A mixture model for foamy oil flow in porous media[1]

D. D. Joseph, A.M. Kamp, P.H. Hammond , M. Huerta[2]

Working document

> *Revision 1, November 6, 1997*
> *[Arjan Kamp] Some remarks: I have somewhat rewritten the introduction, trying to centre the work more in a general context. I hope that this does no harm to Dan's original ideas. The part on solubility has been rewritten as well, introducing Henry's law. I added a short appendix to justify the neglect of gas phase compressibility compared to solubility. I somewhat changed the notation and the terminology in order to be more in line with the usual practices in oil industry.*
> *Some recent thought: should we take into account the effect of the dissolved gas concentration on mobility through the viscosity dependence?*
> *Left to be done:*
> *-insert the numerical method and solutions for the equilibrium model*
> *-perform and insert numerical solutions of the complete model*
> *-write conclusions and put a clear link to field applications*

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[1] Originally based on *"Foamy oil flow in porous media" by D. D. Joseph*
[2] Order of names arbitrarily

Abstract

The processes of nucleation, bubble growth and finally the creation of connected gas during the pressure draw-down in an oil reservoir are discussed. From these mechanisms, possible descriptions of foamy oil production appear. One of these mechanisms is the creation of bubbles, small compared to pore sizes, which move with the oil phase and which push the oil out of the reservoir. If this mechanism occurs, a very simple model can describe the production of foamy oil. This model is obtained by establishing an equilibrium relation between gas saturation and pressure, which then is modified into a constitutive equation relating gas saturation and pressure in a dynamic context where super-saturation is present. Together with a mixture mass conservation equation and Darcy's law, a two equation description of foamy oil production is obtained, depending on only two dimensionless numbers. It is thought that this is the most simple possible way to describe some particular cases of anomalous behavior of foamy oil. The model is tested against laboratory experiments and a satisfactory agreement is obtained. The implications of the model for the production of foamy oil reservoirs are eventually discussed.

1 Introduction

Some heavy oil reservoirs, most of which are located in Canada and Venezuela, produce at the well-head an oil with obvious foaminess, so called foamy oil. Comparing its production to conventional solution gas drive, most of these reservoirs exhibit anomalous *high production rates* and *high primary recovery factors*. The *gas-oil ratio* is often low, and stays low in time, whereas light oil reservoirs show a rapidly increasing gas-oil ratio when the pressure decreases below the bubble point pressure (Maini, 1996). Some of the foamy oil reservoirs show anomalous high sand production and so-called worm holing (Wang, 1997, Geilikman et al., 1995, Smith, 1986) although this does not seem to be a general rule. Especially in Venezuela such anomalous sand production is rarely observed in reservoirs producing foamy oil. The solubility of gas in foamy oils does however not appear to be much higher than that in light oils (comparison of for example the Methane / Decane system with Methane / Athabasca bitumen, the solubility in the latter is found to be approximately 1.5 times higher, see Reamer et al, 1942 and Svrcek & Mehrotra, 1982).

Although the reasons for the favorable response of foamy oils in solution gas drive are not completely understood and tentative explanations which have been put forward are somewhat controversial (see Maini, 1996 and Pooladi-Darvish & Firoozabadi, 1997 for recent reviews), some general concepts of foamy oils have emerged.

Oils contain components of low molecular weight, which at initial reservoir pressures are dissolved in the oil. If the reservoir pressure is above the bubble point pressure, the oil is called *sub-saturated*. If the pressure in the reservoir is however drawn down, and the bubble point pressure is crossed, the light component can no longer be dissolved in the oil. The oil becomes *super-saturated* and starts to release gas by nucleation of bubbles and subsequent diffusion of gas into these bubbles.

Initially the bubbles are small compared to the average pore size. It is then probable that most of the bubbles move with the liquid phase, although some bubble trapping potentially might occur. The bubble sizes however gradually increase due to diffusion of gas into the bubbles and due to coalescence of several bubbles into larger bubbles. Eventually, the bubble diameter will become of the order of magnitude of average pore size. It is quite possible that if this happens, the gas and liquid can no longer be treated as

a mixture and the bubbles move at a velocity which differs from the liquid velocity, because of the different relative permeability. Finally recombination of many large bubble might lead to connected gas which flows, due to its low viscosity, much more rapidly than the liquid phase. The gas saturation at which this happens is called the *critical gas saturation*. Once connected gas is created, the oil reservoir stops producing oil without artificial stimulation.

This explains the tight connection between critical gas saturation and primary recovery. When the gas becomes connected, the end of primary recovery becomes in sight. The (critical) gas saturation is then approximately equal to the fraction of oil pushed out of the reservoir, which is the primary recovery.

The distinct phases in the process of nucleation and bubble growth are schematically depicted in figure 1.

