic fluids over bodies. For example, in the flow of a second order fluid the effects of inertia proportional to ρu^2 and viscoelastic normal stresses proportional to $\Psi \dot{\gamma}^2$ are in opposition and estimating $\dot{\gamma} \approx u/a$ where a is the particle size, we find that viscoelasticity dominates near the body when $\Psi_1/\rho a^2$ is large.

The same scaling follows from analysis of Oldroyd B fluids made dimensionless by $[a, U, a/U, \eta U/a] = [\text{length, velocity, time, stress}]$. The dimensionless equations for momentum and stress then take the form

$$\begin{aligned} Re\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) &= -\nabla p + \nabla \cdot \mathbf{T}, \\ \mathbf{T} + De \overset{\nabla}{\mathbf{T}} &= \mathbf{A} + \frac{\lambda_2}{\lambda_1} De \overset{\nabla}{\mathbf{A}} \end{aligned}$$

where $\nabla \cdot \mathbf{u} = 0$ and

$$\overset{\nabla}{\mathbf{T}} = \frac{\partial \mathbf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{T} - \nabla \mathbf{u}, \mathbf{T} - \mathbf{T} \cdot \nabla \mathbf{u}^T.$$

The parameters are:

$$\begin{aligned} Re &= Ua\rho/\eta \quad (\text{Reynolds number}), \\ De &= U\lambda_1/a \quad (\text{Deborah number}), \\ \lambda_2/\lambda_1 &\quad (\text{retardation / relaxation time ratio}). \end{aligned}$$

The maximum elastic effects are for $\lambda_2 = 0$ (Maxwell model). The fluid is Newtonian when $\lambda_2 = \lambda_1$ and $0 \leq \lambda_2/\lambda_1 \leq 1$.

Huang, Hu and Joseph [1997] used these equations to study the sedimentation of an ellipse with semi-major axis a in an Oldroyd B fluid. They found that the results correlated better with

$$E = De/Re = \lambda_1 \eta / \rho a^2 \quad (\text{elasticity number})$$

and

$$M = \sqrt{ReDe} = U/c \quad (\text{viscoelastic Mach number})$$

where $c = \sqrt{\eta/\lambda_1\rho}$ is the speed of shear waves. They found that when the elasticity number is smaller than critical the fluid is essentially Newtonian with broadside-on falling at the centerline of the channel. For larger elasticity numbers the settling turns the long side of the particle along the stream in the channel center for all velocities below a critical one; identified with a critical Mach number of order one.

The elasticity number depends on the fluid and the particle size and not on the velocity. It says that small particles can experience elastic response in circumstances in which larger particles have essentially Newtonian response, as in experiments on cavitating flows over bodies. On the other hand, the Mach number does not depend on the particle size. The elastic response of small particles is thus rather firmly implied by direct and elementary study of the equations governing the motion of Oldroyd B fluids.

14 Discussion

It is perhaps useful to think of cavitation as a topic in a more general theory of breaking and mixing of incompressible materials in which cavitation and fracture of amorphous solids are different, but perhaps related phenomena. Such a theory might lead to better understandings of the dynamics of mixing and mixers which is an important but as yet undeveloped subject.

The breaking of an incompressible continuum is obviously controlled by breaking stresses and cavitation is not different. The fluid knows the state of stress at a point but it cannot decompose this stress into a pressure determined by the flow dynamics and a constitutively determined extra stress. It is however useful, and is always possible, to define the pressure as the mean normal stress, even when, as in many viscoelastic fluids, it is not originally so defined; in this case the extra stress is deviatoric, with plus and minus entries on its leading diagonal.

It may be assumed that in each liquid or solid there is a field of breaking strengths defined at each and every point of the continuum; this strength field could be discontinuous with singular values at nucleation sites. The breaking of a liquid requires that we compare some function of the principal stresses with the strength criterion; for instance it would be consistent with the literature on cavitation to look for the tensile strength of liquids. In this case we look at the magnitude of the maximum value in the diagonalized extra stress, which must be positive and compare it to the breaking strength (cavitation inception) of the liquid. Our comparison requires that we distinguish also the direction of the maximum tension in the moving continuum, which is determined by diagonalization of the stress tensor; we could look at equi-tension lines and seek the locus of maximum values. It is not completely clear that the maximum tension is the right criterion, though it

is in some sense conventional and does appear to agree with experiments on the tensile strength of liquids. One interesting consequence is that we could test the criterion and the directionality of this criterion by setting the pressure level slightly above the breaking strength in a pure shear flow between parallel plates in which the maximum tension is 45° from the flow direction.

The state of stress in a flowing polymeric liquid could in principle have strong effects on cavitation. The evidence from experiments with dilute solutions used for drag reduction weighs strongly for cavitation inhibition, though neutral inhibition results for cavitation on large bodies and a reduction in the breaking strength of liquids with polymer concentration have been reported. It is argued that potential flow approximations are not appropriate for cavitation studies except possibly in the case of acoustic cavitation. An analysis of cavitation inhibition of polymeric liquid based on the second order fluid model, in which the original pressure in the model is embedded in a more appropriate decomposition based on the mean normal stress, shows that inhibition will not arise in potential flows, increased cavitation is predicted. It was stressed that the definition of pressure in non-Newtonian fluids is not unique and differs from one constitutive equation to another, but the mean normal stress is uniquely defined and sets an appropriate level against which one can measure the differences between principal compressive and tensile stresses.