(a)   τ₁   (b)   τ₂   (c)   τ₃   (d)
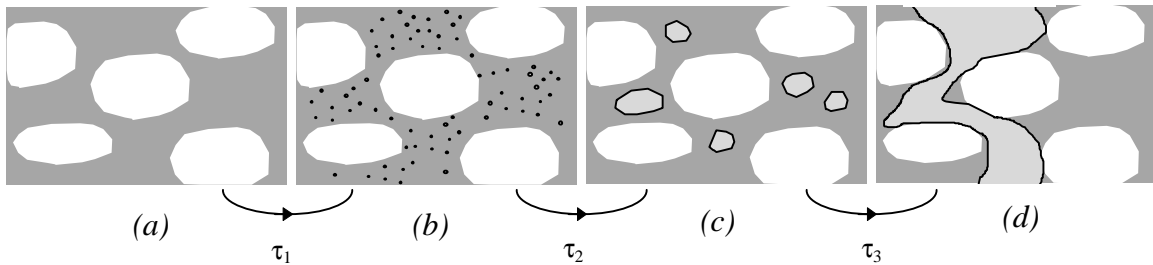
Figure 1   Schematical representation of nucleation and bubble growth in a porous media (white is rock, dark grey is oil and light grey is gas): (a) only dissolved gas; (b) bubbles small compared to average pore size; (c) bubbles of size comparable to average pore size; (d) fully connected gas. Typical time scales of transformation between the stages are indicated as τ_1, τ_2 and τ_3.

It is thought that the precedent description applies to any oil reservoir, whether the oil is light or heavy and independent of composition. The difference between foamy and conventional oil reservoirs is however that the characteristic time scales at which the processes occur are very different. In normal light oil reservoirs it is thought that the transformation of small bubbles, through larger bubbles to connected gas, happens on a time scale which is short compared to a characteristic time scale of the pressure draw-down of the reservoir. In foamy oils however, this is probably not the case. Possible reasons are the following

1. The viscosity of heavy oil is much higher than the viscosity of light oil and the diffusion coefficients of dissolved gas molecules are much smaller. This means that if the pressure is decreased, the transfer of gas from a dissolved state into a small bubbles occurs slower in a heavy oil than in a light oil. Therefore super saturation might at a same pressure decline be higher in heavy oil.

2. The transformation of small bubbles into large bubbles is governed by the coalescence between two bubbles, which increases the mean bubble size. The coalescence process involves the drainage of an oil film which is created between two bubbles which are in near contact. This drainage takes more time when the viscosity of the oil is higher (Chesters, 1991).

3. Also the presence of surface active components such as, for example, asphaltenes or other large chain molecules, might significantly increase drainage time by the so-called *Marangoni* effect.
4. The coalescence of bubbles is however not only governed by a film drainage process but also by a collision process which has to occur before film drainage can take place. It might be expected that the collision rate between bubbles is lower when the viscosity of the fluid is higher.

Thus the transformation of small bubbles into large bubbles and of large bubbles into connected gas, will be slower in heavy oils than in light oils.

Within this framework, some of the typical foamy oil observations can be explained. The low gas-oil ratio follows from the fact that the creation of connected gas is very slow, so that gas moves with the oil, or even slower than the oil if some bubbles are trapped. The high critical gas saturation is simply explained by the fact that coalescence between bubbles is more difficult in a heavy than in a light oil, so that higher gas saturation can be achieved without creating connected gas. The high production rates might be directly the result of the fact that all the dissolved gas helps to push the oil out, and that none of this gas escapes the reservoir by creating channels of connected gas.

The terminology "foamy oil" might however be somewhat misleading. At actual reservoir pressures, gas saturation can be as low as 5 to 10%, far below the gas saturation value of 70% at which bubbles are arranged in close packed arrays, something usually associated with the onset of foaming. It might be possible that this bubbly liquid, that we call foamy oil, actually behaves like a foam, when bubble sizes become large. In this case, the thickness of the "oil lamellae" between the bubbles would be small compared to the bubble sizes, which might be seen as a typical characteristic of foam.

It is also not quite clear from the previous description whether foamy oils display a foaming threshold. That is to say, whether for example the rate of pressure draw-down is determining for whether the oil "foams" or not. Eventually bubbles will always be generated, but it might be that the actual bubble size depends on the pressure decline by for example differences in achieved super saturating. This change in bubble size might then influence the critical gas saturation and so couple back on achievable primary production.

1.1 A model for bubbly oils

The creation of a model for the processes of nucleation and bubble growth assumes that we know at which rates they take place. A detailed mathematical description appears however to be extremely difficult. Nucleation modeling involves often quite a lot of undetermined parameters, which are difficult to obtain in practically relevant situations. Activation energies are not well known and neither are the role and distribution of nucleation sites. Nucleation modeling often introduces extra time scales which make reservoir simulation numerically more difficult to achieve. Film drainage and coalescence between bubbles are processes on which still much work has to be done. Also the dependence of relative permeabilities on bubble sizes and gas saturation are not fully understood.

For these reasons, it seems wise to restrict potential models to the description of situations where one transfer rate is much slower than all other transfer rates. Three possibilities are indicated in table 1.

determining time scale	observed behavior
τ_1	few bubbles, oil heavily supersaturated
τ_2	small bubbles (possibly moving with the oil)
τ_3	large (possibly trapped) bubbles

Table 1   Different observed behaviours for different determinant time constants

In this document we will try to set up a simple description for the case where the transfer rate from small bubbles to large bubbles is much lower than the transfer rate for dissolved gas to small bubbles. In this case only bubbles small compared to pore sizes are present and it seems plausible to assume that these bubble move with the liquid flow. In this case it is thus possible to relate gas saturation to the local concentration of dissolved gas.

The proposed model does not require any microscale information about nucleation, bubble growth, compressibility or forces which produce relative velocity. We put up a mixture theory in which the dispersed gas is described by a gas saturation field in a single fluid in which the viscosity, density and mobility in Darcy's law all depend on the gas saturation. This fluid satisfies the usual Darcy law, and the continuity equation together with a kinetic (constitutive) equation required by the condensation and outgassing of the heavy crude. The theory depends only on three parameters which can be measured in a PVT cell and sand pack experiments: a mobility, a solubility and a time constant for mass transfer from a dissolved state to free gas. The virtue of the model is simplicity, but it can only work if the gas is dispersed and moves with the oil. If the gas phase does not move with the oil, no relation between local concentration of dissolved gas and local gas fraction can be derived.

Certainly the theory could not be expected to give rise to a percolation threshold or even to a critical gas fraction, but it might describe many features of solution gas drive of foamy oils in the dispersed bubbly mixture regime.

It is our idea that the increased recovery and production are generated by the pumping of nucleating and growing gas bubbles. The gas saturation increases the volume of the composite fluid and it acts as a pump, the gas coming out of solution pumping the fluid outward. This pumping action is well described by the continuity equation (2.22) which implies that in a closed volume V with boundary S containing dispersed bubbles of gas saturation S_g

$$\int_V \frac{1}{1-S_g} \frac{DS_g}{Dt} dV = \oint_S \mathbf{q}_m \cdot \mathbf{n} dS \tag{1.1}$$

Where $\mathbf{n}$ is the outward normal on S and $\mathbf{q}_m$ is the velocity of our composite fluid.

2 Model description

2.1 Solubility of gas in crude oil

In the following an equilibrium relation between pressure and void fraction is derived by considering a hypothetical PVT experiment with a foamy crude oil. The experiment consists of pulling out the piston of a PVT cell, which initially contains live oil

at saturation pressure p_{sat}. In this process a volume V_g of gas is generated by dissolved gas coming out of solution. Since the total mass of gas which comes out of solution is low compared to the total liquid mass, it is supposed that in this process of nucleation and bubble growth, the volume V_o and the density ρ_o of the crude oil do not change.

In order to estimate how the gas saturation depends on pressure, it is necessary to know something about the solubility of gas in heavy oil. The concentration of a dissolved gas in a liquid is normally characterized in terms of mole or mass fraction of gas in that liquid. Note that these are not linearly related. For lightly soluble gases, the solubility is characterized by Henry's law, which states that the partial pressure p of a dissolved gas is proportional to its mole fraction x in the liquid (see Denbigh, 1981, Chapter 8)

$$p = K_H x = K_H \frac{N_{g,d}}{N} \tag{2.1}$$

where K_H is Henry's constant, $N_{g,d}$ the number of moles of dissolved gas in the liquid and N the total number of moles in the liquid phase.

Denoting the molar mass of the live oil as M_o, the total number of moles N in the liquid phase is expressed as

$$N = \frac{\rho_o V_o}{M_o} \tag{2.2}$$

Inserting this in equation (2.1)

$$N_{g,d} = \frac{\rho_o V_o}{K_H M_o} p \tag{2.3}$$

The number of moles in the gas phase can be expressed by the ideal gas law as

$$N_{g,f} = \frac{1}{RT} V_g p \tag{2.4}$$

where R is the ideal gas constant and T the temperature. Combining the fact that the total number of moles in the system is independent of pressure and that at saturation pressure no free gas is present, we obtain summing (2.3) and (2.4)

$$p + \frac{K_H M_o p}{\rho_o RT} \frac{V_g}{V_o} = p_{sat} \tag{2.5}$$

The gas saturation S_g of gas is defined as

$$S_g = \frac{V_g}{V_g + V_o} \tag{2.6}$$

At low gas saturation ($S_g \ll 1$) equation (2.6) approximates

$$\frac{V_g}{V_l} = S_g \tag{2.7}$$

Furthermore we suppose the gas is incompressible (see Appendix A) so that

$$\frac{p M_g}{RT} = \rho_g \tag{2.8}$$

where M_g is the molar mass of the gas and ρ_g the gas density. Substituting (2.7) and (2.8) in (2.5) results

$$p_{sat} - p - \gamma S_g = 0 \tag{2.9}$$

where γ is a reciprocal solubility constant defined by

$$\gamma = \frac{K_H M_o \rho_g}{M_g \rho_o} \tag{2.10}$$

For slightly soluble gases, it follows thus that the gas fraction is a linear function of the pressure. This is valid as well in an oil reservoir, under the assumption that the local molar composition of the mixture of live oil and gas bubbles does not change. The bubbles thus have to move with the oil.