Another positive feature of the second order fluid model for studies of cavitation is that a pressure function can be derived a-priori for flows which perturb Stokes flow as well as for potential flows. In the Stokes flow perturbation the no-slip condition is applied so that the analysis is appropriate for discussions of cavitation which originate from nucleation on solid boundaries. An additional normal stress T_{nn} equal to $-\Psi_1 \dot{\gamma}^2/4$ which is always compressive and proportional to the square of the shear rate at each point on the boundary of a solid body is equivalent to a viscoelastic pressure which is large at places where the flow is fast, the opposite of inertia. Though the speeds of cavitating flows are larger than those for which analysis based on second order models is valid, estimates of the size effects do give rise to additional pressures which could suppress nucleation at solids. An important additional effect is that the variation of “backpressure” which controls the position of points of separation would be changed by the presence and variation of this additional viscoelastic pressure.

All evidence from experiments indicates that cavitation inhibition is much more pronounced on small than on large bodies. This observation is consistent with the analysis of response Oldroyd B fluids. The controlling parameters of these fluids are the Reynolds number Ua/ν and the Deborah number $U\lambda/a$. The Deborah number shows a size dependence of the required kind, but the number depends on U which also depends on a . The elasticity number E which is the ratio $De/Re =$

$\lambda\nu/a^2$ is independent of U and it indicates enhanced elasticity for small bodies in a given fluid and the size effect is proportional to a^2 rather than a . The predictions of viscoelastic response based on the elasticity number should be robust since it arises from direct analysis of dimensions but it has not yet been tested in cavitation studies.

15 Conclusions

- The pressure in incompressible Newtonian fluids is the mean normal stress. The stress is decomposed into a pressure and stress deviator with a zero trace. The pressure in incompressible Non-Newtonian liquids is given by the constitutive equation and has no intrinsic significance. Cavitation criteria for liquids in motion must be based on the stress and not on the pressure. The liquid cannot average its stresses or recognize the non-unique quantity called pressure in non-Newtonian fluids.
- It is convenient for the study of cavitation of flowing liquids to decompose the stress into a deviator and mean normal stress. The deviator has positive and negative normal stresses, deviating from the average. The most positive value of principal stresses is the maximum tension. The stress in non-Newtonian liquids should also decompose the stress into average and deviator.
- A cavitation bubble will open in the direction of maximum tension in principal coordinates. The angles defining the principal axis determine how a cavity will open; angles are important.
- A liquid can cavitate in shear. However, it is pulled open by tension in the direction defined by principal stresses; Newtonian liquids in pure plane shear will open 45° from the direction.
- Cavitation in a flowing liquid will occur at a nucleation site when the maximum tensile stress in principal axes coordinates is such as to make the total stress smaller than the cavitation pressure.
- Cavitation is a fast, non-equilibrium event resembling fracture in which the cavity first opens and then fills with gas.
- Outgassing is cavitation of liquid gas in solution.
- Analyses of the second order fluid shows that explanations based on irrotational strain fields are not consistent with inhibition but cavitation on solid

bodies can be inhibited by compressive normal stresses due to shear which give rise to increased pressures preventing the nucleation of gas on bodies. Boundary layer separation is controlled by the variation of the total normal stress which has a substantial viscoelastic component. The observed size effects on cavitation in the presence of polymer additives in which inhibition is seen on small but not large bodies is implied by the elasticity number which is independent of velocity and increases with the reciprocal of the square of particle radius.

References

- V.H. Arakeri and A. Acosta, Viscous Effects on the Inception of Cavitation on Axisymmetric Bodies, *ASME, J. Fluids Eng. Ser I* **95**, 519-528, 1973.
- L.A. Archer, D. Ternet, R.G. Larson, "Fracture" phenomena in shearing flow of viscous liquids, *Rheol Acta* **36**:579–584, 1997.
- R.E.A. Arndt, Recent Advances in Cavitation Research, *Advances in Hydro-science*, Academic Press, **12**, 1-78, 1981.
- R.E.A. Arndt, M.L. Billet, J.W. Holl, and C.B. Baker, A note on the inhibition of cavitation in dilute polymer solutions, *ASME Cavitation and Polyphase Flow Forum*, ASME New York, 1976, pp. 1-3.
- G.K. Batchelor, *Fluid Dynamics*, Cambridge University Press, 1967.
- C.E. Brennen, *Cavitation and bubble dynamics*, *Oxford Engineering & Sciences Series 44*, Oxford University Press, 1995.
- M.P. Brenner, J. Eggers, K. Joseph, R. Nagel and X.D. Shi, Breakdown of scaling in droplet fission at high Reynolds numbers, *Phys Fluids*, **9**, 1573-1590, 1997.
- Y. Chen and J. Israelachvili, New mechanism of cavitation damage, *Science*, **252**, 1157-1160.
- A.T. Ellis and R.Y. Ting, Non-Newtonian effects on flow-generated cavitation and on cavitation in a pressure field, *Fluid Mechanics, Acoustics, and Design of Trubomachinery*, Part I, NASA SP-304, Supt. of Documents, U.S. Government Printing Office, Washington, D.C., No 3300-00583, 1974, pp. 403-421.
- J.P. Franc and J.M. Michel, Attached cavitation and the boundary layer; experimental investigation and numerical treatment. *J. Fluid Mech.* **154**, 63–90, 1985.
- H. Giesekus, Die simultane Translations - und rotations - bewegung einer Kugel in einer elastoviskosen Flüssigkeit, *Rheol. Acta* **3**, 59-71, 1967.
- J.W. Hoyt and J.J. Taylor, A photographic study of cavitation in jet flow, *Transactions of the ASME*, **103**, 14-18, 1981.

- P.Y. Huang, H. Hu and D.D. Joseph, Direct simulation of the sedimentation of elliptic particles in Oldroyd-B fluids, Accepted for publication in *J. Fluid Mech.*, 1998.
- T.T. Huang, Comments on Cavitation Inception: The influence of roughness, turbulence and polymer additives, *Trans. 16th Towing Tank Conference*, Sao Paulo, **1**, 6.10, 1971.
- D.D. Joseph, Bernoulli equation and the competition of elastic and inertial pressures in the potential flow of a second-order fluid. *J. Non-Newtonian Fluid Mech*, **42**, 385–389, 1992.
- D.D. Joseph, Cavitation in a flowing liquid, *Phy. Rev. E*, **51** (3), 1649-1650, 1995.
- D.D. Joseph and J. Feng, A note on the forces that move particles in a second-order fluid, *J. Non-Newtonian Fluid Mech.*, **64**, 299-302, 1996.
- D.D. Joseph, A. Huang, and G. Candler, Vaporization of a liquid drop suddenly exposed to a high speed air stream. *J. Fluid Mech*, **310**, 223–236, 1996.
- R. Knapp, J.W. Daily, F. Hammit, Cavitation, Mc Graw Hill, N.Y., 1970.
- T. Kuhl, M. Ruths, Y.L. Chen, J. Israelachvili, Direct visualization of cavitation and damage in ultrathin liquid films. *The Journal of Heart Valve Disease*, **3**, (suppl. I) 117-127, 1994.
- J.L. Lumley, Centerline-Bernoulli equation for Quasisteady dilute polymer flow, *Phys Fluids*, **15**(2), 217-219, 1972.
- T.S. Lundgren & D.D. Joseph, Capillary collapse & rupture, Submitted to *Phys. Rev. Letters*, 1997.
(see <http://www.aem.umn.edu/people/faculty/joseph/papers/downcap.html>).
- M.E. Mackay, and M.E., Paulaitis, Solid Solubilities of Heavy Hydrocarbons in Supercritical Solvents, *Ind. Engng. Chem. Fundam.*, **18**(2), 149-153, 1979.
- J.H.H. van der Meulen, Cavitation suppression by polymer injection, *1973 Cavitation and Polyphase Flow Forum*, ASME, pp. 48-50.
- J.H.H. van der Meulen, Holographic study of polymer effects on cavitation, *ASME Cavitation and polyphase flow forum*. ASME New York, 4-5, 1976.
- L.M. Milne-Thompson, Theoretical Hydrodynamics, The Macmillan Co., N.Y. 1960.

- Y.D. Peng, C.T. Fu, G. Bird, and C. Hsi, Effects of gas components on thermodynamic properties of Alberta heavy crudes and bitumen. *5th UNITAR Heavy Crude and International Conference*, Caracas, **1**, 47–55, 4–9 Aug 1991.
- M. Plesset, Tensile strength of liquids. *Office of Naval Research*, Report no. 85-4, April 1969.
- M. S. Plesset, and A. Prosperetti, Bubble dynamics and cavitation, *Ann. Rev. Fluid Mech.*, **9** 145–185, 1977.
- S.A. Sedgewick, D.H. Trevena, Breaking tensions of dilute polyacrylamide solutions, *J. Phys. D.*, **11**, 1978.
- W.Y. Svrcek, and A.K. Mehrotra, Gas solubility, viscosity and density measurements for Athabasco bitumen, *J. Canadian Petroleum Technology*, **21**,(4) 31–38, 1982.
- G.I. Taylor, The formation of emulsions in definable fields of flow, *Proc. R. Soc. London Ser. A*, **146**: 501–23, 1934.
- R. Tanner, Plane creeping flows of incompressible second-order fluids, *Phys. Fluids*, **9**, 1246-1247, 1966.
- M.H. Wagner, V. Schulze and A. Göttfert, Rheotens – Mastercurves and Drawability of Polymer melts, *Polymer Eng. Sci*, **36**(7), 925-935, April, 1996.

Acknowledgement. This work was supported NSF/CTS - 9523579 and ARO grant DA/DAAH04. Helpful discussions with Roger Arndt are gratefully acknowledged.