2.2 Dynamic constitutive equation

We propose to describe the evolution of the gas saturation S_g by an evolution equation with linear kinetics

$$a\frac{DS_g}{Dt} + b\frac{Dp}{Dt} = f(p, S_g) \tag{2.11}$$

where a and b are constant parameters and $f(p, S_g)$ is to be determined from experiments. The derivatives in (2.11) are material derivatives defined by

$$\frac{D}{Dt} = \alpha \frac{\partial}{\partial t} + \mathbf{q}_m . \nabla \tag{2.12}$$

where α is the porosity of the medium and $\mathbf{q}_m$ the superficial velocity of mixture flow in the porous medium.

The equilibrium case is defined for

$$f(p, S_g) = 0 \tag{2.13}$$

The equilibrium solubility relation (2.9) can be seen as a degeneration of (2.11). In this case the gas saturation and the dissolved mass fraction of gas are in instantaneous equilibrium with the pressure.

In many cases however, gas cannot come out of solution infinitely fast when pressure is changed, so that equation (2.9) has to be violated. The rate at which gas can come out of solution by nucleation of bubbles and diffusion of gas to these nucleated bubbles might be predicted by a theory of nucleation and diffusion. These theories are however complex and involve several constants which are difficult to quantify. Therefore we suggest the that the dynamic behavior of solubility be described in terms of equation (2.11). In the following we will concentrate on the special case

$$\tau \frac{Dp}{Dt} = f(p, S_g) \tag{2.14}$$

(in appendix B another special case is discussed)

Assuming linear kinetics, we propose then from (2.9) and (2.14)

$$\tau \frac{Dp}{Dt} = p_{sat} - p - \gamma S_g \tag{2.15}$$

which is a rate equation of the Maxwell type.

$Dp/Dt = 0$ when p and S_g are at equilibrium and satisfy (2.9). When p is below this value and S_g at equilibrium, then $dp/dt > 0$ and the pressure will increase to its equilibrium value. When $S_g = 0$, $p - p_{sat}$ relaxes to zero exponentially.

Including of a term proportional to DS_g / Dt would lead to an Oldroyd B type of model (see also appendix B) with a retardation as well as relaxation time.

2.3 Governing equations

In addition to the constitutive equation (2.15), the flow is described by a mass conservation equation for the mixture of gas and liquid

$$\frac{D\rho_m}{Dt} + \rho_m \nabla \cdot \mathbf{q}_m = 0 \tag{2.16}$$

where ρ_m is the mixture density defined by

$$\rho_m = \left(1 - S_g\right)\rho_o + S_g \rho_g \tag{2.17}$$

The mixture velocity $\mathbf{q}_m$, which is the mixture flow rate divided by the total cross sectional area of the porous medium, is supposed to be well represented by Darcy's law

$$\mathbf{q}_m = -\lambda\left(S_g\right)\nabla p \tag{2.18}$$

Here the mobility λ is the ratio between permeability κ and mixture viscosity μ_m

$$\lambda(\phi) = \frac{\kappa}{\mu_m} \tag{2.19}$$

It is not expected that the mixture viscosity μ_m is very different from the oil viscosity μ_o, because the bubbles are assumed small and the gas saturation low. Since the oil viscosity increases as the quantity of dissolved gas decreases, μ_o and thus μ_m are expected to be some increasing function of the gas saturation. Based on this argument, λ would decrease as the gas saturation increases.

If it further assumed that the gas density is small compared to the oil density $\rho_g \ll \rho_o$, then (2.17) can be approximated by

$$\rho_m \approx \left(1 - S_g\right)\rho_o \tag{2.20}$$

After substitution of the definition of the material derivative (2.12), the mixture density (2.20) and the mixture velocity (2.18) in equation (2.15) and (2.16), we obtain

$$\alpha\frac{\partial p}{\partial t} - \lambda|\nabla p|^2 = \frac{p_{sat} - p - \gamma S_g}{\tau} \tag{2.21}$$

$$\alpha\frac{\partial S_g}{\partial t} - \lambda\nabla p \cdot \nabla S_g = \left(1 - S_g\right)\nabla \cdot \left(-\lambda\nabla p\right) \tag{2.22}$$

which are the two fundamental equations of the model.

It is possible to eliminate the gas saturation from the problem, which gives rise to a very non-linear partial differential equation in pressure.

2.4 Dimensionless analysis

In order to make numerical solution of the equations easier and to determine the number of independent constants in the problem, it is convenient to write the equations in

a dimensionless form. Therefore we choose a typical length scale L and a typical pressure difference Δp. The most rigorous choice (the one which does not degenerate for $\tau = 0$) is then given by

$$\nabla^* := L\nabla \tag{2.23a}$$

$$t^* := t\frac{\lambda_0 \Delta p}{\alpha L^2} \tag{2.23b}$$

$$p^* := \frac{p_{sat} - p}{\Delta p} \tag{2.23c}$$

$$S_g^* = \frac{\gamma S_g}{\Delta p} \tag{2.23f}$$

$$\gamma^* = \gamma / \Delta p \tag{2.23d}$$

$$\lambda^*\left(x, S_g\right) = \lambda\left(x, S_g\right) / \lambda_0 \tag{2.23e}$$

where λ_0 is a characteristic mobility, chosen such that $\lambda^*(x) = O(1)$.

The model equations (2.21) and (2.22) can then be written as

$$\frac{\partial p^*}{\partial t^*} + \lambda^* \left|\nabla^* p^*\right|^2 = \frac{1}{N_1}\left(p^* - S_g^*\right) \tag{2.24a}$$

$$\frac{\partial S_g^*}{\partial t^*} + \lambda^* \nabla^* p^* \cdot \nabla^* S_g^* = \left(N_2 - S_g^*\right)\nabla^* \cdot \left(\lambda^* \nabla^* p^*\right) \tag{2.24b}$$

The model involves thus 2 dimensionless numbers:

$$N_1 = \frac{\lambda_0 \Delta p \tau}{L^2} \tag{2.25a}$$

$$N_2 = \frac{\gamma}{\Delta p} \tag{2.25b}$$

the first is a ratio between the time constant for gas mass transfer and a typical transport time and the second is a dimensionless (reciprocal) solubility. Furthermore, the model needs a closure relation for the mobility function λ^*.

In some cases it is convenient to choose

$$t^* = \frac{t}{\alpha\tau} \tag{2.26}$$

which however is not possible for equilibrium solubility considerations when $\tau = 0$. This choice of time scale results in

$$\frac{\partial p^*}{\partial t^*} + N_1 \lambda^* \left|\nabla^* p^*\right|^2 = p^* - S_g^* \tag{2.27a}$$

$$\frac{1}{N_1}\frac{\partial S_g}{\partial t^*} + \nabla^* p^* \cdot \nabla^* S_g^* = \left(N_2 - S_g^*\right)\nabla^* \cdot \left(\lambda^* \nabla^* p^*\right) \tag{2.27b}$$

3 Numerical solution and perturbation analysis

In order to validate the proposed approach we have chosen to perform comparisons between the proposed model, i.e. equations (2.21) and (2.22), and data from laboratory

core flow experiments. It is supposed that the core length is much larger than its diameter and that a one dimensional description in the axial co-ordinate x is sufficient.

A comparison with two types of core flow experiments is performed, which are both described by (2.21) and (2.22), but with different boundary conditions:

1. Core depressurisation of a core closed at one end ($x = 0$) and with given pressure at the other end ($x = L$) (Sheng et al., 1996, Pooladi-Darvish & Firoozabadi, 1997, Firoozabadi, 1992, Huerta et al., 1996);
2. Core through flow with a given pressure gradient over the core (Maini & Sharma, 1994, Maini et al., 1993).

In numerical terms, these two cases correspond to the following boundary conditions:

1. one Neumann boundary condition $\partial p / \partial x|_{x=0} = 0$ and a Dirichlet boundary condition
$$p(L) = p_{sat} - \Delta p(t)$$

2. two Dirichlet boundary conditions: $p(0) = p_{sat}$, $p(L) = p_{sat} - \Delta p$

Several solution strategies are possible, which will de discussed in the following.

3.1 Perturbation analysis

The equations (2.21) and (2.22) admit constant state solutions $p = p_0$ and $S_g = S_{g,0}$ which are solutions of the equilibrium solubility relation (2.9)

$$p^0 + \gamma S_g^0 = p_{sat} \tag{3.1}$$

If we perturb these equilibrium solutions

$$p = p^0 + p' \tag{3.2a}$$

$$S_g = S_g^0 + S_g' \tag{3.2b}$$

and we neglect the terms which are quadratic in the perturbation variables, we obtain

$$\alpha\tau\frac{\partial p'}{\partial t} + p' + \gamma S_g' = 0 \tag{3.3a}$$

$$\alpha\frac{\partial S_g'}{\partial t} + \left(1 - S_g^0\right)\lambda\nabla^2 p' = 0 \tag{3.3b}$$

where $\lambda = \lambda\left(S_g^0\right)$. The gradient cross term $\nabla\lambda \cdot \nabla p'$ has herein bee neglected. We may eliminate p' or S_g' from (3.3); in both cases we find a telegrapher's equation

$$\frac{\partial^2 p'}{\partial t^2} + \frac{1}{\alpha\tau}\frac{\partial p'}{\partial t} = \frac{\left(1 - S_g^0\right)\lambda\gamma}{\tau\alpha^2}\nabla^2 p' \tag{3.4}$$

The same equation is satisfied by S_g'. Equation (3.4) gives rise to a wave equation with a wave speed

$$c = \left(\frac{\left(1 - S_g^0\right)\lambda\gamma}{\tau\alpha^2}\right)^{1/2} \tag{3.5}$$

The waves are damped which is a result of the second term in (3.4). If the relaxation time $\tau \to 0$, then the effects of wave propagation gives way to diffusion

$$\frac{\partial p'}{\partial t} = \frac{(1 - S_g^0)\lambda\gamma}{\alpha} \nabla^2 p' \tag{3.6}$$

Let us consider the solution of the telegraph equation (3.4) for "Stokes 1st problem" (see Joseph, 1990, pp. 582-584). The pressure p' at the boundary $x = 0$ of a semi-infinite region is suddenly raised and held at value p'_0. We must solve (3.4) in one dimension

$$\frac{\partial^2 p'}{\partial t^2} + \frac{1}{\alpha\tau} \frac{\partial p'}{\partial t} = \frac{(1 - S_g^0)\lambda\gamma}{\tau\alpha^2} \frac{\partial^2 p'}{\partial x^2} \tag{3.7}$$

satisfying the initial conditions

$$p'(x,t) = 0 \quad ,t \le 0 \text{ and } x \ge 0 \tag{3.8}$$

and the boundary conditions

$$p'(0,t) = \Delta p H(t) \tag{3.9}$$

where $H(t)$ is Heaviside's step function, and

$$p'(x \to \infty, t) \to 0$$

Equation (3.7) can be written in a dimensionless form, using the wave speed (3.5), a length scale L, time scale $\alpha\tau$ and pressure scale Δp

$$\frac{\partial^2 p^*}{\partial t^{*2}} + \frac{\partial p^*}{\partial t^*} = \left(\frac{c\alpha\tau}{L}\right)^2 \frac{\partial^2 p^*}{\partial x^{*2}} \tag{3.10}$$

Denoting

$$\hat{p}(s,t) = L\left[p^*(x,t)\right] \tag{3.11}$$

and applying the Laplace transform L to (3.7)

$$\left(\frac{c\alpha\tau}{L}\right)^2 \frac{\partial^2 \hat{p}}{\partial x^2} = s(s+1)\hat{p} \tag{3.12}$$

which has solution

$$\left(\frac{c\alpha\tau}{L}\right)^2 \hat{p}(x,s) = \frac{1}{s}\exp\left[-\left\{s(1+s)\right\}^{1/2} x\right] \tag{3.13}$$

From inverse Laplace transform

$$p^*(x^*,t^*) = \left(\frac{L}{c\alpha\tau}\right)^2 L^{-1}\left(\frac{1}{s}\exp\left[-\left\{s(1+s)\right\}^{1/2} x\right]\right) \tag{3.14}$$

A full analytical solution of (3.14) is difficult, although not impossible. Numerical evaluation of (3.14), using the technique of de Hoog et al (1982) can however easily be done by a readily available routine in the MATLAB programming language.

3.2 Steady state solution

The one-dimensional steady state equations obtained from (2.21) and (2.22) are

$$-\lambda\left(\frac{dp}{dx}\right)^2 = \frac{p_{sat} - p - \gamma S_g}{\tau} \tag{3.15}$$

$$\frac{d}{dx}\left\{(1 - S_g)\left(-\lambda\frac{dp}{dx}\right)\right\} = 0 \tag{3.16}$$

The second equation integrates to

$$\frac{dp}{dx} = -\frac{q_0}{\lambda(1 - S_g)} \tag{3.17}$$

where q_0 is a integration constant.

A second integration gives

$$p_{sat} - p = \int_0^x \frac{q_0}{\lambda(1 - S_g)} dx \tag{3.18}$$

Substituting (3.17) and (3.18) in (3.15) we obtain

$$\gamma S_g - \frac{\tau q_0^2}{\lambda(1 - S_g)^2} = \int_0^x \frac{q_0}{\lambda(1 - S_g)} dx \tag{3.19}$$

which differentiates to

$$\left\{ \gamma - \frac{2\tau q_0^2}{\lambda(1 - S_g)^3} \right\} \frac{dS_g}{dx} = \frac{q_0}{\lambda(1 - S_g)} \tag{3.20}$$

Using the oil saturation instead of the gas saturation $S_o = 1 - S_g$ and integrating

$$\int_1^{S_o} \left\{ \gamma \lambda S_o - 2\tau q_0^2 S_o^{-2} \right\} dS_o = -\int_0^x q_0 dx \tag{3.21}$$

which leads to the result

$$S_0^3 - \left\{ 1 - \frac{4q_0^2 \tau}{\lambda \gamma}(1 + x^*) \right\} S_o + \frac{4q_0^2 \tau}{\lambda \gamma} = 0 \tag{3.22}$$

where $x^* := x / (2\tau q_0)$

This third order polynomial has an analytical solution.

3.3 Equilibrium solution

If the gas can come out of solution infinitely fast, then $\tau = 0$ and the equilibrium solubility (2.9) replaces the constitutive equation (2.14). Equation (2.9) states that the gas saturation is high where the pressure is low. The bubbles and the oil flow from regions with higher pressure to regions with lower pressure and thus to regions with higher gas saturation. From (2.9) the gas saturation can be expressed in the pressure as

$$S_g = \frac{p_{sat} - p}{\gamma} \tag{3.23}$$

Substituting this expression in (2.22) a non-linear diffusion equation is obtained

$$-\alpha \frac{\partial p}{\partial t} + \lambda |\nabla p|^2 = (\gamma + p - p_{sat}) \nabla \cdot (-\lambda \nabla p) \tag{3.24}$$

Prof. G.I. Barenblatt has noted that (3.24) can be rearranged in

$$\alpha \frac{\partial p}{\partial t} = \nabla \cdot (D(p)\nabla p) \tag{3.25}$$

with diffusion coefficient

$$D(p) = \lambda(\gamma + p - p_{sat}) \tag{3.26}$$

If we define

$$\hat{p} := \gamma + p - p_{sat} \tag{3.27a}$$

$$\psi := \int_{\hat{p}} \lambda(x) \hat{p} d\hat{p} \tag{3.27b}$$

then (3.25) can be written as

$$\alpha \frac{\partial \hat{p}}{\partial t} = \nabla^2 \psi \tag{3.28}$$

which for constant λ simplifies to the well-known Boussinesq equation (Kalashnikov, 1987)

$$\alpha \frac{\partial \hat{p}}{\partial t} = \frac{\lambda}{2} \nabla^2 \hat{p}^2 \tag{3.29}$$

... complete numerical method description and insert solutions ...

3.4 Full numerical solution

... all this work still has to be done....

4 Results

4.1 Perturbation analysis

The solutions of the perturbation equations in the dimensionless co-ordinates $p^* = (p / \Delta p)(L / c\alpha\tau)^2$ and $x^* = x / L$ do not depend on any parameters other than those embedded in the non-dimenzionalisation. The solution of (3.14) is shown in figure 5.1.
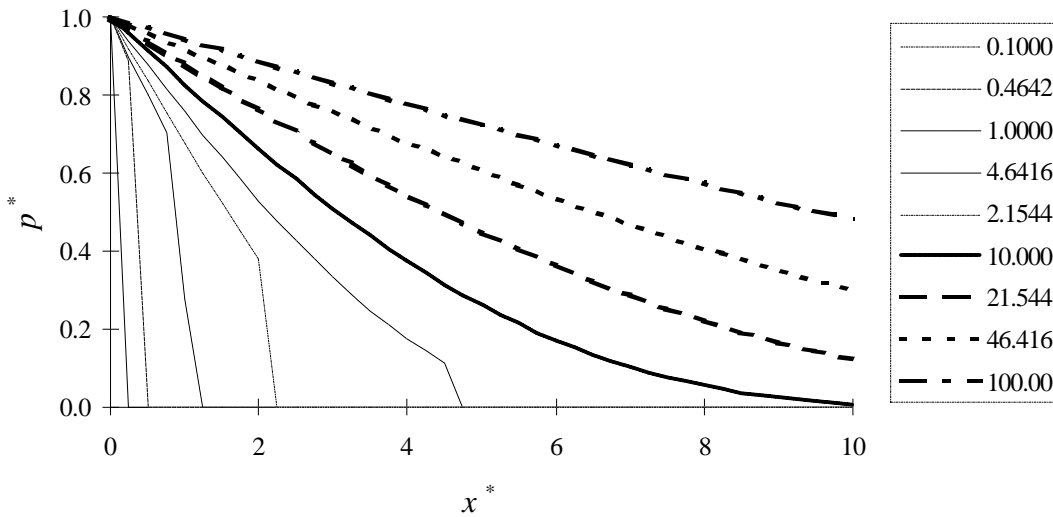

Figure 5.1   Numerical evaluation of equation (3.14): pressure versus distance in the core at different times; a discontinuity in the derivative propagates into the core

The pressure disturbance propagates into the core, resulting in a discontinuity in the space derivative of the pressure which propagates at a speed c and an attenuation given by

$$p'(x, x / c) = p'_0 \exp\left(\frac{\alpha\tau x}{2c}\right) \tag{4.1}$$

If the result is plotted versus a similarity co-ordinate $x^*/\sqrt{t^*}$, we observe that at large times a similarity solution can be found.
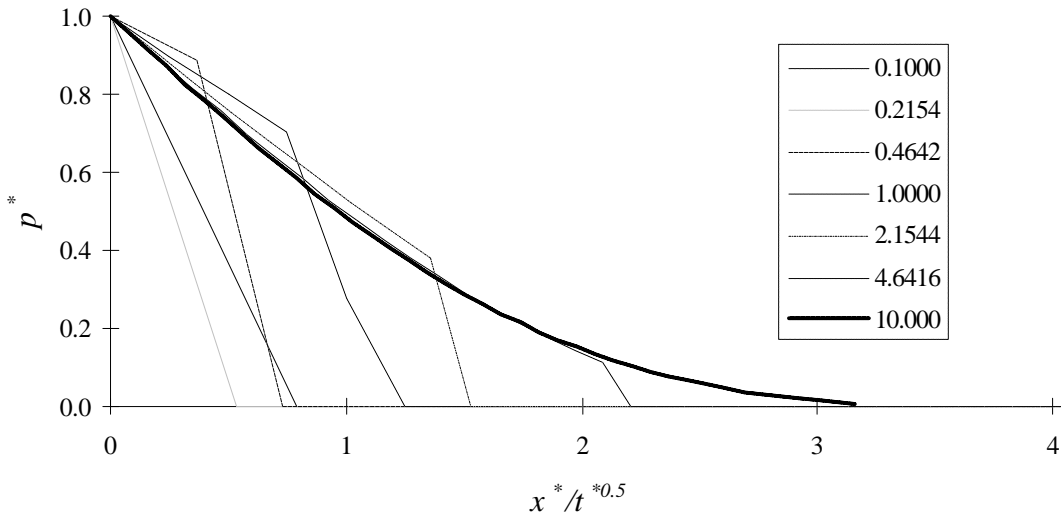

Figure 5.2 Numerical evaluation of equation (3.14): pressure versus similarity co-ordinate $x^*/\sqrt{t^*}$ at different times; at large times a similarity solution can be observed

4.2 Steady state solution

4.3 Equilibrium solution

4.4 Full numerical solution

5 Conclusions

6 Appendix A: Neglect of gas compressibility

If gas compressibility is taken into account, then (2.5) writes for low gas saturation

$$S_g = \frac{\rho_o RT}{K_H M_o p}(p_{sat} - p) \tag{A.1}$$

It follows that the relative change in void fraction, if pressure is changed, is

$$\frac{\partial S_g / \partial p}{S_g} = -\frac{1}{p} - \frac{1}{p_{sat} - p} \tag{A.4}$$

The first term on the right hand side gives the influence of compressibility and the second term on the right hand side the influence of gas coming out of solution. It follows thus that the criterion for neglect of compressibility is

$$\frac{p_{sat} - p}{p} \ll 1 \tag{A.5}$$

so that the pressure should be $p = O(p_{sat})$.

7 Appendix B: an alternative form of the constitutive equation

A general constitutive law of type

$$a\frac{DS_g}{Dt} + b\frac{Dp}{Dt} = f(p, S_g) \tag{B.1}$$

has been assumed in section 2.2, where the special case $a = 0$ has been investigated in detail. If we choose $b = 0$, equation (B.1) can be written as

$$\tau\gamma\frac{DS_g}{Dt} + \gamma S_g = p_{sat} - p \tag{B.2}$$

which is a statement that the volume fraction relaxes toward the equilibrium value, with time constant τ. Using definition (2.12) of the material derivative in (B.2) yields

$$\tau\gamma\frac{\partial S_g}{\partial t} - \tau\gamma\lambda\nabla p \cdot \nabla S_g + \gamma S_g = p_{sat} - p \tag{B.2}$$

Equations (B.2) and (2.22) are the basic governing equations in this form of the theory. Clearly, when $\tau = 0$, we recovery exactly the same equilibrium theory as was derived above. It is also clear that this version of the theory supports steady uniform states as solution, again exactly as found above. The governing equations for small perturbations around these steady states are however different, as we shall now show. As before, introduce small perturbation quantities defined by (3.2) where equilibrium demands that (3.1) holds. Substituting these quantities into (B.2) and (2.22), neglecting quantities of second order in perturbations and treating the mobility as a constant, we obtain

$$\alpha\tau\gamma\frac{\partial S_g'}{\partial t} + \gamma S_g' = -p' \tag{B.3}$$

$$\alpha\frac{\partial S_g'}{\partial t} + (1 - S_{g,0})\lambda_1\nabla^2 p' = 0 \tag{B.4}$$

where $\lambda_1 = \lambda_1(S_g^0)$. If we now take the time derivative of (B.3), and use (B.4) to replace $\partial S_g'/\partial t$ by the Laplacian of p', we find

$$\alpha\left\{1 - \tau(1 - S_g^0)\lambda_1\gamma\nabla^2\right\}\frac{\partial p'}{\partial t} = (1 - S_g^0)\lambda_1\gamma\nabla^2 p' \tag{B.5}$$

The perturbation volume fraction satisfies an identical equation. Initial and boundary conditions on the perturbation pressure are the same here as for the model presented above.

The evolution equation for the perturbation pressure, (B.5), is very different in character from the telegraph equation found for the pressure relaxation form of the constitutive law. It is lower order in time, and, from Laplace transform solutions, appears not to exhibit wave propagation; rather, the solutions are diffusive in character although similarity solutions in $x/t^{1/2}$ do not exist (except as an asymptotic state at large t). The rate of advance of the pressure perturbation front is found from these Laplace transform solutions to be faster in early time than would be the case for classical diffusion.

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